OPTIMAL OBSERVATION DURING CONTROL OF DYNAMIC SYSTEMS UNDER UNCERTAINTY

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Abstract: A linear uncertain system with bounded unknown initial states, disturbances and errors of a sensor is investigated. Optimal observation problems are introduced the solutions of which give estimates sufficient to construct in real-time optimal guaranteeing controls to the terminal control problem under uncertainty. The algorithms suggested are based on the previous results of the authors on fast methods of optimization and develop them on new situations. Results are illustrated on the example of optimal control for the forth order system.

Keywords: uncertain linear systems, optimal estimation, algorithms, on-line control.

1. INTRODUCTION

Theory of optimal guaranteed control of systems under uncertainties is one of the actual and important for applications fields of control theory (see the monographs and the paper by N.N. Krasovskij (1985) and references in them). This paper emphasizes on optimal observation problems arising during optimal control of linear time-varying systems under set-membership uncertainty. Linear, piecewise linear optimal observation problems are introduced to be used in control processes. Finite algorithms of the solution of optimal observation problems are justified. The scheme of solving the optimal on-line control problem to linear time-varying systems is suggested. The problem of complexity of algorithms is under discussion. The algorithms suggested are based on the results of the authors and their collaborators (Gabasov *et al.*, 1995, 2000a) on constructive theory of optimization and develop them on new problems.

The paper is organized as follows. Optimal control problems under uncertainty and classical approach to construction of optimal guaranteeing feedbacks are discussed in Section 2. Optimal online control principle for the problem under consideration is introduced in Section 3. It is shown that the problem with uncertainties decomposes into a determined problem of optimal control and optimal observation problems. The algorithm for solving the latter is presented in Section 4. The methods of reducing the large a priory uncertainty are discussed in Section 5. Section 6 presents a numerical example for the forth order optimal control problem under uncertainty.

2. OPTIMAL CONTROL PROBLEMS UNDER UNCERTAINTY

On the interval $T = [t_*, t^*]$ consider the system

$$\dot{x} = A(t)x + B(t)u + M(t)w. \qquad (1)$$

Here $A(t) \in R^{n\times n}$, $B(t) \in R^{n\times r}$, $M(t) \in R^{n\times n_w}$, $t \in T$, are given piecewise continuous matrix functions; $x = x(t) \in R^n$ is a state of the control system at an instant t; $u = u(t) \in U \subset R^r$ is a value of a discrete control with a quantization period h: $u(t) \equiv u(\tau), t \in [\tau, \tau + h[, \tau \in T_h = \{t_*, t_* + h, \ldots, t^* - h\}$, $(h = (t^* - t_*)/N$, $(N > 0)$, $U = \{u \in R^r : u_* \leq u \leq u^*\}$ is a bounded set of

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admissible values of controls; $w = w(t)$, $t \in T$, is a disturbance function

$$w(t) = \Lambda(t)v, t \in T,$$

where $\Lambda(t) \in R^{n_w \times n_v}$ is a given piecewise continuous matrix function; $v \in R^{n_v}$ is a vector of disturbance parameters taking values from a bounded set $V = \{v \in R^{n_v} : w_* \le v \le w^*\}$.

Assume the initial state $x(t_*)$ of system (1) is an unknown element of a bounded set $X_0 \subset R^n$. Let

$$X_0 = x_0 + GZ,$$

where $x_0 \in R^n$, $G \in R^{n \times n_z}$ are known vector and matrix; $Z = \{z \in R^{n_z} : \ d_* \le z \le d^*\}$ is a set of unknown parameters z of the initial state $x(t_*)$.

Let a sensor (measurer) be of the form

$$y(\theta) = \int\limits_{\theta-h}^{\theta} C(t)x(t)dt + \xi(\theta), \ \theta \in T_h \setminus t_*, \quad (2)$$

where $C(t) \in R^{q\times n}$, $t \in T$, is a given piecewise continuous matrix function; $y(\theta) \in R^q$, $\theta \in T_h \setminus t_*$, is an output signal of the measurer; $\xi(\theta) \in R^q$, $\theta \in T_h \setminus t_*$, are measurement errors satisfying

$$\xi_* \le \xi(\theta) \le \xi^*, \theta \in T_h \setminus t_*; \quad 0 < \|\xi^* - \xi_*\| < \infty.$$

Discrete closed-loop control of system (1) is performed in the following way. On the interval $[t_*, t_* + h[$ the control function $u(t) \equiv u(t_*)$, $t \in [t_*, t_* + h[$, is fed to the input of system (1), where $u(t_*) \in U$ is chosen upon a priori information. At the instant $\tau = t_* + h$ measurer (2) measures the first signal $y(t_* + h)$, generated by the realized initial state $x(t_*)$, the error $\xi(t_*)$ and the disturbance $w(t)$, $t \in [t_*, t_*+h[$. Based on the signal $y(t_*+h)$ a vector $u(t_*+h) = u(t_*+h, y(t_*+h)) \in U$ is chosen following the rules selected in advance (before the process starts). The control function $u(t) \equiv u(t_* + h)$ is fed into system (1) for $t \in [t_* + h, t_* + 2h[$. This control and the realized disturbance $w(t)$, $t \in [t_* + h, t_* + 2h[$, steer the system into the state $x(t_* + 2h)$, and together with the error $\xi(t_* + 2h)$ generate the output signal $y(t_* + 2h)$. At an arbitrary moment $\tau \in T_h \setminus t_*$ based on the measured output signal $y(\tau)$, a vector $u(\tau) = u(\tau, y_\tau(\cdot)) \in U$ is chosen and the control function $u(t) \equiv u(\tau)$, $t \in [\tau, \tau + h[$, is fed into the control system. Here $y_\tau(\cdot) = (y(\theta), \theta \in T_h(\tau))$, $T_h(\tau) = \{t_* + h, t_* + 2h, \ldots, \tau\}$.

Let Y_τ be a set of all output signals $y_\tau(\cdot)$ of (2) that can be obtained by the moment τ.

Definition. A functional

$$u = u(\tau, y_\tau(\cdot)), \ y_\tau(\cdot) \in Y_\tau, \ \tau \in T_h \setminus t_*, \quad (3)$$

and the control functions $u(t, y_t(\cdot)) \equiv u(\tau, y_\tau(\cdot))$, $t \in [\tau, \tau + h[$, $\tau \in T_h \setminus t_*$, generated by (3), are said to be (discrete) feedback controls.

Let $X(t^*|u_{t^*}(\cdot, y_{t^*}(\cdot)))$ be a set of all terminal states of the closed system

$$\dot{x} = A(t)x + B(t)u(t_*) + M(t)w, \ t \in [t_*, t_* + h[;$$

$$\dot{x} = A(t)x + B(t)u(t, y_t(\cdot)) + M(t)w, \ t \in [t_* + h, t^*];$$

with all possible initial states $x(t_*)$, disturbances $w(t)$, $t \in T$, and errors $\xi(\tau)$, $\tau \in T_h \setminus t_*$, able to generate the signal $y_{t^*}(\cdot)$.

Introduce a terminal set $X^* = \{x \in R^n : g_* \le Hx \le g^*\}$, where $H \in R^{m\times n}$, $g_* < g^*$ are given.

Feedback control (3) is called admissible if $X(t^*|u_{t^*}(\cdot|y_{t^*}(\cdot)) \subset X^*$.

Let the quality of admissible control (3) is evaluated by a functional

$$J(u) = \min c'x, x \in X(t^*|u_{t^*}(\cdot|y_{t^*}(\cdot)) \quad (c \in R^n).$$

Definition. An admissible feedback $u^0(\tau, y_\tau(\cdot))$, $y_\tau(\cdot) \in Y_\tau$, $\tau \in T_h \setminus t_*$, is said to be optimal if $J(u^0) = \max J(u)$, where maximum is calculated over all admissible feedbacks (3).

According to the definition, optimal feedback provides the best result under the worse conditions (optimal guaranteed result).

3. OPTIMAL ON-LINE CONTROL FOR SYSTEMS UNDER UNCERTAINTY

Describe optimal on-line control principle for a concrete control process where a signal $y^*(\theta)$, $\theta \in T_h$, would realize. The control process starts at the moment $\tau = t_*$ with the control $u^{**}(t) = u^0(t_*)$, $t \ge t_*$, where $u^0(t)$, $t \in T$, is an optimal open-loop control constructed on the a priori information (rules are presented below). At the instant $\tau = t_* + h$ the first measurement $y^*(t_* + h)$ is obtained. On the base of this measurement a control signal $u^*(t_* + h)$ is calculated (see below) in time $s(t_* + h) < h$. The control function $u^{**}(t) = u^0(t_*)$ is fed into the control system on the interval $[t_* + h, t_* + h + s(t_* + h)[$. Starting from the moment $t_* + h + s(t_* + h)$ the control function switches on $u^{**}(t) = u^*(t_* + h)$.

At arbitrary τ the control function

$$u^{**}(t) = u^0(t_*), \ t \in [t_*, t_* + h + s(t_* + h)[;$$

$$u^{**}(t) = u^*(\vartheta), \ t \in [\vartheta + s(\vartheta), \vartheta + h + s(\vartheta + h)[,$$

$$\vartheta \in \{t_* + h, t_* + 2h, \ldots, \tau - 2h\};$$

$$u^{**}(t) = u^*(\tau - h), \ t \in [\tau - h + s(\tau - h), \tau[;$$

has been fed into the input of (1) and a current measurement $y^*(\tau)$ is obtained. The calculation of the control signal $u^*(\tau) = u^{00}(\tau, y^*_\tau(\cdot))$ is required to be made in time $s(\tau) < h$. Before it is calculated the previous signal $u^*(\tau - h)$ is fed into the system.

To describe the rules for the calculation of $u^*(\tau)$ let us present the signal $y^*(\tau)$ in the form

$$y(\tau) = \int_{\tau-h}^{\tau} C(t)(x_w(t) + x_u(t))dt + \xi(\tau),$$

where $x_w(t)$, $t \in [t_*, \tau]$, is a trajectory of

$$\dot{x} = A(t)x + M(t)w,\ x(t_*) = Gz, \quad (4)$$

$x_u(t)$, $t \in [t_*, \tau]$, is a trajectory of

$$\dot{x} = A(t)x + B(t)u,\ x(t_*) = x_0, \quad (5)$$

with $u(t) \equiv u^{**}(t), t \in [t_*, \tau[$.

Subtract the known trajectory $x_u(t)$, $t \in [\tau-h, \tau]$, from the signal $y^*(\tau)$:

$$y_0^*(\tau) = y^*(\tau) - \int_{\tau-h}^{\tau} C(t)x_u(t)dt.$$

Due to the performance of that operation at every moment $\theta \in T_h(\tau)$ the signal $y_{0\tau}^*(\cdot) = (y_0^*(\theta), \theta \in T_h(\tau) \setminus t_*)$ is available by the moment τ. It coincides with the signal that would be obtained by measurer (2) for (4). The signal $y_{0\tau}^*(\cdot)$ represents additional information about the parameter vector realized in the process. This information is contained in an a posteriori distribution set.

Definition. A set $\hat{\Gamma}(\tau) = \hat{\Gamma}(\tau; y_{0\tau}^*(\cdot))$ is said to be the a posteriori distribution set of parameters if and only if it consists of vectors $\gamma = (z, v) \in \Gamma = Z \times V$, to which there correspond the initial state $x(t_*) = Gz$ of (4) and the disturbance $w(t) = \Lambda(t)v$, $t \in [t_*, \tau[$, able together with some errors $\xi(\theta)$, $\theta \in T_h(\tau)$, to produce the signal $y_{0\tau}^*(\cdot)$.

Definition. A control $u^\tau(\cdot) = (u(t), t \in [\tau, t^*])$ is said to be an admissible open-loop control if for every $\gamma \in \hat{\Gamma}(\tau)$ at the moment t^* it together with $u^{**}(t)$, $t \in [t_*, \tau[$, steers system (1) to X^*, i.e.

$$g_{*i} \le \min h'_{(i)}(x_w(t^*) + x_u(t^*)); \quad (6)$$
$$\max h'_{(i)}(x_w(t^*) + x_u(t^*)) \le g_i^*;\ i = \overline{1, m};$$

where $h_{(i)}$ is the i-th row of the matrix H, g_i^*, g_{*i} are the i-th components of g^*, g_*; $x_u(t^*)$ is a terminal state of (5) under $u(t) = u^\tau(t)$, $t \in [\tau, t^*[$.

Let $\hat{X}_w^*(\tau)$ be a set of all terminal states $x_w(t^*)$ of system (4) generated by $(z, v) \in \hat{\Gamma}(\tau)$.

Extremal problems arising in (6)

$$\chi_i^*(\tau) = \max h'_{(i)}x,\ x \in \hat{X}_w^*(\tau),\ i = \overline{1, m}; \quad (7)$$
$$\chi_{*i}(\tau) = \min h'_{(i)}x,\ x \in \hat{X}_w^*(\tau),\ i = \overline{1, m};$$

are called optimal observation problems accompanying the optimal control problem under uncertainty (*the accompanying optimal observation problems*).

Thus, for the control $u^\tau(\cdot)$ to be admissible in the position $(\tau, y_{0\tau}^*(\cdot))$ it is necessary and sufficient that at the moment it steers determined system (5) with the initial condition $x(\tau) = x_u(\tau)$ to the set $X^*(\tau) = \{x \in R^n : g_*(\tau) \le Hx \le g^*(\tau)\}$, where $g_*(\tau) = g_* - \chi_*(\tau)$, $g^*(\tau) = g^* - \chi^*(\tau)$. The quality of the admissible control $u^\tau(\cdot)$ is evaluated by $I(u) = \min c'x(t^*)$, $\gamma \in \hat{\Gamma}(\tau)$. Thus, the optimal open-loop control $u^{\tau 0}(\cdot) = u^0(t|\tau, y_\tau^*(\cdot))$, $t \in [\tau, t^*]$, is a solution to the problem

$$c'x(t^*) \to \max, \quad (8)$$
$$\dot{x} = A(t)x + B(t)u,\ x(\tau) = x_u(\tau),$$
$$x(t^*) \in X^*(\tau),\ u(t) \in U, t \in [\tau, t^*].$$

Problem (8) is called the determined problem of optimal control accompanying the optimal control problem under uncertainty (*the accompanying optimal control problem*).

Let $u^*(\tau) = u^0(\tau|\tau, y_\tau^*(\cdot))$. On the interval $[\tau, \tau + h[$ the following control function is fed into the input of system (1):

$$u^{**}(t) = u^*(\tau - h),\ t \in [\tau, \tau + s(\tau)[;$$

$$u^{**}(t) = u^*(\tau),\ t \in [\tau + s(\tau), \tau + h[.$$

The optimal open-loop control $u^0(t)$, $t \in T$, introduced above, is a solution to

$$c'x(t^*) \to \max,$$

$$\dot{x} = A(t)x + B(t)u,\ x(t_*) = x_0,$$

$$g_* - \chi_{*i}(t_*) \le Hx(t^*)g^* - \chi_i^*(t_*);$$

where

$$\chi_{*i}(t_*) = \min h'_{(i)}x,\ x \in X_w^*(t_*), i = \overline{1, m};$$

$$\chi_i^*(t_*) = \max h'_{(i)}x,\ x \in X_w^*(t_*), i = \overline{1, m};$$

$X_w^*(t_*)$ is a set of all terminal states $x_w(t^*)$ of system (4) for all possible parameters $(z, v) \in \Gamma$.

According to the scheme presented, to construct a control signal $u^*(\tau)$ one has to solve: 1) 2m accompanying optimal observation problems (7); 2) one accompanying optimal control problem (8).

Definition. A device solving the accompanying optimal observation problem is called an optimal estimator; a device solving the accompanying optimal control problem is called an optimal controller.

If the time $s(\tau)$ needed by optimal estimators and the optimal controller to solve problems (7) and (8) is less than h, then they are suitable for optimal on-line control for the system under uncertainty. The algorithm for operating the optimal controller is elaborated in (Gabasov *et al.*, 2000b).

4. SOLUTION OF THE ACCOMPANYING OPTIMAL OBSERVATION PROBLEM

Consider the problem

$$\chi_*(\tau) = \min p'x, \ x \in \hat{X}^*_w(\tau), \tag{9}$$

which includes accompanying optimal observation problems (7) and is called a *linear* optimal observation problem.

Problem (9) is equivalent to the linear programming problem

$$p'_z z + p'_v v \to \max, \tag{10}$$
$$\xi_* \le y^*_0(\theta) - D(\theta)z - H(\theta)v \le \xi^*, \ \theta \in T_h(\tau),$$
$$d_* \le z \le d^*, \quad w_* \le v \le w^*;$$

where $p'_z = p'F(t^*)$, $p'_v = p'P(t^*)$,

$$D(\theta) = \int_{\theta-h}^{\theta} C(t)F(t)dt, \ H(t) = \int_{\theta-h}^{\theta} C(t)P(t)dt;$$
$$F(t), t \in T : \dot{F} = A(t)F, \ F(t_*) = G; \tag{11}$$
$$P(t), t \in T : \ \dot{P} = A(t)P + M(t)\Lambda(t), P(t_*) = 0.$$

Problem (10) has $q(\tau - t_*)/h + q$ general constraints and $n_z + n_v$ variables. Taking into account that number of the general constraints tends to infinity at $h \to 0$, one can call problem (10) a semi-large extremal problem (Kortanek *et al.*, 1993). The algorithm for operating the suitable optimal estimator is based on a realization of the dual adaptive method (Gabasov *et al.*, 1995) of linear programming where maximal attention is paid to the peculiarities of problem (10) arising due to its dynamical nature.

The main tool of the dual method is a support which is a totality $K_b = K_b(\tau) = \{Q_b; J_b, L_b\}$ such that for the non-empty subsets $Q_b \subset Q = K \times T_h(\tau)$, $K = \{1, 2, \ldots, q\}$; $J_b \subset J = \{1, 2, \ldots, n_z\}$, $L_b \subset L$, $|Q_b| = |J_b| + |L_b|$ the matrix (D_b, H_b) is nonsingular:

$$D_b = \begin{pmatrix} -d_{kj}(\theta), & j \in J_b \\ \{k, \theta\} \in Q_b & \end{pmatrix},$$
$$H_b = \begin{pmatrix} -h_{kl}(\theta), & l \in L_b \\ \{k, \theta\} \in Q_b & \end{pmatrix},$$

where $d_{kj}(\theta)$ is a k,j-th element of the matrix $D(\theta)$, $h_{kl}(\theta)$ is a k,l-th element of the matrix $H(\theta)$. If Q_b, J_b, L_b are empty subsets, then K_b is an empty support by definition.

Along with the support K_b the accompanying elements are employed:
1. The function of the Lagrange multipliers $\nu(\theta) \in R^q$, $\theta \in T_h(\tau)$: $\nu_k(\theta) = 0$, $\{k, \theta\} \in Q_n = Q \setminus Q_b$; $\nu_b = (\nu_k(\theta), \{k, \theta\} \in Q_b)$ is a solution to

$$\nu'_b D_b = p'_{x\ b}, \ \nu'_b H_b = p'_{w\ b};$$

where $p_{x\ b} = (p_{xj}, j \in J_b)$, $p_{w\ b} = (p_{wl}, l \in L_b)$; ($\nu(\theta) = 0$, $\theta \in T_h(\tau)$, for the support $K_b = \emptyset$).

2. The support gradient vectors

$$\delta'_z = p'_z + \sum_{\theta \in T_h(\tau)} \nu'(\theta)D(\theta),$$
$$\delta'_v = p'_v + \sum_{\theta \in T_h(\tau)} \nu'(\theta)H(\theta).$$

The support components of the support gradient vectors are zeroes: $\delta_{zj} = 0, j \in J_b$; $\delta_{vl} = 0, l \in L_b$.

3. The vector of pseudoparameters of the initial state æ and the vector of pseudoparameters of the disturbance ω. Define nonsupport components $æ_j, j \in J_n = J \setminus J_b$; $\omega_l, l \in L_n = L \setminus L_b$ as

$$æ_j = d_{*j}, \text{if } \delta_{zj} < 0; æ_j = d^*_j, \text{if } \delta_{zj} > 0;$$
$$æ_j \in [d_{*j}, d^*_j], \text{if } \delta_{zj} = 0; \ j \in J_n;$$
$$\omega_l = w_{*l}, \text{if } \delta_{vl} < 0; \omega_l = w^*_l, \text{if } \delta_{vl} > 0;$$
$$\omega_l \in [w_{*l}, w^*_l], \text{if } \delta_{vl} = 0; \ l \in L_n.$$

The support components $æ_b = (æ_j, j \in J_b)$, $\omega_b = (\omega_l, l \in L_b)$ are obtained from the equation

$$D_b æ_b + H_b \omega_b = (\zeta_k(\theta) - \zeta_{0k}(\theta), \ \{k, \theta\} \in Q_b),$$

where

$$\zeta_k(\theta) = \xi_{*k}, \text{if } \nu_k(\theta) < 0; \ \zeta_k(\theta) = \xi^*_k, \text{if } \nu_k(\theta) > 0;$$
$$\zeta_k(\theta) \in [\xi_{*k}, \xi^*_k], \text{if } \nu_k(\theta) = 0; \ \{k, \theta\} \in Q_b,$$
$$\zeta_0(\theta) = y^*_0(t) - \int_{\theta-h}^{\theta} C(t)æ_0(t)dt, \ \theta \in T_h(\tau),$$

$æ_0(t)$, $t \in [t_*, \tau]$, is a trajectory of (4) with $z = (z_j = 0, j \in J_b; z_j = æ_j, j \in J_n)$; $v = (v_l = 0, l \in L_b; v_l = \omega_l, l \in L_n)$.

4. The function of pseudoerrors

$$\zeta(\theta) = y^*_0(\theta) - \int_{\theta-h}^{\theta} C(t)æ(t)dt, \ \theta \in T_h(\tau),$$

where $æ(t)$, $t \in [t_*, \tau]$, is a trajectory of (4) with $\gamma = (æ, \omega)$.

Definition. A support K_b is said to be optimal if there exist accompanying elements such that

$$d_{*j} \le æ_j \le d^*_j, \ j \in J_b; \quad w_{*l} \le \omega_l \le w^*_l, \ l \in L_b;$$
$$\xi_{*k} \le \zeta_k(\theta) \le \xi^*_k, \{k, \theta\} \in Q_n.$$

These accompanying elements give a solution to problem (9): $z^0 = æ$, $v^0 = \omega$. The optimal estimator constructs the estimate $\chi^* = p'_z æ + p'_v \omega$, which is used by the optimal controller.

The dual method elaborated is a finite iterative constructing the optimal support $K^0_b = K^0_b(\tau)$ to optimal observation problem (9). The main operations follow the scheme proposed in (Gabasov *et al.*, 2002). According to it, in the course of every iteration the extreme points of the function $\zeta(t)$,

$t \in T_h(\tau)$, move. On the intervals of these movements primal systems (11) are integrated and the length of the intervals define a complexity of the iteration (following Fedorenko R.P. we consider that one integration on the whole T as a unit of complexity).

Take the optimal support $K_b^0(\tau - h)$, obtained at the moment $\tau - h$ ($K_b^0(t_*) = \emptyset$), as an initial support for solving problems (7) at the moment $\tau \in T_h$. Due to the optimality of $K_b^0(\tau - h)$ its accompanying elements satisfy the inequalities $d_* \leq æ \leq d^*$, $w_* \leq \omega \leq w^*$, $\xi_* \leq \zeta(\theta) \leq \xi^*$, $\theta \in T_h(\tau - h)$.

Having obtained $y^*(\tau)$ calculate $\zeta(\tau)$. If $\xi_* \leq \zeta(\tau) \leq \xi^*$, then $K_b^0(\tau - h)$ is optimal also for the moment τ. Otherwise the violation of the constraint at the moment τ is small and small corrections (small movements of the extreme points of $\zeta(\theta)$, $\theta \in T_h(\tau)$) for $K_b^0(\tau - h)$ are to be performed to construct an optimal support $K_b^0(\tau)$. The results on complexity of such corrections for a numerical example are given in Section 6.

5. TWO-STAGE ALGORITHM OF ON-LINE CONTROL AND PIECEWISE LINEAR OPTIMAL OBSERVATION PROBLEMS

If a priori uncertainty of system (1) is large, it is impossible to perform the procedure of optimal on-line control from Section 3 as there are no admissible controls in (8) until some moment τ_*. Keeping in mind that the uncertainty diminishes in the course of control process, for $\tau < \tau_*$ it is reasonable to use some auxiliary problem that has admissible controls. This idea can be realized by two stage algorithm. The aim of the first stage is to construct admissible controls. The optimal real-time control to the initial problem are calculated at the second stage.

The first stage ($\tau \in T_h$, $\tau < \tau_*$) solves a problem

$$\rho(\tau) = \min \rho, \tag{12}$$
$$\dot{x} = A(t)x + B(t)u,\ x(\tau) = x_0^*(\tau),$$
$$x(t^*) \in X_\rho^*(\tau),\ u(t) \in U, t \in [\tau, t^*].$$

where $X_\rho^*(\tau) = \{x \in R^n : g_*(\tau) - \rho e \leq Hx \leq g^*(\tau) + \rho e\}$ ($e = (1, \ldots, 1) \in R^m$).

Problem (12) is equivalent to

$$\rho(\tau) = \min \rho,\ q + \hat{X}_w^*(\tau) \subset X_\rho^*(\tau), \tag{13}$$

where $q \in R^n$ is a vector from a set of reachability of(8).

Problem (13) is called *piecewise linear* optimal observation problem. The algorithm for its solution combines solving $2m$ linear optimal observation problems (Section 4) and optimal control problem (Gabasov *et al.*, 2000b).

If $\rho(\tau) \leq 0$ is a solution to (12), then $\tau = \tau_*$ and a switch for the second stage where problem (8) is solved, is performed. Otherwise the problem

$$c'x(t^*) \to \max, \tag{14}$$
$$\dot{x} = A(t)x + B(t)u,\ x(\tau) = x_0^*(\tau),$$
$$x(t^*) \in X_{\rho+\varepsilon}^*,\ u(t) \in U, t \in [\tau, t^*],$$

is solved. The first part of its optimal open-loop control $u_{\rho+\varepsilon}^0(t)$, $t \in [\tau, t^*]$, is used by the optimal controller on the interval $[\tau + s(\tau), \tau + h + s(\tau + h)[$: $u^{**}(t) = u_{\rho+\varepsilon}^0(\tau)$, $t \in [\tau + s(\tau), \tau + h + s(\tau + h)[$.

Note that in (14) $\varepsilon > 0$ is a small value such that the time of correction of the solution of (14) to optimal open-loop control of (8) is less than h.

Remark. To improve the observation process (so that the moment τ_* happens earlier) the additional information about measurements can be used. In the paper we investigated the cases such as inertial errors ($\xi_*^1 \leq (\xi(t + h) - \xi(t))/h \leq \xi^{*1}, t \in T_h \setminus t^*$) and finite parametric noisy errors ($\xi(\theta) = \sum_{s=1}^{s^*} \xi_s \chi_s(\theta) + \xi_0(\theta)$, where ξ_s, $s \in S = \{1, 2, \ldots, s^*\}$, are unknown parameters; $\chi_s(\theta)$, $s \in S$, $\theta \in T_h$, are known functions; $\xi_0(\theta)$, $\theta \in T$, is an unknown piecewise continuous function which characterizes noise, $\xi_{*s} \leq \xi_s \leq \xi_s^*, s \in S$; $\xi_{*0} \leq \xi_0(\theta) \leq \xi_0^*, \theta \in T_h$). In both situations the solution of optimal observation problems results in linear estimates for the a posteriori distribution of unknown parameters. The algorithms for solving these problems can be constructed as dynamical realization of the adaptive method. The new elements such as points of inflection of the function $\zeta(t)$, $t \in T_h(\tau)$, in case of inertial errors, or new parameters ξ_s, $s \in S$, are to be taken into account.

6. EXAMPLE

On the interval $T = [0, 15]$ consider the system

$$\ddot{x} = -2.1x + 0.31\varphi - u_1 + u_2 + w_1, \tag{15}$$
$$\ddot{\varphi} = 0.93x + 6.423\varphi + 1.1u_1 + 0.9u_2 + w_2,$$

with $x(0) = 0.1$, $\varphi(0) = 0$ and unknown $\dot{x}(0) = z_1$, $\dot{\varphi}(0) = z_2$: $(z_1, z_2) \in Z = \{z \in Z : |z_1| \leq 0.1, |z_2| \leq 0.33\}$, and disturbances of the form $w_1(t) = v_1 \sin(4t)$, $w_2(t) = v_2 \sin(3t)$, $t \in T$: $(v_1, v_2) \in V = \{v \in R^2 : |v_i| \leq 0.01, i = 1, 2\}$.

Let the measurer at moments $t \in T_h = \{0, h, \ldots, 15 - h\}$, $h = 0.02$, returns values

$$y_1 = -x + 1.1\varphi + \xi_1,\ y_2 = x + 0.9\varphi + \xi_2,$$

where $\xi_i = \xi_i(t)$, $|\xi_i(t)| \leq 0.01$, $t \in T_h$, are bounded errors.

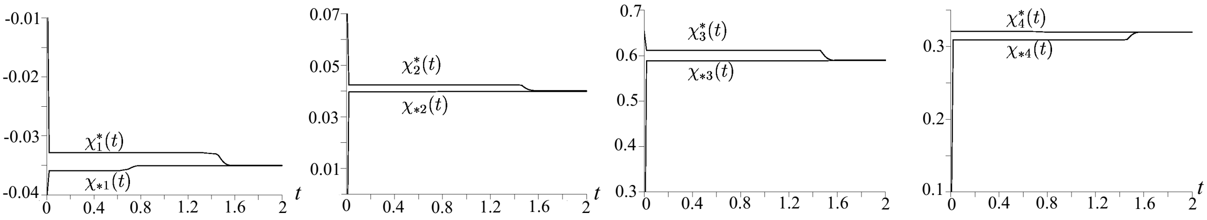

Fig. 1. Linear estimates $\chi_i(t)$, $t \in T_h$, $i = \overline{1,4}$.

The aim of the control process is to steer system (15) at the moment $t^* = 15$ to the sets $X^* = \{x \in R^2 : |x_1| \le 0.05, |x_2| \le 0.1\}$; $\Phi^* = \{\varphi \in R^2 : |\varphi_1| \le 0.05, |\varphi_2| \le 0.2\}$; $(0 \le u_i(t) \le 0.02)$, $i = 1, 2$, $t \in T$; minimizing the functional

$$J(u) = \int_0^{15} (u_1(t) + u_2(t))dt.$$

Let in a concrete control process $z_1^* = -0.1$; $z_2^* = 0.33$; $v_1^* = -0.005; v_2^* = 0.01$; $\xi_1^*(t) = 0.01\cos(2t)$, $\xi_2^*(t) = -0.01\cos(4t), t \in T_h$. The problem was solved using the algorithms from Section 4 and (Gabasov *et al.*, 2000b). The optimal value of the cost function turned out to be equal to 0.1046290478. The complexity of iterations at an arbitrary $\tau \in T_h$ did not exceed 0.042667.

Figure 1 presents the plots for the linear estimates $\chi_{*i}(\tau)$, $\chi_i^*(\tau)$, $i = \overline{1,4}$; $t \in [0, 2]$. When $t > 1.52$ values $\chi_{*i}(\tau)$, $\chi_i^*(\tau)$, $i = \overline{1,4}$, almost coincide.

Figure 2 shows the projections on the phase planes $x\dot{x}$ and $\varphi\dot{\varphi}$ of the optimal trajectories under optimal on-line controls (see figure 3).

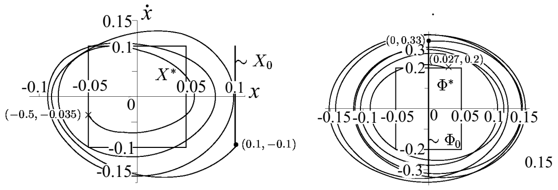

Fig. 2. The projections of the optimal trajectories.

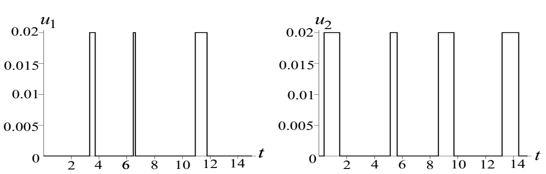

Fig. 3. Optimal on-line controls.

To demonstrate the two-stage method a new terminal sets $X^* = \{x \in R^2 : |x_i| \le 0.001\}$, $\Phi^* = \{\varphi \in R^2 : |\varphi_i| \le 0.001\}$ were chosen as well as new bounds on controls: $0 \le u_i(t) \le 0.05$, $t \in T$, $i = 1, 2$. Under these conditions accompanying optimal control problem (5) on the interval [0,1.52] has no admissible controls and the first stage described in Section 5 is performed. At $t = 0$ a value $\rho(0) = 0.16212$ of the piecewise linear estimate was calculated; the values $\rho(t)$, $t \in]0, 1.52]$, are presented in Figure 4.

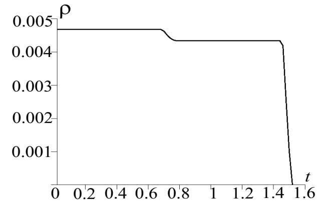

Fig. 4. Piecewise linear estimate $\rho(t)$, $t \in T_h$.

CONCLUSION

In the paper a linear control problem with unknown but bounded initial state, disturbances and measurement errors is considered. The finite algorithms of the solution of optimal observation problems are justified. An algorithm for constructing the optimal guaranteed on-line control is presented. Observation problems with different types of constraints are discussed. The scheme suggested can be applied to nonlinear systems via the linear and piecewise linear approximations (Balashevich *et al.*, 2002).

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