PERFORMANCE ANALYSIS FOR DIRECTIONAL RESIDUAL BASED FAULT ISOLATION

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Abstract: Generalized likelihood ratio test (GLRT) for directional residual based fault isolation is studied in this paper. The conditional probability of correct fault isolation is used to measure the performance of the test. The explicit form of this probability is derived with the help of two GLRT invariant properties and the standard fault isolation subspace. The effect of linear transformation is studied as well. *Copyright © 2005 IFAC*

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1. INTRODUCTION

A broad class of model-based fault detection and diagnosis methods is built on the concept of analytical redundancy (Willsky, 1976). Discrepancies of measured and predicted plant outputs are expressed as residuals which can be enhanced by algebraic manipulations to facilitate the isolation of faults. The directional residual method is one of the enhancement techniques that makes residuals always point in a specific direction in response to a particular fault (Gertler, 1998).

If noise is present and the fault is deterministic but unknown, Generalized Likelihood Ratio Test (GLRT) (Van Trees, 1968) is usually used to isolate the fault. Although GLRT is a well-established method, its performance is seldom discussed (Scharf, 1994). In this paper, the conditional probability of correct isolation (PCI) is presented as the measure of the GLRT performance. Two invariant properties of the GLRT statistic are investigated, which reveal the fact that the explicit form of the PCI can be obtained in a simple and standardized fault isolation subspace.

Since linear transformation can be applied to optimize fault isolation (Basseville, 1997, 2003), its effect on the GLRT performance is further studied. The problem is simplified to a situation where the linear transformation can be decomposed as rotation and scaling operations. The PCI for the transformed residual is expressed as a function of the rotation and scaling factors.

2. PROBLEM STATEMENT

Consider the case of two possible faults in a linear static system. The residual is expressed as:

$$r = Lf + Nv\,, \qquad (1)$$

where

$L = [l_1, l_2]$ contains two vectors representing two fault response directions;

$f = [f_1, f_2]'$ contains the size of each fault;

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$v \sim N(0, I)$ is the noise;

$\text{cov}(r) = NN' = \Sigma$, Σ^{-1} exists.

It is assumed that a fault has been detected; the task of fault isolation is to decide which type of fault occurred. The hypotheses are:

$$H_1 : r = l_1 f_1 + Nv$$
$$H_2 : r = l_2 f_2 + Nv \qquad (2)$$

where $f_1, f_2 \in R$

Because f_1 and f_2 in eq. (1) are deterministic and unknown, the maximum likelihood method can be applied first to estimate the faults; then the estimates can be used to perform the likelihood ratio test as if they were true fault magnitudes. This procedure is called the generalized likelihood ratio test (GLRT), namely,

$$\Lambda_g(R) = \frac{\max_{f_2} p_{r|H_2}(R \mid f_2)}{\max_{f_1} p_{r|H_1}(R \mid f_1)} \underset{H_1}{\overset{H_2}{\gtrless}} \eta_1 \qquad (3)$$

The decision rule for the two-fault case is:

$$g_{21}(R) = R'(Q_1 - Q_2)R \underset{H_1}{\overset{H_2}{\gtrless}} \eta_2 \qquad (4)$$

where

$$Q_1 - Q_2 = \Sigma^{-1}\left[l_2\left(l_2'\Sigma^{-1}l_2\right)^{-1}l_2' - l_1\left(l_1'\Sigma^{-1}l_1\right)^{-1}l_1'\right]\Sigma^{-1} \qquad (5)$$

The threshold η_2 is normally chosen to be 0.

3. PCI DEFINITION

With the GLR test, once the fault response directions and the GLRT threshold are determined the fault decision regions are fixed. Under the condition that the *i*-th fault is present with known magnitude f_i, define the probability of the correct isolation (PCI) as:

$$P_{I|H_i}(f_i) = \int_{\Omega_i} p_{r|H_i}(R \mid f_i)\,dR \qquad (6)$$

where Ω_i is the decision region of the *i*-th fault in the residual space; *R* is the observation in the residual space.

Calculating the PCI directly from the PDF of g_{21} is difficult. The following sections will show another way to obtain the explicit form of the PCI, with the help of the standard fault isolation subspace.

4. GLRT INVARIANT PROPERTIES

Lemma 1: For any residual vector

$$r = Lf + Nv$$

and its linear transformation

$$r_T = T'r = T'Lf + T'Nv \qquad (7)$$

if T^{-1} exists, then the GLRT statistic remains invariant.

This can be proved by verifying $g_{21}(R) = g_{21}(T'R)$ for any residual sample *R*.

Now denote

$$\bar{r} = N^{-1}r = N^{-1}Lf + v = \bar{L}f + v \qquad (8)$$

From Lemma 1 we have

$$g_{21}(\bar{R}) = g_{21}(R) \qquad (9)$$

This means we can always consider the residual only with the identity covariance matrix.

For a subspace $<\bar{L}>$ defined by matrix $\bar{L} = \left[\bar{l}_1, \bar{l}_2\right]$, the corresponding projection matrix is

$$P_{\bar{L}} = \bar{L}\left(\bar{L}'\bar{L}\right)^{-1}\bar{L}' \qquad (10)$$

Considering eq. (5) with $\Sigma = I$, examine the GLRT statistic of the projection of residual $\bar{R}$ on space $\bar{L}$ *:*

$$g_{21}(P_{\bar{L}}\bar{R}) = \bar{R}'P_{\bar{L}}'P_{\bar{l}_2}P_{\bar{L}}\bar{R} - \bar{R}'P_{\bar{L}}'P_{\bar{l}_1}P_{\bar{L}}\bar{R}$$
$$= \bar{R}'P_{\bar{l}_2}\bar{R} - \bar{R}'P_{\bar{l}_1}\bar{R} = g_{21}(\bar{R}) \qquad (11)$$

Eq. (11) means the subspace defined by $\bar{L} = \left[\bar{l}_1, \bar{l}_2\right]$ contains the full fault isolation information. We call the subspace $<\bar{L}>$ *fault isolation space*. The result of eq. (11) can be concluded as:

Lemma 2: For any residuals with identity covariance matrix, the GLRT is invariant under transformations that preserve the fault isolation space.

5. STANDARD FAULT ISOLATION SUBSPACE

Regardless of the dimension of the original residual space, we can always work in the fault isolation subspace, because, as stated in Lemma 2, all fault related information is contained in this subspace, and the rest of the residual space contains nothing but noise. For the cases where there are only two faults in the system, this subspace is a plane defined by $\bar{L} = [\bar{l}_1, \bar{l}_2]$.

If the noise covariance matrix is the identity matrix, the decision lines for the GLRT consist of the points with equal distances to the fault response lines. Because the fault response lines are straight, it follows that the decision lines are straight as well. They are always bisectors of $\bar{l}_1, \bar{l}_2$ and are always orthogonal.

Thanks to the orthogonal property of the decision lines, the residual space can always be rotated so that the decision lines become the coordinate axes. With the identity matrix being the noise covariance matrix, the noise-induced part of the residual does not change under pure rotation. We call the fault isolation subspace *standard fault isolation subspace* when the noise covariance matrix is the identity matrix and the decision lines are the axes. The standard fault isolation subspace is shown in Fig. 1.

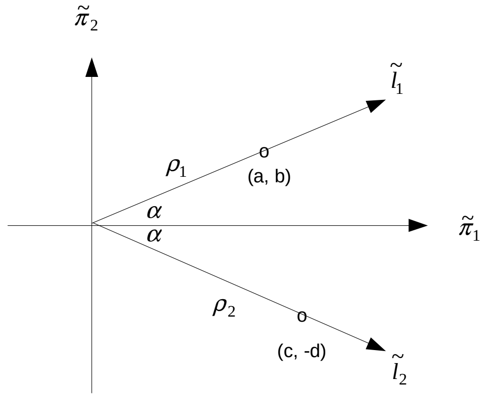

Fig. 1 Fault response directions and decision lines in standard fault isolation subspace
(note: the decision lines, π_i in the original fault isolation subspace, are denoted $\tilde{\pi}_i$ in the standard fault isolation subspace)

In the standard fault isolation subspace, the fault response directions are symmetrical about the axes. We can always rotate the primary residual space and place $\bar{l}_1$ in Quadrant I of the fault isolation subspace, and $\bar{l}_2$ in Quadrant IV of the fault isolation subspace. Thus the rotated $\bar{l}_1$ and $\bar{l}_2$ become

$$\tilde{l}_1 = [a \quad b \quad 0 \quad \cdots \quad 0]', \qquad a,b>0$$
$$\tilde{l}_2 = [c \quad -d \quad 0 \quad \cdots \quad 0]', \qquad c,d>0 \qquad (12)$$

Since the coordinates are the bisectors of $\tilde{l}_1, \tilde{l}_2$, it is always true that

$$\frac{a}{b} = \frac{c}{d} \qquad (13)$$

The fault-induced residual in the fault isolation plane can be expressed as:

$$\tilde{r}_f = \begin{bmatrix} af_1 & cf_2 \\ bf_1 & -df_2 \end{bmatrix} = \begin{bmatrix} \rho_1 \cos(\alpha) & \rho_2 \cos(\alpha) \\ \rho_1 \sin(\alpha) & -\rho_2 \sin(\alpha) \end{bmatrix} \quad (14)$$

where $\rho_i = f_i \left\| \tilde{l}_i \right\|$, i=1,2 are fault to noise ratio; $\alpha, -\alpha$ are fault response angles for fault 1 and fault 2 respectively; $\alpha \in \left(0, \frac{\pi}{2}\right)$

6. PCI CALCULATION

In the fault isolation subspace with normalized noise distribution, the decision region for fault 1 is Quadrant I and Quadrant III of the subspace. The probability of correct isolation (PCI) for fault 1 becomes:

$$P_{I|H_1}(f_1) = \int_{\Omega_1} p_{r|H_1}(R \mid f_1) dR$$

$$= \int_0^{+\infty}\int_0^{+\infty} \frac{1}{2\pi} \exp\left(-\frac{(x-af_1)^2 + (y-bf_1)^2}{2} \right) \cdot dx \cdot dy$$
$$+ \int_{-\infty}^{0}\int_{-\infty}^{0} \frac{1}{2\pi} \exp\left(-\frac{(x-af_1)^2 + (y-bf_1)^2}{2} \right) \cdot dx \cdot dy$$
$$= \left(\int_{-af_1}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2} \right) dx \right) \cdot \left(\int_{-bf_1}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2} \right) dy \right)$$
$$+ \left(\int_{-\infty}^{-af_1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2} \right) dx \right) \cdot \left(\int_{-\infty}^{-bf_1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2} \right) dy \right)$$
$$(15)$$

Define

$$erfc^*(t) = \frac{1}{\sqrt{2\pi}} \int_t^{+\infty} \exp\left(-\frac{x^2}{2} \right) dx, \quad (16)$$

the conditional probability of correct isolation for fault 1 becomes

$$P_{I|H_1}(f_1)$$
$$= erfc^*(-af_1) \cdot erfc^*(-bf_1) + erfc^*(af_1) \cdot erfc^*(bf_1)$$
$$= erfc^*(-\rho_1 \cos(\alpha)) \cdot erfc^*(-\rho_1 \sin(\alpha))$$
$$+ erfc^*(\rho_1 \cos(\alpha)) \cdot erfc^*(\rho_1 \sin(\alpha)) \qquad (17)$$

7. PCI UNDER LINEAR TRANSFORMATION

Once the probability of correct isolation is obtained, it is interesting to know how it can be changed under linear transformation. Assume the residual is transformed from the r_0 space to the r_1 space. According to the GLRT invariant properties expressed in Lemma 1 and 2, the general linear transformation can be narrowed down to a simplified situation:

We can assume that the original residual space r_0 contains the standard fault isolation subspace. Any other residual space with the same dimension can be transformed to it without changing the GLRT statistics.

We can also assume that the new residual space r_1 has identity noise covariance matrix, and it contains the fault isolation subspace only. The removal of the noise-only subspace will not change the PCI. This is a two-dimensional space for pair-wise fault isolation.

Define the original residual $\tilde{r}_0$ as

$$\tilde{r}_0 = \tilde{L}_0 f + v_0 \qquad (18)$$

where

$\tilde{r}_0$ is an $n_0 \times 1$ residual vector; $n_0 \geq 3$;

$\tilde{L}_0 = [\tilde{l}_{01}, \tilde{l}_{02}]$ is the standard fault isolation subspace containing two fault response directions;

f is the fault vector;

$v_0 \sim N(0, I)$ is the noise vector.

Denote the transformed residual $\bar{r}_1$ as:

$$\bar{r}_1 = \bar{T}'\tilde{r}_0 = \bar{T}'\tilde{L}_0 f + \bar{T}' v_0 \quad (19)$$

where

r_1 is a 2-dimensional residual vector;

$\mathrm{cov}(r_1) = I$

$\bar{T}$ is an $n_0 \times 2$ linear static transformation matrix.

Starting from this simplified situation, we will first check the changes in the fault isolations subspace after the linear transformation and then express the new PCI as a function of the linear transformation.

Lemma 3: Given the original residual defined in eq. (18) and the transformed residual defined in eq. (19), the effect of linear transformation in the fault isolation subspace is equivalent to rotating the fault response directions with an angle β and applying a pure scaling matrix

$$K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \quad (20)$$

where

$0 < k_1 \le 1,\ 0 < k_2 \le 1$

Proof:

Decompose $\bar{T} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$ (21)

where is T_1 a 2x2 matrix.

According to the $\bar{r}_1$ definition in eq. (19), we have

$$\bar{T}'\bar{T} = I \Rightarrow T_1'T_1 + T_2'T_2 = I \quad (22)$$

$$\Rightarrow T_1'T_1 = I - T_2'T_2 \quad (23)$$

Applying principal component decomposition, we have

$$T_1'T_1 = U_1\Lambda_1 U_1' \quad (24)$$

$$T_2'T_2 = U_2\Lambda_2 U_2' \quad (25)$$

where

$$\Lambda_1 = \begin{bmatrix} \lambda_{1,1} & 0 \\ 0 & \lambda_{1,2} \end{bmatrix} \quad (26)$$

$$\Lambda_2 = \begin{bmatrix} \lambda_{2,1} & 0 \\ 0 & \lambda_{2,2} \end{bmatrix} \quad (27)$$

U_1, U_2 are orthonormal matrices; λ_i are non-negative real numbers. Then eq. (23) becomes

$$T_1'T_1 = I - U_2\Lambda_2 U_2' = U_2(I - \Lambda_2)U_2' \quad (28)$$

Both eq. (24) and eq. (28) are principal component decompositions of $T_1'T_1$. Since such decomposition is unique, apart from the order of the eigenvectors and eigenvalues, we know that Λ_1 and $(I - \Lambda_2)$ contain the same set of eigenvalues of $T_1'T_1$. By adjusting the sequence of column vectors in U_1, we can have

$$\Lambda_1 = I - \Lambda_2 \quad (29)$$

$$\Rightarrow 0 \le \lambda_{1,1} \le 1,\ 0 \le \lambda_{1,2} \le 1 \quad (30)$$

Note that

$$\bar{T}'\tilde{L}_0 = [T_1', T_2']\tilde{L}_0 = T_1'\begin{bmatrix} a_0 & c_0 \\ b_0 & -d_0 \end{bmatrix} + T_2' \cdot 0 \quad (31)$$

If $\lambda_{1,1} = 0$ or $\lambda_{1,2} = 0$, then T_1 has rank defect, then $\bar{T}'\tilde{L}_0$, the fault response matrix after transformation, has rank defect, which means there is only one fault response direction. This is not acceptable for directional residual based fault diagnosis. Thus we only consider the situation where

$$0 < \lambda_{1,1} \le 1,\ 0 < \lambda_{1,2} \le 1 \quad (32)$$

Let

$$V_1 = \Lambda_1^{-1/2} U_1' T_1' \quad (33)$$

From eq. (24), we have

$$V_1 V_1' = \Lambda_1^{-1/2} U_1' U_1 \Lambda_1 U_1' U_1 \Lambda_1^{-1/2} = I \quad (34)$$

Thus V_1 is an orthonormal matrix. From eq. (33) we know T_1' can be written as

$$T_1' = U_1 \Lambda_1^{1/2} V_1 = U_1 \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} V_1 = U_1 K V_1 \quad (35)$$

where U_1, V_1 are orthonormal matrices, and

$$K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_{1,1}} & 0 \\ 0 & \sqrt{\lambda_{1,2}} \end{bmatrix} \quad (36)$$

From eq. (32), we know

$$0 < k_1 \le 1,\ 0 < k_2 \le 1 \quad (37)$$

Bringing eq. (14), the fault response matrix for residual $\bar{r}_1$ becomes

$$\bar{T}'\tilde{L}_0 f = [T_1', T_2']\tilde{L}_0 f = [U_1 K V_1, T_2']\tilde{L}_0 f = U_1 K V_1 \cdot \tilde{r}_{f,0} + T_2' \cdot 0 = U_1 K V_1 \cdot \tilde{r}_{f,0} \quad (38)$$

Bringing eq. (38) to eq. (19), the residual $\bar{r}_1$ becomes

$$\bar{r}_1 = \bar{T}'\tilde{L}_0 f + v_1 = U_1 K V_1 \tilde{r}_{f,0} + v_1 \quad (39)$$

Compare $\bar{r}_1$ with

$$\bar{r}_2 = U_1'\bar{r}_1 = K \cdot V_1 \cdot \tilde{r}_{f,0} + U_1' v_1 \quad (40)$$

The full-rank rotation matrix U_1 will not affect the *PCI;* we are interested in the effects of K and V_1 only.

Since V_1 is an orthonormal matrix, it can be expressed as

$$V_1 = \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix} \quad (41)$$

Thus

$$\bar{r}_2 = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \cdot \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix} \cdot \tilde{r}_{f,0} + v_1 \quad (42)$$

This effectively shows the transformation as the combination of rotation and scaling (Fig. 2 shows a typical linear transformation decomposition.). End of proof.

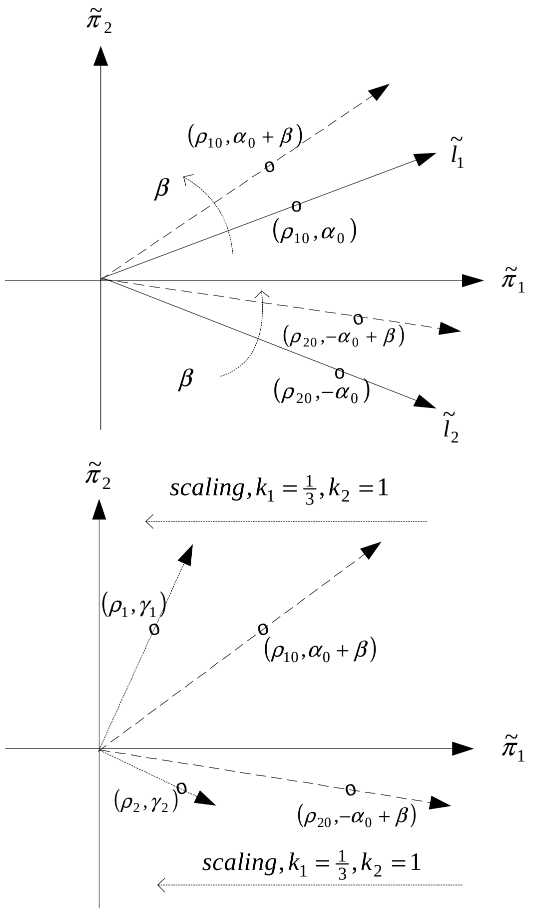

Fig. 2 Linear Transformation Decomposition in the Fault Isolation Subspace
(In the first figure, the fault response directions are rotated with an angle β ; in the second figure, a scaling operation with $k_1 = \frac{1}{3}, k_2 = 1$ is applied to the rotated fault response directions)

Now we express the PCI of $\bar{r}_1$ in terms of the rotation factor β and the scaling matrix K. According to eq. (42), the fault induced residual in the $\bar{r}_2$ space is

$$\bar{r}_{f,2} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \cdot \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix} \cdot \begin{bmatrix} a_0 & c_0 \\ b_0 & -d_0 \end{bmatrix} f \quad (43)$$

Bring (14) to (43)

$$\bar{r}_{f,2} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \cdot \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix} \cdot \begin{bmatrix} \rho_{10}\cos(\alpha_0) & \rho_{20}\cos(-\alpha_0) \\ \rho_{10}\sin(\alpha_0) & \rho_{20}\sin(-\alpha_0) \end{bmatrix} \quad (44)$$

$$\bar{r}_{f,2} = \begin{bmatrix} k_1\rho_{10}\cos(\alpha_0 + \beta) & k_1\rho_{20}\cos(-\alpha_0 + \beta) \\ k_2\rho_{10}\sin(\alpha_0 + \beta) & k_2\rho_{20}\sin(-\alpha_0 + \beta) \end{bmatrix} \quad (45)$$

Rewrite $\bar{r}_{f,2}$ as

$$\bar{r}_{f,2} = \begin{bmatrix} \rho_1\cos(\gamma_1) & \rho_2\cos(\gamma_2) \\ \rho_1\sin(\gamma_1) & \rho_2\sin(\gamma_2) \end{bmatrix} \quad (46)$$

and rotate it to standard fault isolation position where

$$\tilde{r}_{f,2} = \begin{bmatrix} \rho_1\cos(\alpha) & \rho_2\cos(\alpha) \\ \rho_1\sin(\alpha) & -\rho_2\sin(\alpha) \end{bmatrix} \quad (47)$$

we obtain

$$\rho_1 = \rho_{10} \cdot \sqrt{k_1^2\cos^2(\alpha_0 + \beta) + k_2^2\sin^2(\alpha_0 + \beta)} \quad (48)$$

$$\rho_2 = \rho_{20} \cdot \sqrt{k_1^2\cos^2(\alpha_0 - \beta) + k_2^2\sin^2(\alpha_0 - \beta)} \quad (49)$$

$$\alpha = \frac{\gamma_1 - \gamma_2}{2} \quad (50)$$

where

$$\gamma_1 = \arctan\left(\frac{k_2}{k_1}\tan(\alpha_0 + \beta)\right) \quad (51)$$

$$\gamma_2 = -\arctan\left(\frac{k_2}{k_1}\tan(\alpha_0 - \beta)\right) \quad (52)$$

γ_1 and $(\alpha_0 + \beta)$ belong to the same quadrant; γ_2 and $(-\alpha_0 + \beta)$ belong to the same quadrant.

The new PCIs after the linear transformation are:

$$PCI_1 = erfc*(-\rho_1\cos(\alpha)) \cdot erfc*(-\rho_1\sin(\alpha)) + erfc*(\rho_1\cos(\alpha)) \cdot erfc*(\rho_1\sin(\alpha)) \quad (53)$$

$$PCI_2 = erfc*(-\rho_2\cos(\alpha)) \cdot erfc*(\rho_2\sin(\alpha)) + erfc*(\rho_2\cos(\alpha)) \cdot erfc*(-\rho_2\sin(\alpha)) \quad (54)$$

8. NUMERICAL EXAMPLES

Consider a 3-residual 2-fault linear static system

$$r = Lf + Nv$$

where

$$L = \begin{bmatrix} 7 & 2 \\ 6 & 8 \\ 5 & 2 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \quad v \sim N(0, I)$$

Assume that when a fault happens, the actual sizes of each type of fault are $f_1 = 3$, $f_2 = 5$; assume there is a residual observation $R = [-1,1,2]'$.

The following are examples of calculating the PCI and of the linear transformation decomposition.

Example 1: PCI Calculation

Let $\bar{r} = N^{-1}r = \bar{L}f + \bar{v}$.

From *Lemma 1* we know $g_{21}(R) = g_{21}(\overline{R})$.

According to the discussion about the standard fault isolation subspace in Section 5, linear transformation matrices (such as T_r shown below) can be applied to the uncorrelated residual $\overline{r}$ to obtain the standard fault isolation subspace.

$$T_r = \begin{bmatrix} 0.7903 & 0.0368 & 0.6116 \\ 0.6123 & -0.0820 & -0.7863 \\ 0.0212 & 0.9959 & -0.0874 \end{bmatrix}$$

The new transformed residual becomes

$$\tilde{r}_0 = T_r'\overline{r} = T_r'\overline{L}f + T_r'\overline{v} = \tilde{L}_0 f + v_0$$

where the new fault response matrix

$$\tilde{L}_0 = T_r'\overline{L} = \begin{bmatrix} a_0 & c_0 \\ b_0 & -d_0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5.0195 & 6.3797 \\ 0.8970 & -1.1401 \\ 0 & 0 \end{bmatrix}$$

defines the standard fault isolation space and where $v_0 \sim N(0, I)$.

According to eq. (14), the fault-to-noise ratios and the fault response direction angle are derived as follows:

$$\alpha_0 = \arctan\left(\frac{b_0}{a_0}\right) = 0.1768 = 0.0563\pi$$

$$\rho_{10} = \sqrt{a_0^2 + b_0^2} \cdot f_1 = 15.2971$$

$$\rho_{20} = \sqrt{c_0^2 + d_0^2} \cdot f_2 = 32.4037$$

$$\begin{aligned} &PCI_{\tilde{r}_0, H_1}(f_1) \\ &= erfc*(-af_1) \cdot erfc*(-bf_1) + erfc*(af_1) \cdot erfc*(bf_1) \\ &= 0.8151 \end{aligned}$$

Example 2: Linear Transformation Decomposition

Applying an arbitrary linear transformation matrix

$$\overline{T} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 0.1361 & 0.1156 \\ -0.0158 & 0.9933 \\ 0.9906 & 0 \end{bmatrix}$$

to $\tilde{r}_0$, where

$$T_1 = \begin{bmatrix} 0.1361 & 0.1156 \\ -0.0158 & 0.9933 \end{bmatrix}$$

From eq. (24), eq. (33) and eq. (36),

$$T_1'T_1 = U_1\Lambda_1U_1' = I \cdot \begin{bmatrix} 0.0188 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V_1 = \Lambda_1^{-1/2}U_1'T_1' = \begin{bmatrix} 0.9933 & -0.1156 \\ 0.1156 & 0.9933 \end{bmatrix}$$

$$K = diag\{\sqrt{\lambda_{1,1}}, \sqrt{\lambda_{1,2}}\} = diag\{0.1370, 1\}$$

According to *Lemma 3*, the effect of $\overline{T}$ on the fault isolation subspace is a combination of rotation and scaling. From eq. (41) and (36), the rotation angle and scaling factors are:

$$\beta = 0.0369\pi,\ k_1 = 0.1370,\ k_2 = 1$$

9. CONCLUSION

The performance of the generalized likelihood ratio test (GLRT) for pair-wise directional residual based fault isolation is studied in this paper. Two invariant properties of the GLRT statistics are addressed, along with the idea of standard fault isolation subspace, to obtain the explicit form of the conditional probability of correct isolation (PCI). The effect of linear transformation is studied as well. It is shown to be a combination of rotation and scaling on the fault isolation subspace. The new PCI's are expressed as functions of the rotation angle and scaling factors.

The result about the linear transformation's effect on the PCI can be used to optimize the fault isolation performance. When applying a linear static matrix to the primary residual space, the equivalent rotation and scaling operations will usually increase the PCI of one fault while decreasing that of the other, thus the overall fault isolation performance changes under linear transformation. An optimization method has been developed (Hu, 2004) to find the point where the overall fault isolation performance reaches its best. It is applicable to any linear dynamic system that uses the pair-wise directional residual method for fault isolation. Details will be presented in a later paper.

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