ZENO MODE CONTROL OF UNDERACTUATED MECHANICAL SYSTEMS WITH APPLICATION TO PENDUBOT STABILIZATION AROUND THE UPRIGHT POSITION

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Abstract: Switched control synthesis is developed for underactuated mechanical systems. In order to locally stabilize an underactuated system around an unstable equilibrium, its output is specified in such a way that the corresponding zero dynamics is locally asymptotically stable. Once such an output has been chosen, the desired stability property of the closed-loop system is provided by applying a Zeno mode controller, driving the system to the zero dynamics manifold in finite time. Although the present synthesis exhibits the Zeno behavior with an infinite number of switches on a finite time interval, it does not rely on the generation of sliding modes, while providing robustness features similar to those possessed by their sliding mode counterparts. Theoretical results are supported by a simulation study made for a Pendubot.

Keywords: Periodic motion, finite-time convergence, underactuated systems.

1. INTRODUCTION

Stabilization of underactuated systems, forced by fewer actuators than degrees of freedom, presents a challenging problem. These systems, as well known (see, e.g., (Berkemeier and Fearing, 1999; Zhang and Tarn, 2002)), possess nonholonomic properties, caused by nonintegrable differential constraints, and therefore, they can not be stabilized, even locally, by means of smooth feedback.

In the present paper a switched control synthesis procedure is developed to locally stabilize underactuated mechanical systems with unstable equilibria. In order to locally stabilize such a system around an unstable equilibrium, an output of the system is specified to ensure that the corresponding zero dynamics is locally asymptotically stable. Once such an output has been chosen, the desired stability property of the closed-loop system is provided by applying a switched controller, driving the system to the zero dynamics manifold in finite time.

The structure of the switched controller constructed is inspired from that of (Orlov, 2005; Orlov *et al.*, 2003c), stabilizing a one-link manipulator in finite time. Although that controller exhibited the so-called Zeno behavior (Liberzon, 2003; Lygeros *et al*., 2003) with an infinite number of switches on a finite time interval, it did not rely on the generation of sliding modes, while providing robustness features similar to those possessed by their sliding mode counterparts. The so-called second order sliding mode (Levant, 1993) appeared in the equilibrium point only. In contrast to standard sliding mode control algorithms which are capable of providing the closed-loop manipulator with the ultimate boundedness property only (Lu and Spurgeon, 1997), the Zeno mode-based controller stabilized the manipulator in finite time, thus constituting an interesting alternative to standard sliding mode controllers.

Capabilities of the Zeno mode synthesis procedure, developed in the paper, are illustrated in a simulation study made for a simple underactuated Pendulum robot, typically abbreviated as Pendubot.

The rest of the paper is organized as follows. Section 2 is focused on the Zeno mode stabilization of a simple one degree-of-freedom manipulator, operating under uncertainty conditions. In Section 3, the proposed Zeno mode synthesis procedure is extended to underactuated mechanical systems and its effectiveness is then illustrated by the numerical simulation of balancing the Pendubot around its inverted equilibrium. Section 4 presents some conclusions.

2. ROBUST FINITE TIME STABILIZATION OF ONE LINK MANIPULATOR

The Zeno mode controller design is first illustrated with a simple one degree-of-freedom mechanical manipulator, operating under uncertainty conditions. The dynamics of the manipulator is governed by

$$\ddot{y} = \omega(y, \dot{y}, t) + u \tag{1}$$

where y is the position of the manipulator, $\dot{y}$ is the velocity of the manipulator, u is the controlled input, $\omega(y, \dot{y}, t)$ is a piece-wise continuous nonlinearity that captures all forces (viscous and Coulomb frictions, gravitation, etc.), affecting the manipulator.

Operating under uncertainty conditions implies imperfect knowledge of the nonlinearity $\omega(y, \dot{y}, t)$. This possibly destabilizing term

$$\omega(y, \dot{y}, t) = \omega^{nom}(y, \dot{y}, t) + \omega^{un}(y, \dot{y}, t) \tag{2}$$

typically contains an *apriori* known nominal part $\omega^{nom}(y, \dot{y}, t)$ to be handled through nonlinear damping and an uncertainty $\omega^{un}(y, \dot{y}, t)$ to be rejected. It is assumed that $\omega^{un}(y, \dot{y}, t)$ is locally bounded

$$|\omega^{un}(y, \dot{y}, t)| \leq N \; for \; all \; t \geq 0 \tag{3}$$

by an *apriori* known constant $N > 0$. Apart from this, both functions $\omega^{nom}(y, \dot{y}, t)$ and $\omega^{un}(y, \dot{y}, t)$ are assumed to be piece-wise continuous.

The following control law

$$\begin{aligned} u = -\omega^{nom}(y, \dot{y}, t) - asign(y) - bsign(\dot{y}) \\ -hy - p\dot{y} \end{aligned} \tag{4}$$

subject to

$$N < b < a - N, \; h, p \geq 0 \tag{5}$$

appears to stabilize the uncertain system (1)-(3) in finite time.

Until recently, finite time stability of asymptotically stable homogeneous systems has been well-recognized for only continuous vector fields (Bhat and Bernstein, 1997; Hong *et al.*, 2001). Extending this result to switched systems has required proceeding differently (Orlov, 2005) because a smooth homogeneous Lyapunov function, whose existence was proven in (Rosier, 1992) for continuous asymptotically stable homogeneous vector fields, can no longer be brought into play.

The novel techniques that was developed in (Orlov, 2005) has established that the finite time stability of a switched homogenous system persists regardless of some inhomogeneous perturbations. In particular, it has been shown that the inhomogeneous system (1)-(5) is globally finite time stable whenever condition (3) holds globally. The local version of this result is as follows.

Theorem 1. Let a one link manipulator (1)-(3) be driven by a switched controller (4), (5). Then the closed-loop system (1)-(5) is locally finite time stable, uniformly in the admissible uncertainties (2), (3).

Proof of Theorem 1 follows the line of reasoning used in the proof of Theorem 4.2 of (Orlov, 2005). This line is applicable to the closed-loop system (1)-(5) because the stabilizing controller (4), (5) consists of the nonlinear damping $-\omega^{nom}(y, \dot{y}, t)$, the homogeneous switching part $-asign(y) - bsign(\dot{y})$ and the remainder $-hy - p\dot{y}$ that vanishes in the origin $y = \dot{y} = 0$. The detailed proof of Theorem 1 is nearly the same as that of Theorem 4.2 of (Orlov, 2005) and it is therefore omitted.

The qualitative behavior of the one link manipulator (1)-(3), driven by the switched controller (4), (5), is as follows. While approaching the origin $y = \dot{y} = 0$, the system trajectories rotate around it. Since by Theorem 1, the closed-loop system is locally finite time stable, the switching times of the controller have a finite accumulation point.

Thus, system (1)-(5) does exhibit Zeno behavior with an infinite number of switches in a finite amount of time. This system does not generate sliding motions everywhere except the origin. If a trajectory starts there at any given finite time, the so-called sliding mode of the second order appears (Levant, 1993). In a particular case, when the uncertainty $\omega(y, \dot{y}, t) = \omega^{un}(y, \dot{y}, t)$ has no nominal part and the control gains h, p are set to zero, the proposed control law (4) degenerates to the well-known homogeneous twisting algorithm (Fridman and Levant, 2002).

3. STABILIZATION OF UNDERACTUATED MECHANICAL SYSTEMS

In the present section, the Zeno mode control synthesis is developed for stabilization of underactuated systems of the form

$$\ddot{q} = M^{-1}(q)[B\tau - C(q,\dot{q})\dot{q} - G(q)] \quad (6)$$
$$z = \Omega(q,\dot{q}). \quad (7)$$

In the above equation, $q \in \mathbb{R}^n$ is the joint position vector, $\tau \in \mathbb{R}^m,\ m < n$ is the input torque, $z \in \mathbb{R}^m$ is the output vector, $\dot{q}$ and $\ddot{q}$ are the velocity and acceleration vectors, respectively, $M(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $C(q,\dot{q})\dot{q}$ represents centrifugal and Coriolis terms, $G(q)$ is the gravity vector, and B of rank m is the input matrix that maps the torque input τ of dimension m to the joint coordinates space of dimension n.

Under certain conditions system (6), (7) has m-vector relative degree $(2,\dots,2)^T$ at $x = 0$ and the distribution, spanned by the columns of the control matrix, is involutive. Just in case, it can be represented, by means of a nonlinear change of state coordinates and a feedback transformation (see (Byrnes and Isidori, 1991) for details), in the form

$$\begin{aligned}\dot{x} &= g(x,\xi,\dot{\xi})\\ \ddot{\xi} &= f(x,\xi,\dot{\xi}) + u.\end{aligned} \quad (8)$$

If in addition, this system is locally minimum phase and sufficiently smooth, it can locally be stabilized by a Zeno mode controller similar to (4). Throughout, the following assumptions are imposed on system (8).

(1) The functions $g(x,\xi,\dot{\xi})$ and $f(x,\xi,\dot{\xi})$ are piece-wise continuous in all the arguments, and in addition, the function $g(x,\xi,\dot{\xi},t)$ is continuous in $(\xi,\dot{\xi})$ locally around $(\xi,\dot{\xi}) = 0$ for almost all x.

(2) Given piece-wise differentiable functions $\xi(t),\dot{\xi}(t)$ of sufficiently small magnitudes, an arbitrary solution of the system

$$\dot{x} = g(x,\xi(t),\dot{\xi}(t)) \quad (9)$$

is bounded on any finite time interval.

(3) The system

$$\dot{x} = g(x,0,0) \quad (10)$$

has 0 as a locally asymptotically stable equilibrium.

Solutions of the state and zero dynamics differential equations (8)-(10) with piece-wise continuous right-hand sides, are defined in the sense of Filippov (Filippov, 1988) as that of a certain differential inclusion with a multi-valued right-hand side. Under Assumption 1 the existence of a solution (possibly non-unique) of either equation with an arbitrary initial condition is guaranteed by Theorem 8 of (Filippov, 1988, p. 85).

Other assumptions are made for technical reasons. Assumption 2 is introduced to avoid the destabilizing effect of the peaking phenomenon and particularly it holds whenever the function $g(x,\xi,\dot{\xi})$ satisfies a corresponding linear growth condition in x (cf. that of (Sussman and Kokotovic, 1991)). Assumption 3 means that (8), specified with the output $z = \xi$, is a locally minimum phase system. The role of this notion is well-known from the theory of smooth fields (Byrnes and Isidori, 1991) and it is now under study for switched nonautonomous systems.

As in the manipulator case (1), system (8) is operating under uncertainty conditions. The nonlinear gain g can not destabilize the closed-loop system because of the minimum phase hypothesis, which is why no more information is required for this gain. The destabilizing term

$$f(x,\xi,\dot{\xi}) = f^{nom}(x,\xi,\dot{\xi}) + f^b(x,\xi,\dot{\xi}) \quad (11)$$

is partitioned into a nominal part f^{nom}, known *apriori*, and an uncertain bounded gain f^b whose components $f_j^b,\ j = 1,\dots,m$ are locally upper estimated

$$|f_j^b(x,\xi,\dot{\xi})| \le N_j \quad (12)$$

by *apriori* known constants $N_j > 0$. Apart from this, both functions f^{nom} and f^b are assumed to be piece-wise continuous.

Being inspired from the Zeno mode controller (4), (5), the following switched control law

$$\begin{aligned}u(x,\xi,\dot{\xi}) = &-f^{nom}(x,\xi,\dot{\xi}) - \alpha sign\,\xi\\ &-\beta sign\,\dot{\xi} - H\xi - P\dot{\xi}\end{aligned} \quad (13)$$

with the parameter gains

$$\begin{aligned}H = diag\{h_j\},\ P = diag\{p_j\},\\ \alpha = diag\{\alpha_j\},\ \beta = diag\{\beta_j\}\end{aligned} \quad (14)$$

subject to

$$\begin{aligned}N_j < \beta_j < \alpha_j - N_j,\\ h_j, p_j \ge 0,\ j = 1,\dots,m,\end{aligned} \quad (15)$$

is proposed to locally stabilize the uncertain system (8), (11), (12) whose state $(x,\xi,\dot{\xi})$ is available for measurements. Hereafter, the notation $diag$ is used to denote a diagonal matrix of an appropriate dimension; $sign\,\xi$ with a vector $\xi = (\xi_1,\dots,\xi_m)^T$ stands for the column vector $(sign\,\xi_1,\dots,sign\,\xi_m)^T$.

In what follows, the switched control law (13), (15) is shown to drive the uncertain system (8) to the zero dynamics manifold $\xi = \dot{\xi} = 0$ in finite time thereby yielding desired stability properties of the closed-loop system.

Theorem 2. Let Assumptions 1-3 be satisfied and let the uncertain system (8), (11), (12) be driven by the state feedback (13) such that condition (15) holds. Then the closed-loop system (8), (13)-(15) is locally asymptotically stable, uniformly in the admissible uncertainties (11), (12).

Proof of Theorem 2 is similar to that of Theorem 5.1 of (Orlov, 2005) and it is omitted because of space limitations.

Summarizing, the following Zeno mode stabilization procedure is proposed for underactuated systems. First, an output of the system is specified in such a way that the corresponding zero dynamics is locally asymptotically stable. Once such an output has been chosen, the underactuated system is transformed into the normal form (8), whose stabilization is achieved by applying the Zeno mode controller (13), (15).

In the sequel, the effectiveness of the proposed procedure is illustrated in a simulation study of the Pendubot stabilization.

4. CASE OF STUDY: PENDUBOT

A pendubot is a simple underactuated mechanical manipulator, whose first link (shoulder) is actuated whereas the second one (elbow) is not actuated (see Fig. 1). The Pendubot state $q = (q_1, q_2)^T$ is governed by equation (6) subject $n = 2,\ m = 1$ and specified with (Utkin *et al.*, 1999, p.55):

$$B = \begin{bmatrix} 1 & 0 \end{bmatrix}^T,\ M(q) = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} =$$

$$\begin{bmatrix} J_m + J_1 + m_1 l_1^2 + m_2 L_1 L_1 & m_2 L_1 l_2 \cos(q_1 - q_2) \\ m_2 L_1 l_2 \cos(q_1 - q_2) & m_2 l_2^2 + J_2 \end{bmatrix},$$

$$C(q, \dot{q}) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & m_2 L_1 l_2 \sin(q_1 - q_2)\dot{q}_2 \\ -m_2 L_1 l_2 \sin(q_1 - q_2)\dot{q}_1 & 0 \end{bmatrix},$$

$$G(q) = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} -g(m_1 l_1 + m_2 L_1)\sin q_1 \\ -m_2 g l_2 \sin q_2 \end{bmatrix}. \tag{16}$$

By inverting the matrix M, the Pendubot equations (6), (16) are simplified to

$$\begin{aligned} \ddot{q}_1 &= f_1(q, \dot{q}) + \frac{m_{22}}{\Delta}\tau \\ \ddot{q}_2 &= f_2(q, \dot{q}) - \frac{m_{12}}{\Delta}\tau \end{aligned} \tag{17}$$

where

$$f_1 = \frac{m_{12}(C_{21}\dot{q}_1 + G_2) - m_{22}(C_{12}\dot{q}_2 + G_1)}{\Delta} \tag{18}$$

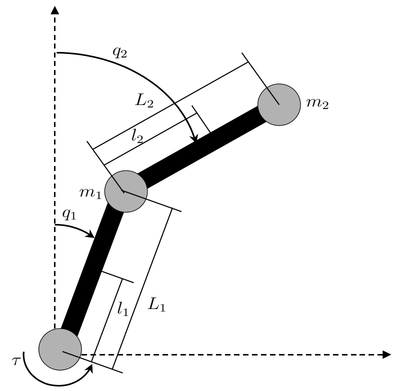

Fig. 1. Schematic diagram of Pendubot.

$$f_2 = \frac{m_{12}(C_{12}\dot{q}_2 + G_1) - m_{11}(C_{21}\dot{q}_1 + G_2)}{\Delta} \tag{19}$$

and

$$\Delta = m_{11}m_{22} - m_{12}^2 > 0 \tag{20}$$

because the inertia matrix M is positive definite. The physical sense of the parameters g, J_m, m_i, l_i, L_i, J_i $i = 1, 2$ is given in Table I.

Table 1. Pendubot parameters.

Description	**Notation**	**Value**	**Units**
lenght of link 1	L_1	0.2032	m
lenght of link 2	L_2	0.2540	m
center of mass 1	l_1	0.1574	m
center of mass 2	l_2	0.1109	m
mass of link 1	m_1	0.132	Kg
mass of link 2	m_2	0.088	Kg
inertia 1	J_1	3.62×10^{-3}	Kg m^2
inertia 2	J_2	1.14×10^{-3}	Kg m^2
motor inertia	J_m	6×10^{-5}	Kg m^2
gravity acceleration	g	9.8	m/seg^2

The Pendubot has four equilibria, one of them $(q_1, q_2) = (\pi, \pi)$ is stable and the others $(q_1, q_2) = (0, \pi)$, $(q_1, q_2) = (\pi, 0)$, $(q_1, q_2) = (0, 0)$ are unstable. We are interested in the Zeno mode-based local stabilization of the Pendubot around the upright position $(q_1, q_2) = (0, 0)$, the most difficult case for feedback stabilization among all the equilibria.

Our study is confined to the case

$$\gamma = \frac{m_2 l_2^2 + J_2}{m_2 L_1 l_2} > 1. \tag{21}$$

As reported in (Utkin *et al.*, 1999, p. 58), under this assumption the Pendubot motion, being driven along the manifold

$$\sin q_2 + k_1 w + k_2 \dot{w} = 0 \tag{22}$$

where $k_1 > 0$, $k_2 > 0$,

$$w(q_1, q_2) = q_2 - \varphi(q_1 - q_2), \tag{23}$$

$$\varphi(\nu) =$$

$$-\nu-\frac{2\gamma}{\sqrt{\gamma^2-1}}\tan^{-1}\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\tan\left(\frac{\nu}{2}\right)\right),\quad(24)$$

is locally asymptotically stable at the upright position. Thus, system (6)-(16) becomes locally minimum phase if the system output (7) is specified as follows

$$\mu=\sin q_2+k_1w+k_2\dot{w}. \quad (25)$$

For later use, let us denote

$$A=\frac{m_2gl_2}{m_2L_1l_2},\ \nu=q_1-q_2,\ \eta=\frac{1}{\gamma+\cos\nu}, \quad (26)$$

$$\delta=\eta\sin\nu\left[(\dot{w}+\gamma\eta\dot{\nu})^2-\gamma\eta\dot{\nu}^2\right]+A\eta\sin(q_2), \quad (27)$$

where

$$\dot{\nu}=\dot{q}_1-\dot{q}_2,\ \dot{\eta}=\eta^2\sin(\nu)(\dot{q}_1-\dot{q}_2) \quad (28)$$

and by virtue of (23), (24)

$$\dot{w}=\dot{q}_2-\eta\cos(\nu)(\dot{q}_1-\dot{q}_2). \quad (29)$$

Then differentiating (22) along the solutions of (17) yields

$$\dot{\mu}=\cos(q_2)\dot{q}_2+k_1\dot{w}+k_2\delta, \quad (30)$$
$$\ddot{\mu}=F(q_1,q_2,\dot{q}_1,\dot{q}_2)+u \quad (31)$$

where

$$u=\Phi(q_1,q_2,\dot{q}_1,\dot{q}_2)\tau, \quad (32)$$

$$\begin{aligned}F=&\cos q_2f_2-\sin(q_2)\dot{q}_2^2+k_1\delta\\&+k_2\left[\dot{\eta}\sin(\nu)+\eta\cos(\nu)\dot{\nu}\right](\dot{w}+\gamma\eta\dot{\nu})^2\\&+2k_2\eta\sin(\nu)(\dot{w}+\gamma\eta\dot{\nu})(\delta+\gamma\dot{\eta}\dot{\nu})\\&-2\gamma k_2\eta\sin(\nu)\dot{\eta}\dot{\nu}^2-\gamma k_2\eta^2\cos(\nu)\dot{\nu}^3\\&+Ak_2\dot{\eta}\sin(q_2)+Ak_2\eta\cos(q_2)\dot{q}_2\\&2\gamma k_2\eta^2\sin\nu\left[\dot{w}+\dot{\nu}(\gamma\eta-1)\right](f_1-f_2),\end{aligned}\quad(33)$$

$$\begin{aligned}\Phi=&\ 2\gamma k_2\eta^2\sin\nu\left[\dot{w}+\dot{\nu}(\gamma\eta-1)\right]\\&-\frac{1}{\Delta}m_{12}\cos q_2.\end{aligned}\quad(34)$$

Due to (28), (29),

$$\Phi(q_1,q_2,\dot{q}_1,\dot{q}_2)|_{(0,0,0,0)}\neq 0 \quad (35)$$

which is why the locally minimum phase system (17), (25) has relative degree 2 at the origin. Thus, the Zeno mode control law (13) becomes applicable to the stabilization of the Pendubot around its upright position.

Remark 1. The applicability of Assumption 2 to system (17), (25) is not studied here in details, but only simulation evidences, demonstrating that this is indeed the case, are presented.

While being specified for system (17), (25), the Zeno mode control law (13), (15), corresponding to

$$m=1,\ f^{nom}=F,\ f^b=0,\ N_1=0,\ H=0,\ P=0,$$

takes the form

$$u(q,\dot{q})=-F(q,\dot{q})-\alpha_1 sign\ \mu-\beta_1 sign\ \dot{\mu} \quad (36)$$

where the amplitudes α_1 and β_1 of switching are positive constants subject to

$$\alpha_1>\beta_1, \quad (37)$$

$F(q,\dot{q})$ is governed by (33), and $\mu,\dot{\mu}$ are viewed as functions of $(q,\dot{q})$, which are defined by relations (25)-(30).

4.1 Simulation Results

In the simulation study, performed with SIMNON, the Zeno mode-based controller

$$\tau(q,\dot{q})=\frac{u(q,\dot{q})}{\Phi(q,\dot{q})}, \quad (38)$$

coupled to (36), (37), was applied to the Pendubot to move it from a perturbed state $q_1(0),\ q_2(0)$ to the inverted equilibrium point $q_1=0,\ q_2=0$. Due to (35), the resulting controller (36)-(38) is locally bounded. In order to illustrate the size of the attraction domain of the controller the initial conditions $q_1(0)=0.5rad,\ q_2(0)=0.3rad$ for the simulation were chosen reasonably far from the upright position whereas the initial velocity $\dot{q}(0)$ were set to zero.

To make a physical sense the numerical simulation addressed a real model, presented in (Utkin *et al.*, 1999). The values of the model parameters are listed in Table I. For this set of the parameters restriction (21) is satisfied with $\gamma=1.15$.

Figs. 2 and 3 show simulation results with the controller gains $\alpha_1=10,\ \beta_1=1$ and parameters $k_1=k_2=0.174$. The link positions $q_1(t)$ and $q_2(t)$ are shown in Fig. 2. The control torque $\tau(t)$ is presented in Fig. 3. From these figures, good performance of the Zeno mode controller is concluded for local stabilization of the Pendulum around the upright position.

5. CONCLUSIONS

Zeno mode-based control synthesis is developed to locally asymptotically stabilize underactuated mechanical systems. The stabilizing strategy is to drive the system to the zero dynamics manifold in finite time and

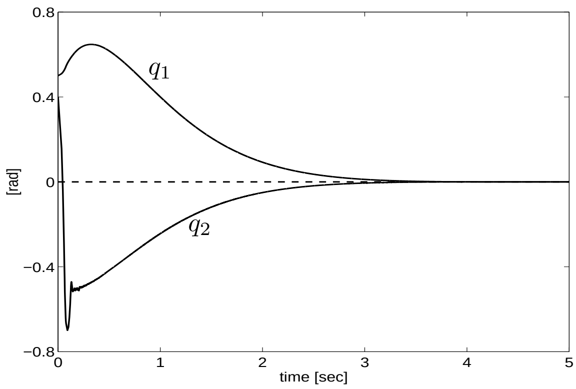

Fig. 2. Pendulum Stabilization at the Upright Position.

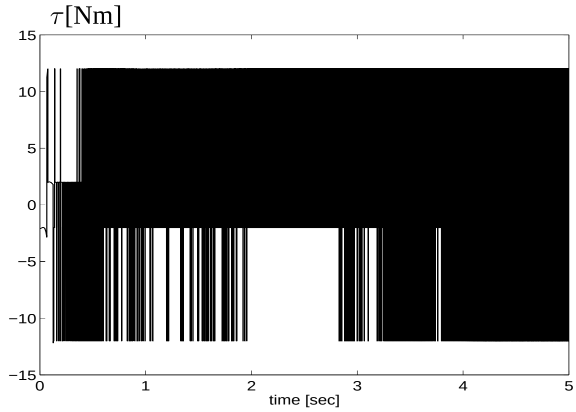

Fig. 3. Zeno Mode Controller.

maintain it there in spite of parameter uncertainties and external disturbances. Desired robustness properties and asymptotic stability of the closed-loop system are thus provided.

The proposed control synthesis presents an interesting alternative to the standard sliding mode control techniques. Although the resulting controllers do exhibit Zeno modes with an infinite number of switches on a finite time interval, however, they do not rely on the generation of sliding motions on the switching manifolds but on their intersections. Performance issues of the proposed synthesis procedure are illustrated in a simulation study made for an underactuated two-link inverted pendulum.

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