MODEL PREDICTIVE CONTROL DESIGN USING NON-MINIMAL STATE SPACE MODEL

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Abstract: This paper examines the design of model predictive control using non-minimal state space models, in which the state variables are chosen as the set of measured input and output variables and their past values. It shows that the proposed design approach avoids the use of an observer to access the state information and, as a result, the disturbance rejection, particularly the system input disturbance rejection, is significantly improved when constraints become activated. In addition, the paper shows that the system output constraints can be achieved in the proposed approach, which provides a significant improvement over the general observer based approach.

Keywords: State space model, constrained control, optimization, disturbance rejection

1. INTRODUCTION

There are three general approaches to MPC design. Each approach uses a unique model structure. In the earlier formulations, *Finite Impulse Response* (FIR) and Step Response models were favoured. FIR model-based design algorithms include *Dynamic Matrix Control* (DMC) (Cutler and Ramaker, 1979) and the quadratic DMC formulation of Garcia and Morshedi (Garcia and Morshedi, 1986). However, they are limited to stable plants and often require large model orders. Transfer function models give a more parsimonious description of process dynamics and are applicable to both stable and unstable plants. Representatives of transfer function model based predictive control include the predictive control algorithm of Peterka (Peterka, 1984) and the *Generalized Predictive Control* (GPC) algorithm of Clarke and colleagues (Clarke *et al.*, 1987). Transfer function model-based predictive control is often considered to be less effective in handling multivariable plants but the present paper will contest this issue: it will show that the non-minimal state space form that is defined most transparently by the transfer function model provides an excellent basis for MPC. GPC has also been analyzed using the framework of state space methods (Bitmead *et al.*, 1990).

The general framework of MPC using state space models follows the fundamentals of state estimation feedback control that exploits the separation (certainty equivalence) theorem, in which the state feedback control law utilizes state estimates generated by a state observer. In conventional linear state feedback control, the observer dynamics are assumed to be faster than the dynamics of state feedback control and, hence, the error signal between the observer and the actual system is assumed to converge to zero at a much faster rate. However, when the system constraints become activated in MPC, nonlinearity dominates the properties of the state feedback control system and the effect of observer dynamics becomes a much more complicated issue.

The presence of the observer is also a problem in more general terms since it can lead to robustness problems and the need for introducing some method of 'loop transfer recovery': see (Bitmead *et al.*, 1990); also the discussion on this topic in (Taylor *et al.*, 2000*a*), which shows how observer-free state space control, based on the non-minimal state space model used in the present paper (see below), avoids these difficulties. Through a simulation example, based on a double integrator system, this present paper examines some of the issues that are associated with the observer and the effect of input disturbances when constraints become activated. It is shown that not only does the closed-loop performance deteriorate significantly in terms of input disturbance rejection when plant input constraints become activated, but it is also difficult to impose plant output constraints in the presence of observer.

To overcome the observer-based obstacles to good MPC design, the present paper discusses the design of MPC systems using the special *Non-Minimal State Space* (NMSS) model structure mentioned above, in which the state variables correspond to the measured plant input, output and their past measured values, as defined by the structure of the system transfer function model. The NMSS structure in this special form was first discussed in detail by the second author and his colleagues (Young *et al.*, 1994; Wang and Young, 1988), including the introduction of the *Proportional Integral Plus* (PIP) control system that provides the practical embodiment of such NMSS control. Since these seminal papers were published, NMSS-PIP analysis has been used successfully as a basis for the design and practical implementation of advanced control systems in many different areas of application (e.g. (Young *et al.*, 1994; Lees *et al.*, 1998; Chotai *et al.*, 1998; Taylor *et al.*, 2000*b*; Taylor *et al.*, 2001). The most recent generalization and unification of the NMSS-PIP concept is discussed in Taylor *et al* (Taylor *et al.*, 2000*a*), which shows how it can exactly mimic other well known control systems, including *Generalized Predictive Control* (GPC) and standard *Linear Quadratic Gaussian* (LQG) control, while providing greater inherent flexibility and power than either of these.

By exploiting a similar NMSS model formulation to that used in the above references, the approach to MPC design proposed in the present paper maintains the simplicity of the framework for previous MPC design using state space models, as developed by the first author. At the same time, it successfully overcomes the difficulties and performance deterioration when plant operational constraints are present and important in design terms.

2. MODEL PREDICTIVE CONTROL DESIGN USING A NON-MINIMAL, INPUT-OUTPUT STATE SPACE MODEL

2.1 Model Structure

We assume that the plant to be controlled has p inputs and q outputs. The discrete mathematical model to be used in the design is captured by the following difference equation relating the uniformly sampled input $u(k)$ and output $y(k)$:

$$\begin{aligned} y(k) + A_1 y(k-1) + A_2 y(k-2) \\ + \ldots + A_n y(k-n) \\ = B_1 u(k-1) + B_2 u(k-2) \\ + \ldots + B_n u(k-n) + \xi(k) \end{aligned} \quad (1)$$

We assume that the disturbance $\xi(k)$ is a random walk type of disturbance; more specifically, $\Delta\xi(k) = \xi(k) - \xi(k-1)$ is zero mean white noise. Equation (1) can also be presented in terms of differenced input and output variables, yielding embedded integral action in the MPC system [1].

$$\begin{aligned} \Delta y(k) + A_1 \Delta y(k-1) \\ + \ldots + A_n \Delta y(k-n) \\ = B_1 \Delta u(k-1) + B_2 \Delta u(k-2) \\ + \ldots + B_n \Delta u(k-n) + \Delta\xi(k) \end{aligned} \quad (2)$$

In a similar manner to that used in the above references on NMSS-PIP control system design, the NMSS state vector $\Delta x_m(k)^T$ is chosen as: $[\Delta y(k)^T \ldots \; \Delta y(k-n+1)^T \; \Delta u(k-1)^T \ldots \; \Delta u(k-n+1)^T]$ where $dim(\Delta x_m) = p \times (n-1) + q \times n = m$. Then the NMSS model is defined as follows,

$$\begin{aligned} \Delta x_m(k+1) &= A_m \Delta x_m(k) + B_m \Delta u(k) + \Omega_m \Delta\xi(k) \\ \Delta y(k+1) &= C_m \Delta x_m(k+1) \end{aligned} \quad (3)$$

where $A_m =$

[1] In previous NMSS-PIP formulations (see the cited references), integral action and the associated type-one servo performance is ensured by adding an additional integral-of-error state, which clearly provides an alternative formulation of the problem.

$$\begin{bmatrix} -A_1 & -A_2 & \ldots & -A_{n-1} & -A_n & B_2 & \ldots & B_{n-1} & B_n \\ I_q & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & I_q & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & I_q & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & I_p & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & I_p & 0 \end{bmatrix}$$

$$B_m^T = \begin{bmatrix} B_1^T & 0 & 0 & \ldots & 0 & I_p & 0 & 0 \end{bmatrix}$$

$$C_m = \begin{bmatrix} I_q & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \end{bmatrix}$$

Choosing a new state variable vector,

$$x(k)^T = \begin{bmatrix} \Delta x_m(k)^T & y(k)^T \end{bmatrix}$$

we have

$$\begin{aligned} x(k+1) &= Ax(k) + B\Delta u(k) + \bar{\xi}(k) \\ y(k) &= Cx(k) + \eta(k) \end{aligned} \quad (4)$$

where A, B and C are matrices corresponding to the forms defined by

$$A = \begin{bmatrix} A_m & 0 \\ C_m A_m & I_q \end{bmatrix}; B = \begin{bmatrix} B_m \\ C_m B_m \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & I_q \end{bmatrix}$$

where 0_1, 0_m and 0_2 are zero matrices with dimensions $m \times q$, $q \times m$ and $q \times 1$ respectively, and $I_{q\times q}$ is a unit matrix with dimension q. In the sequel, the dimensionality of the augmented state space equation is taken to be $\delta = m+q$, and Equation (4) is referred as the 'composite design model'.

There are two points worth mentioning here. The first is related to the eigenvalues of the composite design model. Note that the characteristic equation of the model is

$$\begin{aligned} \rho(\lambda) &= det \begin{bmatrix} \lambda I - A_m & 0_1 \\ -C_m A_m & (\lambda - 1) I_{q\times q} \end{bmatrix} \\ &= (\lambda - 1)^q det(\lambda I - A_m) \end{aligned} \quad (5)$$

Hence, the eigenvalues of the composite model are the union of the eigenvalues of the plant model and the q eigenvalues being on the unit circle of the complex plane. The second point is that it can be verified (Wang, 2001) that the z-transfer function of the composite model is

$$C(zI - A)^{-1}B = \frac{z}{z-1} C_m (zI - A_m)^{-1} B_m \quad (6)$$

Hence, the composite model is detectable and stabilizable if the plant model is detectable and stabilizable, and has no transmission zeros on the unit circle.

2.2 State feedback control using predictive principle

Note that at the sampling instant k_i, $k_i > 0$, the state variable vector $x(k_i)$ is available through measurement and storage of appropriate previous measurements. Following the standard approach in MPC, the future state variables can be expressed as the function of future control trajectory by defining vectors,

$$X = \begin{bmatrix} x(k_i+1/k_i)^T & \ldots & x(k_i+N_p/k_i)^T \end{bmatrix}^T$$

$$\Delta U = \begin{bmatrix} \Delta u(k_i)^T \Delta u(k_i+1)^T \ldots \Delta u(k_i+N_c-1)^T \end{bmatrix}^T$$

in which

$$X = Fx(k_i) + \Phi \Delta U \quad (7)$$

where,

$$F = \begin{bmatrix} A \\ A^2 \\ A^3 \\ \vdots \\ A^{N_p} \end{bmatrix} \Phi = \begin{bmatrix} B & 0 & \ldots & 0 \\ AB & B & \ldots & 0 \\ A^2B & AB & \ldots & 0 \\ \vdots & & & \\ A^{N_p-1}B & A^{N_p-2}B & \ldots & A^{N_p-N_c-1}B \end{bmatrix}$$

where $\bar{C}$ $((q \times N_p) \times (n \times N_p))$ is an N_p-block diagonal matrix with the C matrix in Equation (4) sitting on its diagonal. Assume that the future set-point trajectory is governed by $s(k_i+1)$, $s(k_i+2)$, $s(k_i+3)$, ..., $s(k_i+N_p)$, and write

$$S = \begin{bmatrix} s(k_i+1)^T & s(k_i+2)^T & \ldots & s(k_i+N_p)^T \end{bmatrix}^T \quad (8)$$

Then the cost function for set-point following can be chosen as follows:

$$J_s = (S - Y)^T \bar{Q} (S - Y) + \Delta U^T \bar{R} \Delta U \quad (9)$$

where $\bar{Q}$ has dimension of $(q \times N_p) \times (q \times N_p)$ and $\bar{Q} > 0$. By substituting (8) into (9), we obtain the optimal control trajectory as

$$\Delta U = -(\Phi^T \bar{C}^T \bar{Q} \bar{C} \Phi + \bar{R})^{-1} \Phi^T \bar{C}^T \bar{Q} (\bar{C} F x(k_i) - S) \quad (10)$$

For the quadratic cost function J_s, without hard constraints, the resultant control system is in the form of linear time invariant state feedback control: see Equation (10). This point becomes clearer when we let

$$K_s = (\Phi^T \bar{C}^T \bar{Q} \bar{C} \Phi + \bar{R})^{-1} \Phi^T \bar{C}^T \bar{Q} \bar{C} F$$

$$R_s = (\Phi^T \bar{C}^T \bar{Q} \bar{C} \Phi + \bar{R})^{-1} \Phi^T \bar{C}^T \bar{Q} S$$

Notice that the above matrices only depend on the plant model parameters and the set point trajectory within the prediction horizon (assumed invariant for the majority of applications), thus they are constant matrices for linear, time-invariant systems. From the receding horizon control principle, the actual incremental control signal at time k_i is

$$\Delta u(k_i) = -k_s x(k_i) + r_s \quad (11)$$

where k_s and r_s are the first p rows of the matrices K_s and R_s respectively. Equation (11) is in the standard form of linear, time invariant, state feedback control. The integral action is inherent in this the predictive control system since the control input is calculated as:

$$u(k_i + 1) = u(k_i) + \Delta u(k_i) \quad (12)$$

Note that this state space approach leads to a simple framework for predictive control design. However, with the specific choice of the NNMSS state variables used here, the framework does not require the use of observers for the estimation of the state variables. This particular advantage will be highlighted in the later simulation studies (see Section 3). In comparison with the design approaches using a state observer, the state space model (3) is no longer in the form of a minimal structure and, as a consequence, the A, B and C matrices have higher dimensionality. For large dimensional systems, this might lead to difficulties in forming the F and Φ matrices in Equation (7). This problem could be avoided using the basis function approach proposed in (Wang, 2004), where the predictive control gain matrices are calculated sequentially.

3. MODEL PREDICTIVE CONTROL OF DOUBLE INTEGRATING PLANT

Consider a continuous-time, double integrator plant that is sampled at an interval $\Delta t = 1$ second. This system is used in this example to compare the performance of the model predictive control algorithms using state space design frameworks with and without using observers. A step input disturbance with amplitude of -10 is added to the plant input at the $4th$ second of time in the simulation. In the MPC design, the prediction horizon, N_p, is selected to be 150 samples and the control horizon N_c is chosen to be 3. The weighting on the output error $\bar{Q}$ is the unit matrix and the weighting on Δu is chosen to be the unit matrix. For pedagogical reasons, we choose this 'toy' example with the intention of making the comparative studies more transparent.

The corresponding discrete-time transfer function model is

$$G(z) = \frac{0.5(z+1)}{(z-1)^2} \quad (13)$$

so that the NMSS model is given by:

$$\begin{aligned} x_m(k+1) &= A_m x_m(k) + B_m u(k) \\ y(k) &= C_m x_m(k) \end{aligned} \quad (14)$$

where,

$$A_m = \begin{bmatrix} 2 & -1 & 0.5 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; B_m = \begin{bmatrix} 0.5 \\ 0 \\ 1 \end{bmatrix};$$

$$C_m = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

This NMSS model is augmented to yield the state space model, as shown in Equation (4) for the design of the model predictive controller. For comparative purposes, the conventional MPC state space design using an observer is applied to the same process. In the observer-based design, the system matrices for the double integrator example are:
$A_m = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; $B_m = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$; $C_m = \begin{bmatrix} 1 & 0 \end{bmatrix}$
The augmented model for this plant is given by

$$\begin{aligned} x(k+1) &= Ax(k) + B\Delta u(k) \\ y(k) &= Cx(k) \end{aligned} \quad (15)$$

where $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 0.5 \\ 1 \\ 0.5 \end{bmatrix}$; $C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

A pole-assignment strategy is used to design an observer based on the model (15). The set of observer's poles are chosen to be $(0.01, 0.0105, 0.011)$ corresponding to a fast dynamic response. The resultant observer gain is J_{ob}=[1.968 0.9688 2.9685]^T. (The deadbeat observer for this case has a gain of $J_d = [2 \quad 1 \quad 3]$ the difference is negligible in terms of dynamic responses).The closed-loop eigenvalues for the predictive control systems are identical except that the NMSS-MPC system has an extra pole at the origin of the complex plane. The identical poles are $0.2262 \pm j0.05086$, 0.3938.

3.1 Case A. Input amplitude constraint.

Figure 1 shows the comparison of the results obtained in the case of input amplitude constraints, where the control signal was limited to between -12 and 12. It is clear from the plots that the NMSS-MPC control system produces faster disturbance rejection with much smaller magnitude in the plant response. It is interesting to note that the control signal recovers from the saturation in a shorter time interval .

3.2 Case B. Input rate-of-change constraint.

Figure 2 shows the comparative results for input rate-of-change constraints. In the first case,

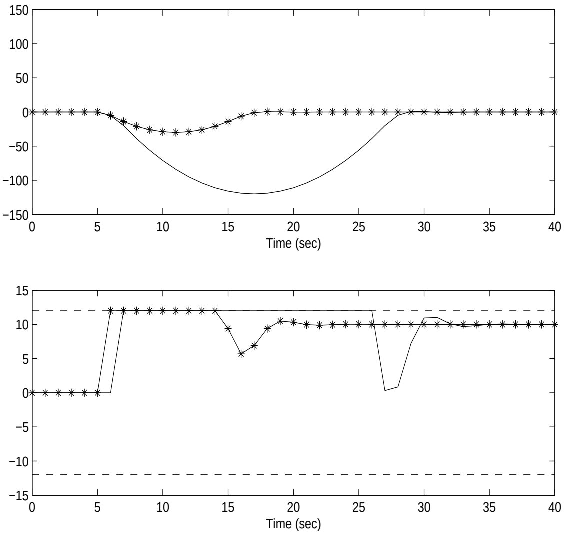

Fig. 1. Closed-loop response with control amplitude constraints ($-12 \leq u(k) \leq 12.$).Top figure: output response; bottom figure: control signal response (Solid line: observer based design; solid-star line: NMSS design).

the input rate-of-change $\Delta u(k)$ was constrained between -15 and 15. It is seen from the plots that the MPC using the observer-based design produces oscillatory control and plant output signals. In contrast, the non-minimal NMSS-MPC structure has no difficulty in coping with the input rate-of-change. In fact, the performance is almost identical to the case where no constraints are imposed. To further investigate this issue, when the limits of rate-of-change are reduced to -8 and 8, the MPC using the observer-based design became unstable, whilst the NMSS-MPC control system produced a closed-loop system with satisfactory performance (see Figure 3).

3.3 Case C. Output constraint.

Figure 4 shows the comparative results in the case of output constraints, where the target is to maintain the plant output signal within the limits of -16 and 6. It is clear again that the model predictive control using observer-based design is unable to maintain the plant output within the specified limits. In contrast, the NMSS-MPC control system successfully maintains the output signal within these limits.

3.4 Discussion

When constraints become activated, MPC becomes a nonlinear control system. As a result, the separation principle that is the cornerstone of state estimation-based feedback control, no longer holds true. From the above case studies, it is seen that the observer errors generated in this

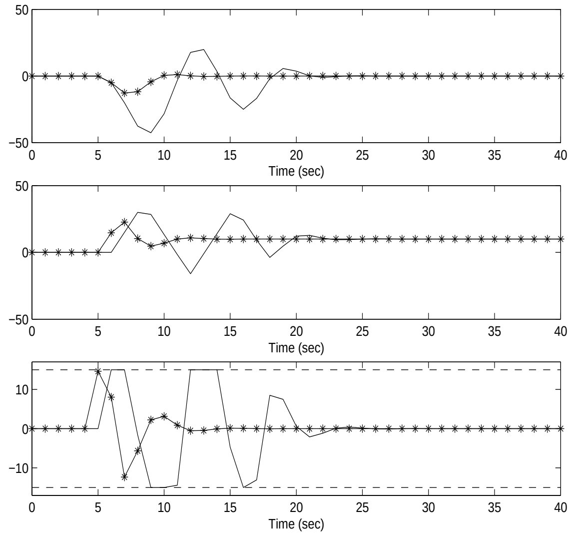

Fig. 2. Closed-loop response with control rate-of-change constraints ($-15 \leq \Delta u(k) \leq 15$). Top figure: output response; middle figure: control signal response; bottom figure: rate-of-change in control signal (Solid line: observer based design; solid-star line: non-minimal NMSS design).

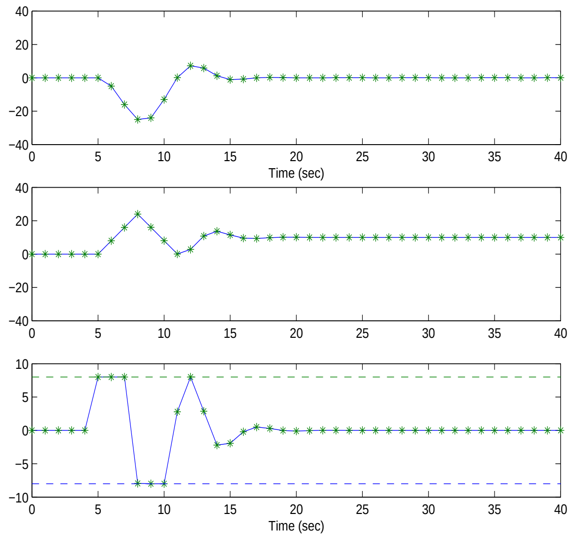

Fig. 3. Closed-loop response with control rate-of-change constraints ($-8 \leq \Delta u(k) \leq 8$). Top figure: output response; middle figure: control signal response; bottom figure- rate-of-change in control signal (observer based design is unstable and not shown; solid-star line: NMSS design)

manner result in a larger deterioration in performance than that experienced in the case of linear, state estimation-based feedback control. In particular, larger observer errors often activate the constraints and produce more deterioration in the closed loop performance. And output constraints will not work at all unless the observer produces almost perfect estimation of the unobserved states. The new NMSS-MPC controller

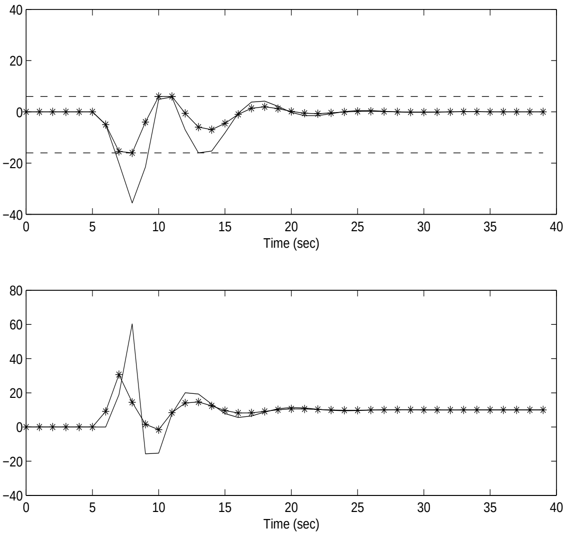

Fig. 4. Closed-loop response with output constraints ($-16 \leq y(k) \leq 6$). Top figure: output response; bottom figure: control signal response (Solid line: observer based design; solid-star line: non-minimal NMSS design).

avoids the use of the observer by directly accessing and exploiting only the measured plant input and output information. As a result, the constraints become activated only when the actual system operation is beyond the specified limits.

4. CONCLUSIONS

This paper has proposed a model predictive control (MPC) algorithm using a non-minimal state space (NMSS) structure. By choosing the set of state variables that corresponds to the measured plant input, output and their past values, as defined by the structure of the transfer function model, a state observer is no longer required as part of this NMSS-MPC design. As a result, the closed-loop performances for disturbance rejection in the presence of constraints are significantly improved, as demonstrated by the simulation example. In addition, plant output constraints can be effectively utilized when required.

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