# THE ATTITUDE DETERMINATION ALGORITHM USING INTEGRATED GPS/INS DATA

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Abstract: Traditionally, the extended Kalman filter (EKF) is a well known approach for general nonlinear data fusion problem. But the unit norm constraint of quaternion vector, which is the most efficient expression of attitude, results in the singularity of error covariance matrix. To overcome this difficulty, the modified extended Kalman filter (MEKF) has been developed by Shuster (1982). In order to improve the computational complexity of MEKF, we introduce the concept of spherical linear interpolation (SLERP). SLERP is a simple approach when two quaternions are averaged according to their accuracy dependent weightings. The SLERP based MEKF is proposed in this paper. However, its shortcoming is that the SLERP based MEKF cannot estimate gyro bias. Hence the proposed approach is not appropriate for the long duration navigation. The trade-off of MEKF and the SLERP based MEKF is discussed also.


Keywords: attitude algorithms, gyros, extended Kalman filters, quaternion, SLERP

## 1. INTRODUCTION

The attitude determination is very important in the moving vehicles, the main topic of this paper is the attitude determination from the information that GPS and gyro provide. The unit quaternion which expressed the attitude can be derived from the angular velocity that gyro measures; also we can use the QUEST algorithm (Shuster and Oh, 1981) to obtain the unit quaternion from double difference of GPS carrier phase. These two data can be combined into a more accurate one by filters.

Since the state equations are nonlinear, the linearization approaches may be used. The EKF is an appropriate method for attitude determination. But the unit norm constraint of quaternion vector results in the singularity of error covariance matrix. To overcome this problem, we can use the MEKF (Shuster, *et al.*, 1982). The MEKF adopts quaternion product which preserves the unit norm of quaternion and reduces the dimension of error covariance matrix by an assumption of infinitesimal rotation.

In addition to the MEKF, we use the concept of SLERP in the filter. The advantage of SLERP is less computation load and its performance is similar to that of MEKF. The method of SLERP can not provide the update estimation of gyro bias, so using this method is restricted by mission duration. Therefore, when selecting the filter, we need consider a tradeoff among mission duration, computation load and gyro bias.

## 2. BACKGROUND FOR QUATERNION

### *2.1 Quaternion algebra*

A quaternion number $\overline{\mathbf{q}}$ is defined as $\overline{\mathbf{q}} = q_1 i + q_2 j + q_3 k + q_4$, where $i$, $j$, $k$ satisfy the following rotations

$$
\begin{aligned}
i^2 = j^2 = k^2 &= ijk = -1 \\
ij &= -ji = k \\
jk &= -kj = i \\
ki &= -ik = j
\end{aligned} \tag{1}
$$

and $q_1$, $q_2$, $q_3$, $q_4$ are real numbers.
Also, it can be represented as following

$$
\overline{\mathbf{q}} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{q}} \\ q_4 \end{bmatrix} = \begin{bmatrix} \vec{n}\sin\dfrac{\theta}{2} \\ \cos\dfrac{\theta}{2} \end{bmatrix} \tag{2}
$$

where $\underline{\mathbf{q}} = [q_1 \quad q_2 \quad q_3]^T$ is referred to as the vector part of the quaternion and $q_4$ is referred to as the scalar part; $\vec{n}$ is the unit vector and $\theta$ is a positive rotation about $\vec{n}$.

From equation (1), we can define the quaternion product of two quaternions $\overline{\mathbf{p}}$, $\overline{\mathbf{q}}$ as following (Mason, 2001; Kuipers, 1999). The notation "$\otimes$" denotes the quaternion product.

$$
\overline{\mathbf{p}} \otimes \overline{\mathbf{q}} = \begin{bmatrix} p_4 & -p_3 & p_2 & p_1 \\ p_3 & p_4 & -p_1 & p_2 \\ -p_2 & p_1 & p_4 & p_3 \\ -p_1 & -p_2 & -p_3 & p_4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} q_4 & q_3 & -q_2 & q_1 \\ -q_3 & q_4 & q_1 & q_2 \\ q_2 & -q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \tag{3}
$$

Quaternion norm is defined to be

$$
|\overline{\mathbf{q}}| = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2} = 1 \tag{4}
$$

Quaternion conjugate is defined as

$$
\overline{\mathbf{q}}^* = -q_1 i - q_2 j - q_3 k + q_4 \tag{5}
$$

The inverse of quaternion is given by

$$
\overline{\mathbf{q}}^{-1} = \frac{\overline{\mathbf{q}}^*}{|\overline{\mathbf{q}}|^2} = \overline{\mathbf{q}}^* \tag{6}
$$

To rotate a vector $\mathbf{r}$ from frame a to frame b, we form the product

$$
\begin{bmatrix} \mathbf{r}^b \\ 0 \end{bmatrix} = \overline{\mathbf{q}}^* \otimes \begin{bmatrix} \mathbf{r}^a \\ 0 \end{bmatrix} \otimes \overline{\mathbf{q}} \tag{7}
$$

where $\mathbf{r}^q, \mathbf{r}^b$ are 3×1 vectors which are represented in frame a and frame b, respectively. Use equation (3), we can simplify equation (7) as

$$
\mathbf{r}^b = \mathbf{A}(\overline{\mathbf{q}})\mathbf{r}^a \tag{8}
$$

where $\mathbf{A}(\overline{\mathbf{q}})$ is the attitude matrix

$$
\mathbf{A}(\overline{\mathbf{q}}) = (q_4^2 - |\underline{\mathbf{q}}|^2)\mathbf{I} + 2\underline{\mathbf{q}}\underline{\mathbf{q}}^T - 2q_4[\underline{\mathbf{q}}\times] \tag{9}
$$

where

$$
[\underline{\mathbf{q}}\times] = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{10}
$$

*2.2 Distance of quaternions*

Before define the distance of quaternions, we first consider a unit circle O in the plane (see Fig. 1). $\mathbf{r}_1$, $\mathbf{r}_2$ are two radiuses of circle O, and $P_1$, $P_2$ are two points on the circle. To find the distance of $P_1$, $P_2$ ( or $\mathbf{r}_1$, $\mathbf{r}_2$ ), there are

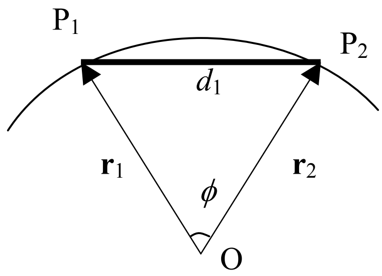


Fig. 1: distance of two radiuses in unit circle

two definitions of the distance. One is defined the chord length between $P_1$ and $P_2$, called Euclidean metric; and the other is defined the arc length between $P_1$ and $P_2$ (or the angle between $\mathbf{r}_1$, $\mathbf{r}_2$), called right metric. So are quaternions. There are two definitions:

a. Euclidean metric:

$$
d_1(\overline{\mathbf{q}}_1, \overline{\mathbf{q}}_2) = |\overline{\mathbf{q}}_1 - \overline{\mathbf{q}}_2| \tag{11}
$$

b. Right metric (Mason, 2001):

$$
d_2(\overline{\mathbf{q}}_1, \overline{\mathbf{q}}_2) = \phi = \cos^{-1} \overline{\mathbf{q}}_1{}^T \overline{\mathbf{q}}_2 \tag{12}
$$

where $0 \le \phi \le \pi$

In this paper, we choose the second definition.

*2.3 Spherical linear interpolation*

The use of linear interpolation between two unit quaternions will produce non-unit quaternion without normalization. To solve this problem, we should find a new interpolation method. We call it spherical linear interpolation.

Suppose that $u$ is the weighting where $0 \leqq u \leqq 1$, $\phi$ is the distance of $\overline{\mathbf{q}}_1$ and $\overline{\mathbf{q}}_2$, and $\overline{\mathbf{q}}$ is the quaternion after SLERP. We assume that

$$
\overline{\mathbf{q}} = \alpha\overline{\mathbf{q}}_1 + \beta\overline{\mathbf{q}}_2 \tag{13}
$$

where $\alpha$, $\beta$ are function of $u$ and to be determined. As the above mentioned, we know that the distance of $\overline{\mathbf{q}}$ and $\overline{\mathbf{q}}_1$ is $u\phi$; the distance of $\overline{\mathbf{q}}$ and $\overline{\mathbf{q}}_2$ is $(1-u)\phi$. Use equation (12) and multiply equation (13) for $\overline{\mathbf{q}}_1^T$ and $\overline{\mathbf{q}}_2^T$ yields

$$
\begin{cases} 1 = \alpha\cos u\phi + \beta\cos(1-u)\phi \\ \cos u\phi = \alpha + \beta\cos\phi \end{cases} \tag{14}
$$

solve the equation (14), we can get

$$
\alpha = \frac{\sin(1-u)\phi}{\sin\phi} \text{、} \quad \beta = \frac{\sin u\phi}{\sin\phi} \tag{15}
$$

Therefore, the spherical linear interpolation, abbreviated as SLERP, is defined by

$$
SLERP(\overline{\mathbf{q}}_1, \overline{\mathbf{q}}_2, u) = \frac{\sin(1-u)\phi}{\sin\phi}\overline{\mathbf{q}}_1 + \frac{\sin u\phi}{\sin\phi}\overline{\mathbf{q}}_2 \tag{16}
$$

where $0 \leqq u \leqq 1$

if $\overline{\mathbf{q}}_1 = \overline{\mathbf{q}}_2$, then

$$
SLERP(\overline{\mathbf{q}}_1, \overline{\mathbf{q}}_2, u) = \overline{\mathbf{q}}_1 = \overline{\mathbf{q}}_2 \tag{17}
$$

Given any two quaternions, $\overline{\mathbf{q}}_1$ and $\overline{\mathbf{q}}_2$, there exists two possible arcs along while we can move. One of these arcs is shorter. A method for finding the shorter interpolation is as follows (Watt, 1992). Check the inner product $\overline{\mathbf{q}}_1^T\overline{\mathbf{q}}_2$. If $>0$, then we are already moving along the smaller arc and nothing needs to be done. However, if $<0$, then we just replace $\overline{\mathbf{q}}_2$ by $-\overline{\mathbf{q}}_2$ and proceed.

## 3. FILTER DESIGN

### *3.1 Modified EKF*

The modified EKF (MEKF) has been developed in some papers (Shuster, *et al.*, 1982), so we will review it briefly in the following subsection.

The kinematic relationship between attitude quaternion $\overline{\mathbf{q}}(t)$ and angular velocity $\boldsymbol{\omega}(t)=[\omega_1 \quad \omega_2 \quad \omega_3]^T$ can be represented as

$$\dot{\overline{\mathbf{q}}}(t)=\frac{1}{2}\overline{\mathbf{q}}(t)\otimes\overline{\boldsymbol{\omega}}(t)=\frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}(t))\overline{\mathbf{q}}(t)=\frac{1}{2}\boldsymbol{\Xi}(\overline{\mathbf{q}}(t))\boldsymbol{\omega}(t) \tag{18}$$

where $\overline{\boldsymbol{\omega}}(t)=[\boldsymbol{\omega}(t) \quad 0]^T$

$$\boldsymbol{\Omega}(\boldsymbol{\omega})=\begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix}$$

$$\boldsymbol{\Xi}(\overline{\mathbf{q}})=\begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix} \tag{19}$$

Here, equation (18) serves as the evolution equation to perform attitude determination, if the angular velocity is known. To measure the angular velocity, a rate gyro may be used, whose model is given in the next subsection.

*Gyro model:*

The gyro output vector $\boldsymbol{\omega}_g$ can be modeled as (Shuster, *et al.*, 1982; Chiang, 2002)

$$\boldsymbol{\omega}_g=\boldsymbol{\omega}+\mathbf{b}+\eta_1 \tag{20}$$

where $\boldsymbol{\omega}$ is the "true" angular velocity, $\eta_1$ is a zero-mean white Gaussian noise and $\mathbf{b}$ denotes the gyro drift, which is assumed to be governed by

$$\frac{d}{dt}\mathbf{b}=\eta_2 \tag{21}$$

where $\eta_2$ is zero-mean Gaussian noise which is independent of $\eta_1$. The gyro measurement is useful for long duration only if the effect of the drift **b** can be properly compensated. Hence, the previous dynamical equation shall be included in the system for the attitude determination algorithm to obtain an estimate of **b**.

*State equations:*

First, we define the state as

$$\mathbf{x}=\begin{bmatrix}\overline{\mathbf{q}} \\ \mathbf{b}\end{bmatrix}_{7\times 1} \tag{22}$$

From equation (20), equation (18) can be rewritten as

$$\begin{aligned}\frac{d}{dt}\overline{\mathbf{q}}&=\frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}_\mathbf{g}-\mathbf{b}-\eta_1)\overline{\mathbf{q}} \\ &=\frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}_\mathbf{g}-\mathbf{b})\overline{\mathbf{q}}-\frac{1}{2}\boldsymbol{\Xi}(\overline{\mathbf{q}})\eta_1\end{aligned} \tag{23}$$

Equation (21) and (23) are the state equations of the state $\mathbf{x}$ (Shuster, *et al.*, 1982; Chiang, 2002). The terms of $\eta_1$ and $\eta_2$ are the process noise. By taking expectation on equation (21) and (23), we can get

$$\begin{aligned}&\frac{d}{dt}\hat{\overline{\mathbf{q}}}=\frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}_\mathbf{g}-\hat{\mathbf{b}})\hat{\overline{\mathbf{q}}}=\frac{1}{2}\boldsymbol{\Omega}(\hat{\boldsymbol{\omega}})\hat{\overline{\mathbf{q}}} \\ &\frac{d}{dt}\hat{\mathbf{b}}=0\end{aligned} \tag{24}$$

where

$$\hat{\boldsymbol{\omega}}=\boldsymbol{\omega}_g-\hat{\mathbf{b}}=[\hat{\omega}_1 \quad \hat{\omega}_2 \quad \hat{\omega}_3]^T \tag{25}$$

Equation (24) is the unbiased state estimation equation in time propagation.

*Propagation of error covariance matrix:*

In order to obtain the propagation equation of the error covariance matrix, we must derive the error state equations first. But due to the unit norm constraint of quaternion vector, the error covariance matrix for the seven-dimensional state vector is singular. To solve this problem, we reduce the dimension of error covariance matrix by an assumption of infinitesimal rotation (Shuster, *et al.*, 1982; Chiang, 2002).

Define the error quaternion $\boldsymbol{\delta}\overline{\mathbf{q}}$ as

$$\boldsymbol{\delta}\overline{\mathbf{q}}=\overline{\mathbf{q}}^*\otimes\hat{\overline{\mathbf{q}}} \tag{26}$$

where $\overline{\mathbf{q}}$ is the true quaternion and $\hat{\overline{\mathbf{q}}}$ is the estimated quaternion. Here, we assume $\boldsymbol{\delta}\overline{\mathbf{q}}$ is a small angle rotation. Differentiate equation (26) with respect to time, and use the equation (18), (26), we have

$$\boldsymbol{\delta}\dot{\overline{\mathbf{q}}}=-\frac{1}{2}\hat{\overline{\boldsymbol{\omega}}}\otimes\boldsymbol{\delta}\overline{\mathbf{q}}+\frac{1}{2}\boldsymbol{\delta}\overline{\mathbf{q}}\otimes\hat{\overline{\boldsymbol{\omega}}}-\frac{1}{2}\boldsymbol{\delta}\overline{\boldsymbol{\omega}}\otimes\boldsymbol{\delta}\overline{\mathbf{q}} \tag{27}$$

where

$$\boldsymbol{\delta}\overline{\boldsymbol{\omega}}=\begin{bmatrix}\boldsymbol{\delta\omega} \\ 0\end{bmatrix}=\begin{bmatrix}\boldsymbol{\omega}-\hat{\boldsymbol{\omega}} \\ 0\end{bmatrix}=\begin{bmatrix}\hat{\mathbf{b}}-\mathbf{b}-\eta_1 \\ 0\end{bmatrix} \tag{28}$$

Use quaternion product and substitute equation (25), (28) into equation (27), we can get

$$\boldsymbol{\delta}\dot{\overline{\mathbf{q}}}=\begin{bmatrix} 0 & \hat{\omega}_3 & -\hat{\omega}_2 & 0 \\ -\hat{\omega}_3 & 0 & \hat{\omega}_1 & 0 \\ \hat{\omega}_2 & -\hat{\omega}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\boldsymbol{\delta}\dot{\overline{\mathbf{q}}}+\frac{1}{2}\boldsymbol{\delta}\overline{\boldsymbol{\omega}} \tag{29}$$

The advantage of the error quaternion vector ($\boldsymbol{\delta}\overline{\mathbf{q}}$) is that the fourth component will be very close to unit since the incremental quaternion corresponds to a small angle rotation. Therefore,

$$\boldsymbol{\delta}\overline{\mathbf{q}}=\begin{bmatrix}\boldsymbol{\delta}\underline{\mathbf{q}} \\ q_4\end{bmatrix}\approx\begin{bmatrix}\boldsymbol{\delta}\underline{\mathbf{q}} \\ 1\end{bmatrix} \tag{30}$$

Subtract equation (24) form (21), we can get

$$\boldsymbol{\delta}\dot{\mathbf{b}} = \dot{\mathbf{b}} - \dot{\hat{\mathbf{b}}} = \eta_2 \tag{31}$$

where define $\boldsymbol{\delta}\mathbf{b} = \mathbf{b} - \hat{\mathbf{b}}$.

Define the error state

$$\boldsymbol{\delta}\mathbf{x} = \begin{bmatrix} \boldsymbol{\delta}\underline{\mathbf{q}} \\ \boldsymbol{\delta}\mathbf{b} \end{bmatrix}_{6\times1} \tag{32}$$

Simplify equation (29), we have

$$\boldsymbol{\delta}\dot{\mathbf{x}}(t) = \mathbf{F}(t)\boldsymbol{\delta}\mathbf{x}(t) + \mathbf{G}(t)w(t) \tag{33}$$

where

$$\mathbf{F}(t) = \begin{bmatrix} -[\hat{\boldsymbol{\omega}}(t)\times] & 1/2\,\mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}_{6\times6}$$

$$\mathbf{G}(t) = \begin{bmatrix} 1/2\,\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}_{6\times6}, \quad w(t) = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}_{6\times1} \tag{34}$$

Use equation (33), (34), we can obtain the propagation of the error covariance matrix.

*Measurement equation:*

We choose measurement equation as following

$$\mathbf{y}(t) = \mathbf{H}_{4\times7}\mathbf{x}(t) + v(t) \tag{35}$$

where

$$\mathbf{H} = \begin{bmatrix} \mathbf{I}_{4\times4} & \mathbf{0}_{4\times3} \end{bmatrix} \tag{36}$$

$\mathbf{y}(t)$ is the quaternion which can be obtained by the QUEST algorithm from GPS signal. The detail about the QUEST algorithm can be found in (Shuster and Oh, 1981; Chiang, 2002). Note that equation (35) is not the only choice for the measurement equation. Recall that in order to solve the problem of the singularity of the error covariance matrix, we reduce the order of the error covariance matrix in section 3.1. So, when we compute the Kalman gain $\mathbf{K}(k)$ and the measurement update of error covariance matrix as following

$$\begin{aligned} \mathbf{K}(k) &= \mathbf{P}^-(k)\mathbf{H}^T(\mathbf{R}(k) + \mathbf{H}\mathbf{P}^-(k)\mathbf{H}^T)^{-1} \\ \mathbf{P}^+(k) &= (\mathbf{I} - \mathbf{K}(k)\mathbf{H})\mathbf{P}^-(k) \end{aligned} \tag{37}$$

where $k$ is the sample time

The dimension of $\mathbf{H}$ and $\mathbf{P}^-(k)$ are not match. To overcome this problem, we multiply $\mathbf{H}$ by the matrix $\mathbf{S}(k)$ (Shuster, *et al.*, 1982)

$$\mathbf{H}'_{4\times6}(k) = \mathbf{H}_{4\times7}\mathbf{S}_{7\times6}(k) \tag{38}$$

where

$$\mathbf{S}_{7\times6}(k) = \begin{bmatrix} \boldsymbol{\Xi}(\hat{\overline{\mathbf{q}}}(k))_{4\times3} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}_{7\times6} \tag{39}$$

Then, we replace $\mathbf{H}$ by $\mathbf{H}'$ matrix in equation (37) and proceed with the measurement update. Section 3.1 is the introduction to MEKF, and we will show its performance at the last section. We also compare this filter with the SLERP based MEKF which will be introduced in the next section.

### 3.2 SLERP based MEKF

The derivations of this method are the same as MEKF except for the measurement update equations. We propose the concept of SLERP in the measurement update. Since SLERP can not provide the gyro bias estimation, use this method is restricted by mission duration.

*Time propagation of SLERP based MEKF:*

As mentioned, it is all the same as MEKF in the time propagation part. We use the derivations and the equations in section 3.1.

*Measurement update of SLERP based MEKF:*

Similarly, we can obtain the quaternion from GPS signal. Besides, the error covariance matrix of the quaternion which GPS measures is also available by using QUEST algorithm (Shuster and Oh, 1981). Now, we define the optimal estimation quaternion $\hat{\overline{\mathbf{q}}}(k)$ at the sample time $k$ as

$$\hat{\overline{\mathbf{q}}}(k) = \frac{\sin(1-u(k))\phi}{\sin\phi}\overline{\mathbf{q}}_1(k) + \frac{\sin u(k)\phi}{\sin\phi}\overline{\mathbf{q}}_2(k) \tag{40}$$

where $\overline{\mathbf{q}}_1(k)$ is the quaternion which is obtained form time propagation; $\overline{\mathbf{q}}_2(k)$ is the quaternion which is obtained form GPS by using QUEST algorithm; $u(k)$ is the optimal weighting value which is to be determined. For simplification, $k$ will be omitted later. Then, multiply equation (40) by $\overline{\mathbf{q}}^*$ and use the definition (26), we have

$$\boldsymbol{\delta}\overline{\mathbf{q}} = \frac{\sin(1-u)\phi}{\sin\phi}\boldsymbol{\delta}\overline{\mathbf{q}}_1 + \frac{\sin u\phi}{\sin\phi}\boldsymbol{\delta}\overline{\mathbf{q}}_2 \tag{41}$$

where

$\boldsymbol{\delta}\overline{\mathbf{q}} = \overline{\mathbf{q}}^* \otimes \hat{\overline{\mathbf{q}}}$, $\boldsymbol{\delta}\overline{\mathbf{q}}_1 = \overline{\mathbf{q}}^* \otimes \overline{\mathbf{q}}_1$, $\boldsymbol{\delta}\overline{\mathbf{q}}_2 = \overline{\mathbf{q}}^* \otimes \overline{\mathbf{q}}_2$ and $\phi$ is the distance between $\overline{\mathbf{q}}_1$ and $\overline{\mathbf{q}}_2$ .Our objective is to find an optimal weighting $u$ to minimize $\mathrm{tr}E[\boldsymbol{\delta}\underline{\mathbf{q}}\boldsymbol{\delta}\underline{\mathbf{q}}^T]$. Here, we assume $\overline{\mathbf{q}}_1$ and $\overline{\mathbf{q}}_2$ are very close, so $\phi$ is a small angle. Thus, equation (41) can be simplified as

$$\boldsymbol{\delta}\overline{\mathbf{q}} = (1-u)\boldsymbol{\delta}\overline{\mathbf{q}}_1 + u\boldsymbol{\delta}\overline{\mathbf{q}}_2 \tag{42}$$

Substitute the vector part of equation (42) into $\mathrm{tr}E[\boldsymbol{\delta}\underline{\mathbf{q}}\boldsymbol{\delta}\underline{\mathbf{q}}^T]$, and let

$$\frac{d}{du}\mathrm{tr}E[\boldsymbol{\delta}\underline{\mathbf{q}}\boldsymbol{\delta}\underline{\mathbf{q}}^T] = 0 \tag{43}$$

solve equation (43) to find $u$, we have

$$u = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \tag{44}$$

$$\begin{aligned} \mathrm{tr}E[\boldsymbol{\delta}\underline{\mathbf{q}}\boldsymbol{\delta}\underline{\mathbf{q}}^T] &= (1-u)^2\,\mathrm{tr}E[\boldsymbol{\delta}\underline{\mathbf{q}_1}\boldsymbol{\delta}\underline{\mathbf{q}_1}^T] + u^2\,\mathrm{tr}E[\boldsymbol{\delta}\underline{\mathbf{q}_2}\boldsymbol{\delta}\underline{\mathbf{q}_2}^T] \\ &= (1-u)^2\sigma_1^2 + u^2\sigma_2^2 \end{aligned} \tag{45}$$

where $\sigma_1^2 = \mathrm{tr}E[\boldsymbol{\delta}\underline{\mathbf{q}_1}\boldsymbol{\delta}\underline{\mathbf{q}_1}^T]$, $\sigma_2^2 = \mathrm{tr}E[\boldsymbol{\delta}\underline{\mathbf{q}_2}\boldsymbol{\delta}\underline{\mathbf{q}_2}^T]$

$\sigma_1^2$ can be obtained form equation (33) and $\sigma_2^2$ can be obtained form GPS by using QUEST algorithm. Equation (45) is a 3×3 submatrix of the 6×6 error covariance matrix. Following, we will derive residual submatrices of it.

$$\begin{aligned} E[\boldsymbol{\delta}\underline{\mathbf{q}}\boldsymbol{\delta}\mathbf{b}^T] &= E[((1-u)\boldsymbol{\delta}\underline{\mathbf{q}_1} + u\boldsymbol{\delta}\underline{\mathbf{q}_2})\boldsymbol{\delta}\mathbf{b}^T] \\ &= (1-u)E[\boldsymbol{\delta}\underline{\mathbf{q}_1}\boldsymbol{\delta}\mathbf{b}^T] + uE[\boldsymbol{\delta}\underline{\mathbf{q}_2}\boldsymbol{\delta}\mathbf{b}^T] \\ &= (1-u)E[\boldsymbol{\delta}\underline{\mathbf{q}_1}\boldsymbol{\delta}\mathbf{b}^T] \end{aligned} \tag{46}$$

In equation (46), the $E[\boldsymbol{\delta}\underline{\mathbf{q}_2}\boldsymbol{\delta}\mathbf{b}]$ term is set to zero, because we can not observe the gyro bias error from GPS signal. Another submatrix term $E[\boldsymbol{\delta}\mathbf{b}\boldsymbol{\delta}\mathbf{b}^T]$ is unchanged, in other words, it will not be updated by filter. This is because this method can not estimate the gyro bias. So using this method is restricted by mission duration. Following is the measurement update equation of SLERP based MEKF

*Measurement update equation*

$$u(k)=\frac{\sigma_1^2(k)}{\sigma_1^2(k)+\sigma_2^2(k)}$$

$$\bar{\mathbf{q}}(k)=\frac{\sin(1-u(k))\phi}{\sin\phi}\bar{\mathbf{q}}^-(k)+\frac{\sin u(k)\phi}{\sin\phi}\bar{\mathbf{q}}_{GPS}(k)$$

$$\mathbf{b}(k)=\mathbf{b}^-(k)$$

$$\mathbf{P}_q(k)_{3\times3}=(1-u(k))^2\mathbf{P}_q^-(k)+u(k)^2\mathbf{P}_{GPS}(k)$$

$$\mathbf{P}_{qb}(k)_{3\times3}=(1-u(k))\mathbf{P}_{qb}^-(k)$$

$$\mathbf{P}_b(k)_{3\times3}=\mathbf{P}_b^-(k)$$

$$\mathbf{P}(k)=\begin{bmatrix}\mathbf{P}_q(k) & \mathbf{P}_{qb}(k)\\ \mathbf{P}_{qb}(k)^T & \mathbf{P}_b(k)\end{bmatrix}_{6\times6}$$

where with superscript '-' denoting the result of time propagation; $\bar{\mathbf{q}}_{GPS}(k)$, $\mathbf{P}_{GPS}(k)$ can be get from GPS by using QUEST algorithm (Shuster and Oh, 1981).

## 4. SIMULATION RESULT

### *4.1 Scenario setup*

We make a list to show our simulation scenario

a. Simulation time = 450 sec
b. The vehicle is a platform which has two degree of freedom
c. Gyro sample rate = 10 Hz; GPS sample rate = 1 Hz
d. The length of GPS baseline is 3m and the error of GPS carrier phase is 5 cm
e. The gyro bias is integrated a white noise which standard deviation is $1\times10^{-5}$ rad/sec$^2$; the standard deviation of the white noise of gyro is $1\times10^{-6}$ rad/sec; the initial attitude error is 0.2 radian

### *4.2 Simulation result*

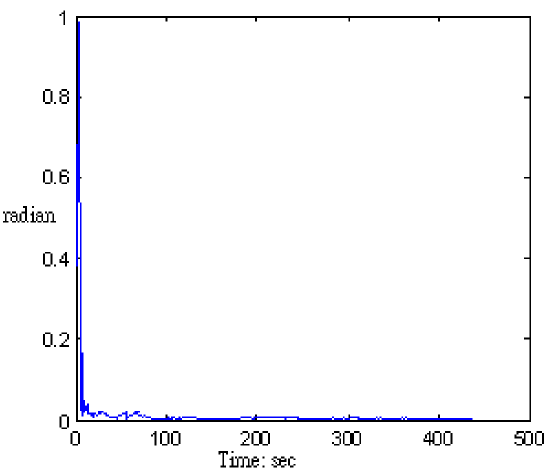


Fig. 2: estimation error of attitude using MEKF

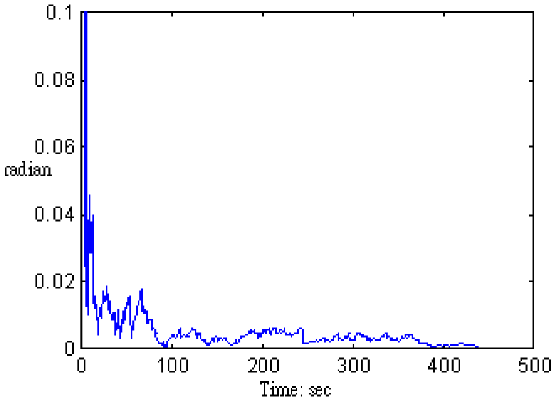


Fig. 3: the enlargement of Fig. 2

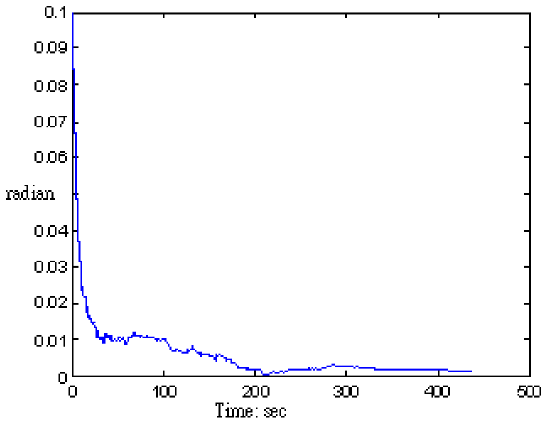


Fig. 4: estimation error of attitude using SLREP based MEKF

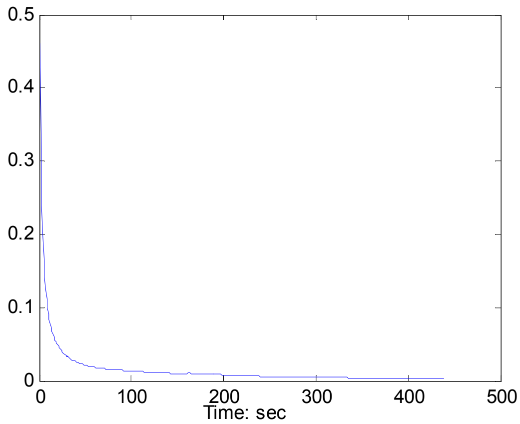


Fig. 5: SLERP weighting

See Fig. 2, we find that a transient state occurred in 0 ~ 5 seconds, then see Fig. 3, we know the steady state error is less than $5\times10^{-3}$ radian.

The SLERP based MEKF (see Fig. 4) nearly has no transient state, and furthermore the steady state error is less than $5\times10^{-3}$ radian (similar to MEKF). Beside, the computation load of SLERP based MEKF is less than the one of MEKF. This is because the former reduces the Kalman gain $\mathbf{K}$ (a matrix) to the weighting $u$ (a scalar).

See Fig. 5, we find the weighting is always close to 0 at the steady state, in other words, the estimation quaternion is close to the quaternion which gyro provide. But it is not always like this for all solutions. Let us enlarge the bias standard deviation to $1\times10^{-4}$

rad/sec$^2$ and increase the simulation time to 4500 seconds. See Fig. 6, while the steady state error converges in the first few steps, it diverges at about 500 seconds. This is because that SLERP based MEKF can not provide the gyro bias estimation so that the attitude estimation error and the SLERP weighting (Fig. 7) increase with time. So, when we use this method, we should consider the mission duration.

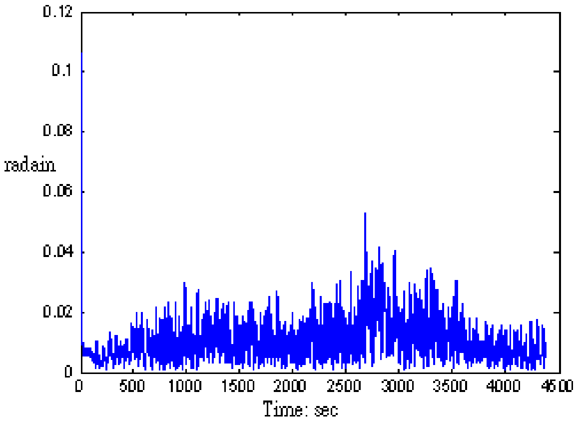


Fig. 6: estimation error of attitude using SLREP based MEKF

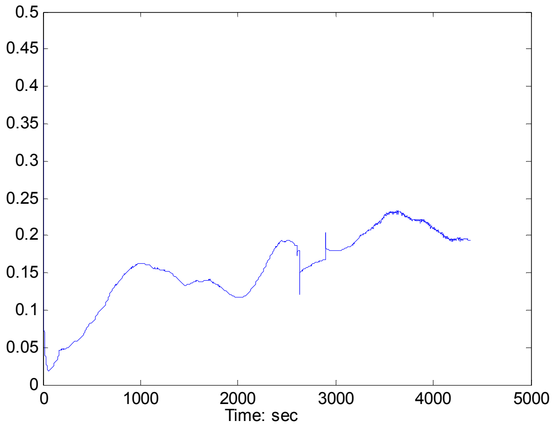


Fig. 7: SLERP weighting

*4.3 Conclusions*

As last section mentioned, SLERP based MEKF has similar performance to MEKF at the steady state while SLERP based MEKF has the better performance of transient state than MEKF. In addition to performance, SLERP based MEKF has less computation load than MEKF. But SLERP based MEKF can not provide the gyro bias estimation, it is restricted by mission duration when using this filter.

Table 1 is the operating time of this filter which consider the mission duration and the standard deviation of gyro bias. The operating time here means that the filter has similar performance to MEKF in using.

Table 1: the operating time of SLERP based MEKF

| Standard deviation of gyro bias | Operating time | Steady state error of attitude |
|---|---|---|
| $1\times10^{-4}$ rad/sec$^2$ | about 500 sec | $< 5\times10^{-3}$ rad |
| $1\times10^{-5}$ rad/sec$^2$ | about 2.7 hr | $< 5\times10^{-3}$ rad |
| $1\times10^{-6}$ rad/sec$^2$ | about 277 hr | $< 5\times10^{-3}$ rad |

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