# INTERVAL TECHNIQUE FOR PARAMETER ESTIMATION UNDER MODEL UNCERTAINTY

**Boris T. Polyak, Sergey A. Nazin**

*Institute for Control Science RAS*
*65 Profsoyuznaya str., 117997 Moscow, Russia*
*E-mails: {boris, snazin}@ipu.rssi.ru*

Abstract: This paper is devoted to the problem of estimation of parameters for linear multi-output models with uncertain regressors and additive noise. The uncertainty is assumed to be described by intervals. Outer-bounding interval approximations of the non-convex feasible parameter set for uncertain system are obtained. The proposed method is based on the calculation of the interval solution for an interval system of linear algebraic equations and provides the parameter estimators for models with large number of measurements.


Keywords: bounded noise, intervals, linear equations, parameter estimation, uncertain linear systems, uncertainty.

## 1. INTRODUCTION

The set-membership estimation framework for uncertain systems has attracted much attention during the past few decades. It is an alternative to the stochastic approach where some prior information on the statistical distribution of errors is needed, because only bounds on uncertainty in system parameters and signals are required. This assumption is often much more acceptable in practice. Various types of compact sets (intervals, polytopes, ellipsoids, etc.) are usually used to characterize these bounds. They are called the membership set constraints on uncertain variables.

The parameter estimation problem for uncertain dynamic systems is one of the most natural in this context. The problem is to determine bounds or set constraints on system parameters based on output measurements, the model structure and bounds on uncertain variables. In this paper we focus on the so-called interval type of uncertainty. This means that each component of an uncertain vector or matrix is assumed to belong to a known finite interval. Although the description is natural and simple (Jaulin, *et al.*, 2001), combinatorial difficulties may become so severe as to make estimation intractable especially in the multidimensional case. The goal of the paper is to construct an effective interval parameter estimator for uncertain multi-output static system that can be used with large sets of data.

Consider a linear regression model under interval uncertainty

$$y = Cx + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is an unknown parameter vector, $y \in \mathbb{R}^m$ denotes a vector of results of measurements, $C \in \mathbb{R}^{m\times n}$ is a matrix of regressors and $w \in \mathbb{R}^m$ is an unknown vector of measurement errors. The classical parameter bounding approach is based on the assumption that matrix $C$ is known precisely while vector $w$ is bounded and lies in the box $w \in [\underline{w}, \overline{w}]$, where $\underline{w}$ and $\overline{w}$ are known. The sequence of measurements $y_1, \dots, y_m$ then provides a convex polytope in the parameter vector space. A number of methods has been developed to characterize this polytope or to construct the techniques for outer-bounding approximation of it (Kostousova, 1998; Kurzhanskii and Valyi, 1997; Milanese, *et al.*, 1996; Norton, 1994, 1995; Walter and Piet-Lahanier, 1989; Walter, 1990).

The present paper deals with more general problem where the matrix of regressors is also uncertain, i.e. $C \in \mathbf{C}$, and $\mathbf{C} \in \mathbb{IR}^{m\times n}$ is an interval matrix (the standard notation $\mathbb{IR}^{m\times n}$ or $\mathbb{IR}^n$ indicates the space of all interval $m \times n$-matrices or $n$-dimensional interval vectors respectively). This situation arises in many real-life problems when we do not have complete information concerning the plant. Furthermore, weakly non-linear systems can be treated in the same manner if non-linearity is replaced by uncertainty. Particular cases of this problem have been considered in the literature (Cerone, 1993; Norton, 1999; Walter, 1990). The presence of matrix uncertainty in the model leads to serious difficulties due to the non-convexity of the resulting set constraints. Ellipsoidal techniques, see (Chernousko and Rokityanskii, 2000; Polyak, *et al.*, 2004), were applied to state and parameter estimation for linear models with matrix uncertainty; the non-convexity of reachable and feasible parameter sets was pointed out. The main purpose of this article is to apply an interval technique to parameter estimation. The approach proposed in previous paper (Polyak and Nazin, 2004) for calculating the interval solution of linear interval systems of equations is taken as a basic tool.

## 2. PROBLEM STATEMENT

Consider a multi-output model with measurement noise and uncertainty in the matrix of regressors

$$y = (C + \Delta C)x + w, \tag{2}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $C \in \mathbb{R}^{m\times n}$ and $w \in \mathbb{R}^m$. The number of measurements is usually much larger than the dimension of the parameter vector, so $m \gg n$. Assume

$$\|\Delta C\|_\infty \le \varepsilon, \quad \|w\|_\infty \le \delta. \tag{3}$$

The infinity norms of matrices and vectors are equal to the maximal absolute value of their elements, i.e.

$$\|\Delta C\|_\infty = \max_{\substack{1\le i\le m,\\ 1\le j\le n}} |(\Delta C)_{ij}|, \quad \|w\|_\infty = \max_{1\le i\le m} |w_i|. \tag{4}$$

Inequalities in (3) describe a particular case of interval uncertainty when all components of uncertain matrix $\Delta C$ or uncertain vector $w$ have the same bounds. Matrix $C$, vector $y$ and scalars $\varepsilon$, $\delta$ are assumed to be known. All vectors $x \in \mathbb{R}^n$ satisfying (2) under the constraints (3) form the *feasible parameter set*

$$X = \{x \in \mathbb{R}^n : \; y = (C + \Delta C)x + w, \\ \|\Delta C\|_\infty \le \varepsilon, \; \|w\|_\infty \le \delta\}.$$

Assume that $x \in \mathbf{X_0}$, where $\mathbf{X_0} \in \mathbb{IR}^n$. The initial approximation $\mathbf{X_0}$ should be taken large enough to guarantee inclusion of all parameter vectors of interest. The problem is to construct a more accurate outer-bounding interval approximation for the vector $x$ in accordance with the large number of measurements $y_1, \ldots, y_m$ and model structure (2)–(3). In other words, we look for an interval vector $\mathbf{X} \in \mathbb{IR}^n$ (preferably of minimal size) containing the intersection $\mathbf{X_0} \bigcap X$.

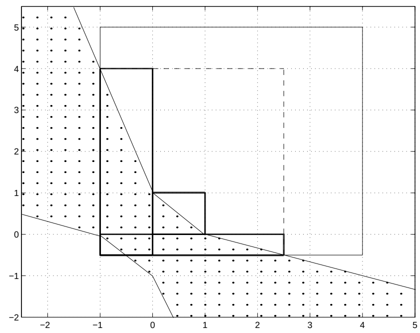


Fig. 1. Single-output interval param. bounding.

## 3. SCALAR OBSERVATION

The single-measurement case ($m = 1$) is a good particular example for the parameter estimation. Set $X$ for the scalar model

$$y = (c + \Delta c)^T x + w \tag{5}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $\|\Delta c\|_\infty \le \varepsilon$ and $|w| \le \delta$ can be explicitly rewritten as

$$X_1 = \{x \in \mathbb{R}^n : \; |y - c^T x| \le \varepsilon \|x\|_1 + \delta\}, \tag{6}$$

where $\|x\|_1 = \sum_{i=1}^n |x_i|$. Figure 1 depicts a typical shape of $X_1$, which is non-convex for any $\varepsilon > 0$ (the region between two solid polygonal lines). This set reduces to a strip as $\varepsilon \to 0$. However it is convex in each orthant of $\mathbb{R}^n$. Let $E^k$ be the $k$-th orthant of the vector space, $k = 1, \ldots, 2^n$. If $x \in E^k$ for any fixed number $k$, the right-hand side of the inequality in (6) becomes a linear function and therefore $X_1 \bigcap E^k$ is a convex set.

The smallest interval vector $\mathbf{X}$ containing $\mathbf{X_0} \bigcap X_1$ can be found by solving a linear programming problem in each orthant of $\mathbb{R}^n$. Indeed, the vector $s^k = \operatorname{sign} x$ for any $x \in E^k$ is uniquely defined with elements $s_i^k$ such that $|s_i^k| = 1$. Thus

$$X_1 \bigcap E^k = \left\{ x : \; |y - c^T x| \le \varepsilon \sum_{i=1}^n x_i s_i^k + \delta \right\} \tag{7}$$

is a convex set given by linear constraints. Denote by $e^j$ the $j$-th ort of $\mathbb{R}^n$, $j = 1, \ldots, n$. Then the $j$-th lower and upper bounds on the intersection $\mathbf{X_0} \bigcap X_1 \bigcap E^k$ are calculated by linear programming as

$$\begin{aligned} \underline{x_j^k} &= \arg \min_{x \in \mathbf{X_0} \bigcap X_1 \bigcap E^k} x^T e^j, \\ \overline{x_j^k} &= \arg \max_{x \in \mathbf{X_0} \bigcap X_1 \bigcap E^k} x^T e^j, \end{aligned} \tag{8}$$

Hence $\mathbf{X^k} = ([\underline{x_1^k}, \overline{x_1^k}], \ldots, [\underline{x_n^k}, \overline{x_n^k}])^T$ gives an interval vector that is the minimal box containing $\mathbf{X}_0 \bigcap X_1 \bigcap E^k$.

Notice however that the intersection of the set $\mathbf{X_0} \bigcap X_1$ with some orthants may be empty. The calculations in these orthants can obviously be omitted as far as the linear programming problem (8) turns out to become infeasible.

Further, let $K = \{k: \ \mathbf{X_0} \bigcap X_1 \bigcap E^k \neq \emptyset\}$. Then $\{E^k : k \in K\}$ represents a family of orthants containing $\mathbf{X_0} \bigcap X_1$. Checking all orthants $E^k$ such that $k \in K$ we obtain the inclusion $\mathbf{X_0} \bigcap X_1 \subseteq \bigcup_{k\in K} \mathbf{X^k}$. Finally, take

$$\underline{x_i} = \min_{k\in K} \left\{ \underline{x_i^k} \right\}, \ \ \overline{x_i} = \max_{k\in K} \left\{ \overline{x_i^k} \right\}, \ \ i = 1, ..., n. \quad (9)$$

$\mathbf{X} = ([\underline{x_1}, \overline{x_1}], ..., [\underline{x_n}, \overline{x_n}])^T$ gives the optimal interval approximation of $\mathbf{X}_0 \bigcap X_1$.

*Example 1.* Let $\mathbf{X_0} = ([-1, 4], [-0.5, 5])^T$ and $y = 0$, $c = (1, 1)^T$, $\varepsilon = \delta = 0.5$. The set $X_1 = \{x \in \mathbb{R}^2 : 2|x_1 + x_2| \leq |x_1| + |x_2| + 1\}$ is shown on Figure 1 (shaded region). The auxiliary interval vectors $\mathbf{X^k}$ are found via linear programming according to (8) in each orthant $E^k$, $k = 1, ..., 4$ (bold boxes). Then $\mathbf{X} = ([-1, 2.5], [-0.5, 4])^T$.

In the multi-output case, one can consider the scalar observations recursively and apply the above linear programming procedure. However this technique becomes unsuitable for models with a large number of measurements ($m \gg n$). The main idea of the present paper is to consider blocks of $n$ measurement equations in (2) and to treat each of them as a system of linear algebraic equations under interval uncertainties. A simple algorithm to obtain an interval solution for this system is described in the next section.

## 4. INTERVAL SYSTEM OF LINEAR ALGEBRAIC EQUATIONS

Let $C \in \mathbb{R}^{n\times n}$, i.e. the number of parameters is equal to the number of observations. Then rewrite (2) in the form

$$(A + \Delta A)\, x = b + \Delta b \quad (10)$$

with $A = C$, $b = y$, $\Delta A = \Delta C$ and $\Delta b = -w$ such that $\|\Delta A\|_\infty \leq \varepsilon$, $\|\Delta b\|_\infty \leq \delta$. The calculation of the interval solution for the square interval system of equations (10) is a challenging problem in numerical analysis and robust linear algebra. This problem was first considered at 1960-th by Oettli and Prager (1964). Since then, the problem has attracted much attention and was developed in the context of the modelling of uncertain systems.

Assume that the matrix family $\mathbf{A} = \{A + \Delta A : \|\Delta A\|_\infty \leq \varepsilon\} \in \mathbb{IR}^{n\times n}$ is nonsingular (it contains no singular matrix) and that the interval vector $\mathbf{b} = \{b + \Delta b : \|\Delta b\|_\infty \leq \delta\} \in \mathbb{IR}^n$. Then for any $A \in \mathbf{A}$ and any $b \in \mathbf{b}$ the ordinary linear system $Ax = b$ has a unique solution. We are interested in a set $\hat{X}$ of all these solutions of the interval system:

$$\hat{X} = \{x \in \mathbb{R}^n : \ Ax = b,\ A \in \mathbf{A},\ b \in \mathbf{b}\}. \quad (11)$$

Our main objective is to find an interval solution of the linear interval system (10), that is, to determine the smallest interval vector $\mathbf{X}^*$ containing all possible solutions (11). In other words, we want to embed the solution set $\hat{X}$ into the minimal box in $\mathbb{R}^n$. This problem is known to be NP-hard (Kreinovich, *et al.*, 1993) and complicated from a computational viewpoint for large-scale systems. The paper by Oettli and Prager (1964) shows how multiple linear programming can be used to obtain $\mathbf{X}^*$; this line of research was continued in (Cope and Rust, 1979; Rust and Burrus, 1972). Iterative approaches have been established at this context as well as direct numerical methods that provide an over-bounding of $\mathbf{X}^*$, see monographs (Higham, 1996; Neumaier, 1990) and papers (Rohn, 1989; Shary, 1995).

In this section we briefly describe a simple approach proposed in (Polyak and Nazin, 2004) for interval approximation of the solution set. Instead of employing linear programming in each orthant it is suggested to deal with a scalar equation. This method is based on Rohn's result (Rohn, 1989) and simplifies his algorithm. In order to find the optimal interval estimate $\mathbf{X}^*$ of the solution set $\hat{X}$, all vertices of its convex hull $Conv\hat{X}$ should be obtained. The search of each vertex is via the solution of a scalar equation. In the case of large-scale systems we also provide a simple and fast procedure for over-bounding of the optimal interval solution.

### 4.1 The solution set

A detailed description of the solution set for the linear interval systems was given in the pioneering work by Oettli and Prager (1964) for a general situation of interval uncertainty. In our case their result is reduced as follows.

*Lemma 1.* The set of all admissible solutions of the system (10) is the non-convex polytope:

$$\hat{X} = \{\, x \in \mathbb{R}^n : \ \|Ax - b\|_\infty \leq \varepsilon \|x\|_1 + \delta \,\}. \quad (12)$$

This result also follows from (6). The set $\hat{X}$ remains bounded as long as the interval matrix $\mathbf{A}$ is regular. This regularity is characterized by a nonsingularity radius. For the interval family $\mathbf{A}$ this radius is equal to

$$\rho(A) = \frac{1}{\|A^{-1}\|_{\infty,1}}, \quad (13)$$

see (Polyak, 2003) for details. Recall that for any matrix $G$ its $(\infty, 1)$-norm is defined as $\|G\|_{\infty,1} = \max_{\|x\|_\infty \leq 1} \|Gx\|_1$. Note also that the calculation of this norm is NP-hard.

While $\varepsilon < \rho(A)$, $\mathbf{A}$ remains regular and $\hat{X}$ is bounded. If the solution set $\hat{X}$ lies in a given orthant of $\mathbb{R}^n$, then it becomes convex, and the search for its interval approximation reduces to convex optimization. However this is no longer the case in most situations, and the problem then meets combinatorial difficulties.

*4.2 Optimal interval estimates of the solution set*

The problem is to determine exact lower $\underline{x}_i$ and upper $\overline{x}_i$ bounds on each component $x_i$ of the vector $x \in \mathbb{R}^n$ under the assumption that $x \in \hat{X}$. The approach is focused on searching for vertices of the convex hull $Conv\hat{X}$ of the solution set $\hat{X}$ instead of employing linear programming in each orthant. The main base for this technique is the paper by Rohn (1989), where a key result defining $Conv\hat{X}$ was proved.

Let $S$ be the set of vertices of the unit cube $S = \{s \in \mathbb{R}^n : |s_i| = 1, i = 1,...,n\}$. Consider a system of equations

$$(a_i^T x - b_i)\, s_i = \varepsilon \|x\|_1 + \delta, \quad i = 1,...,n, \tag{14}$$

for some $s \in S$, where $a_i$ is the $i$-th row of the matrix $A$.

*Lemma 2.* For a given nominal matrix $A$, let the interval family $\mathbf{A} = \{A + \Delta A : \|\Delta A\|_\infty \leq \varepsilon\}$ be regular, i.e. all matrices in $\mathbf{A}$ are nonsingular. Then the nonlinear system of equations (14) has exactly one solution $x_s \in \hat{X}$ for every fixed vector $s \in S$, and $Conv\hat{X} = Conv\{x_s : s \in S\}$.

Proof: see (Rohn, 1989).

To simplify the search for vertices $x_s$, introduce $\hat{y} = Ax - b$. After change of the variables equalities (14) are converted to

$$\hat{y}_i\, s_i = (\varepsilon \|A^{-1}(\hat{y} + b)\|_1 + \delta), \quad i = 1,...,n. \tag{15}$$

Recall that $s_i = \pm 1\ \forall i$. The transformed solution set $\hat{Y} = \{\hat{y} : \|\hat{y}\|_\infty \leq \varepsilon \|A^{-1}(\hat{y} + b)\|_1 + \delta\}$ is the affine image of $\hat{X}$ that is $\hat{Y} = A\hat{X} - b$. Note that $Conv\hat{Y} = A\, Conv\hat{X} - b$. For any positive value $\varepsilon$ the intersection of $\hat{Y}$ with each orthant of $\mathbb{R}^n$ is non-empty. Following Lemma 2 each orthant contains only one vertex of $Conv\hat{Y}$ that gives the solution of the system of equations (15) while the vector $s = (s_1,...,s_n)^T = \text{sign}\hat{y}$ specifies the choice of the orthant under consideration. Taking all vectors $s$ from $S$ we find all vertices for $Conv\hat{Y}$. Moreover, (15) is equivalent to one scalar equation

$$\tau = \varphi(\tau), \tag{16}$$

where $\tau = \hat{y}_i s_i$, $\hat{y}_i = \tau / s_i = \tau s_i$ and $\varphi(\tau) = \varepsilon \|A^{-1}(\tau s + b)\|_1 + \delta$. The function $\varphi(\tau)$ is defined for all $\tau \geq 0$ and it is a convex piecewise linear function of $\tau$.

*Lemma 3.* For any regular interval family $\mathbf{A} \in \mathbb{IR}^{n \times n}$ and any fixed vector $s \in S$ the scalar equation (16) has a unique solution over $[0, \infty)$.

Proof: see (Polyak and Nazin, 2004).

The solution $\tau^*$ of (16) can be obtained using a simple iterative scheme, for example, Newton iterations

$$\tau_{k+1} = \left[\tau_k + \frac{\varphi(\tau_k) - \tau_k}{1 - \varphi'(\tau_k)}\right]_+, \tag{17}$$

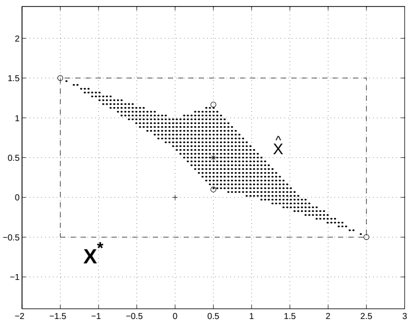


Fig. 2. The original solution set $\hat{X}$.

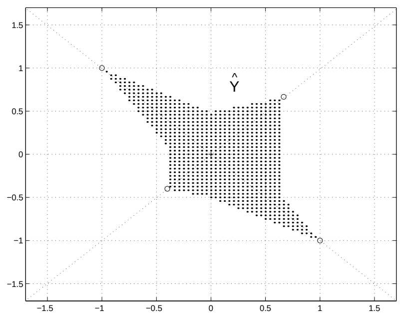


Fig. 3. The transformed solution set $\hat{Y}$.

where we use the notation $[\alpha]_+ = \max\{0, \alpha\}$. Procedure (17) converges to $\tau^*$ for any initial $\tau_0 \geq 0$ in a finite (no more than $n$) number of iterations.

*Theorem 1.* The set $Conv\hat{X}$ has $2^n$ vertices. Each vertex $x_s$ can be found by solving the scalar equation (16) for a given vector $s \in S$ by algorithm (17). Then $x_s = A^{-1}(\hat{y}(\tau^*) + b)$, where $\hat{y}(\tau) = \tau s$ and $\tau^*$ is the solution of (16).

With these vertices we find the optimal lower and upper bounds for each component of $x$ in the solution set $\hat{X}$

$$\underline{x}_i = \min_{s \in S}\{x_{s_i}\}, \quad \overline{x}_i = \max_{s \in S}\{x_{s_i}\}, \quad i = 1,...,n, \tag{18}$$

and finally $\mathbf{X}^* = ([\underline{x}_1, \overline{x}_1],...,[\underline{x}_n, \overline{x}_n])^T$.

*Example 2.* For $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}$, and $\varepsilon = \delta = 0.25$ the solution set $\hat{X}$ is a bounded and nonconvex polytope depicted on Figure 2. Its image after the affine transformation $\hat{y} = Ax - b$ is shown on Figure 3. All vertices of the convex hull $Conv\hat{Y}$ of the solution set with the variables $\hat{y}$ are represented by vector $s^k$ with elements $s_i^k = \pm 1$ and the value of $\tau$ from (16): $y^1 = (2/3,\ 2/3)^T$, $y^2 = (-1,\ 1)^T$, $y^3 = (1,\ -1)^T$ and $y^4 = (-0.4,\ -0.4)^T$. By inverse transformation $x = A^{-1}(\hat{y} + b)$ the vertices of $Conv\hat{X}$ are obtained. And then it is trivial to find the interval bounds on $\hat{X}$ using (18). Finally $\mathbf{X}^* = ([-1.5, 2.5],\ [-0.5, 1.5])^T$.

*4.3 Interval over-bounding technique*

As already mentioned, the calculation of the optimal interval solution $\mathbf{X}^*$ may be hard for large-dimensional problems. Hence, its simple interval over-bounding is of interest. This over-bounding is often said to be an interval solution of the interval system of equations as well. We provide below two such estimates.

According to the inequality $\|\hat{y}\|_\infty \le \varepsilon\|A^{-1}(\hat{y}+b)\|_1 + \delta$ for the set $\hat{Y}$ we write $\|A^{-1}(\hat{y}+b)\|_1 \le \|A^{-1}\hat{y}\|_1 + \|x^*\|_1 \le \|A^{-1}\|_{\infty,1}\|\hat{y}\|_\infty + \|x^*\|_1$, where $x^* = A^{-1}b$. Therefore

$$\|\hat{y}\|_\infty \le \gamma = \frac{\varepsilon\|x^*\|_1 + \delta}{1-\varepsilon\|A^{-1}\|_{\infty,1}}. \qquad (19)$$

All vectors $\hat{y}$ that belong to $\hat{Y}$ thus also lie also inside the ball in $\infty$-norm of radius $\gamma$. This ball is the first over-bounding interval estimate. In most cases (19) is the minimal cube centered at the origin containing $\hat{Y}$. The main difficulty here is to calculate the $(\infty,1)$-norm of $A^{-1}$; this is again NP-hard problem. There exist tractable upper bounds for this norm; we use the simplest one: for any given matrix $G$ the value of $\|G\|_{\infty,1}$ can always be approximated by a 1-norm: $\|G\|_{\infty,1} \le \|G\|_1$. Hence, the inequality (19) is replaced by

$$\|\hat{y}\|_\infty \le \frac{\varepsilon\|x^*\|_1 + \delta}{1-\varepsilon\|A^{-1}\|_1}, \qquad (20)$$

where $\varepsilon$ should be less than $1/\|A^{-1}\|_1$. An interval estimate for $\hat{Y}$ implies an interval estimate for $\hat{X}$. Indeed, $x$ is an affine function of $\hat{y}$: $x = x^* + A^{-1}\hat{y}$ and component-wise optimization for $x_i$ on a cube can be performed explicitly. Then we arrive to the following result.

*Theorem 2.* The box $\mathbf{X} = ([\underline{x}_1,\overline{x}_1],\dots,[\underline{x}_n,\overline{x}_n])^T$ with

$$\underline{x}_i = x_i^* - \gamma\|g_i\|_1,\ \ \overline{x}_i = x_i^* + \gamma\|g_i\|_1,\ \ i = 1,\dots,n \quad (21)$$

contains the solution set $\hat{X}$, where $g_i$ is the $i$-th row of $G = A^{-1}$ while $\gamma$ is the right-hand side of (19) or (20).

Thus the calculation of $\mathbf{X} \supseteq \mathbf{X}^*$ given by (20), (21) is not involved, it does not lead to any combinatorial difficulties and does not require the solution of linear programming problems. Numerous examples confirm that this over-bounding solution is close to optimal. One such example is considered in (Polyak and Nazin, 2004) for the linear system $Hx = b$ with $H$ being a Hilbert matrix. Hilbert matrices are ill conditioned even for small dimensions and for this reason it is a good test example in the framework of interval uncertainty. It was demonstrated in (Polyak and Nazin, 2004) that the over-bounding estimates (20), (21) and (19), (21) coincide in this case and give a very precise approximation of the smallest interval solution.

## 5. LARGE-SCALE INTERVAL PARAMETER BOUNDING

Assume now that $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $m \gg n$. The interval vector $\mathbf{X}_0$ is taken to be a prior approximation containing the parameter vector $x$. Let $c_i$ be the $i$-th row of $C$. Below we describe two recursive algorithms for an outer-bounding interval approximation of the intersection $\mathbf{X_0} \bigcap X$.

In Algorithm 1, for simplicity, we assume $m = Kn$, where $K$ is an integer.

Algorithm 1: Let $k = 1$. Assume $\mathbf{X} = \mathbf{X_0}$ as an initial interval approximation.

- Step 1: Consider the interval system of linear equations from (2) that corresponds to the regressors $c_{(kn-n+1)},\dots,c_{kn}$. Compute the nonsingularity radius $\rho_k$ for the nominal matrix $A$ of this system. If $\varepsilon < \rho_k$, then find its interval solution $\mathbf{X_k}$, else (in particular, if $A$ is singular) $\mathbf{X_k}$ is assumed to be infinitely large and go to step 3.
- Step 2: Find the smallest interval vector $\tilde{\mathbf{X}}$ containing $\mathbf{X} \bigcap \mathbf{X_k}$. Put $\mathbf{X} = \tilde{\mathbf{X}}$.
- Step 3: If $k = K$, then terminate, else set $k = k+1$ and go to step 1.

The interval solution $\mathbf{X_k}$ in step 1 can be calculated as described in Section 4.2. For large-scale systems (e.g., $n > 15$) it can be obtained as a simple interval over-bounding, see Section 4.3. The interval vector $\mathbf{X}$ computed by Algorithm 1 contains $\bigcap_{k=0}^{K}\mathbf{X_k}$. The main benefit of Algorithm 1 is its relatively low complexity. It requires the solution of $K = m/n$ interval systems of equations.

Algorithm 2: Let $k = 1$. Assume $\mathbf{X} = \mathbf{X_0}$ as an initial interval approximation.

- Step 1: Consider the interval system of linear equations from (2) corresponding to the regressors $c_k,\dots,c_{k+n-1}$. Compute the nonsingularity radius $\rho_k$ for the nominal matrix $A$ of this system. If $\varepsilon < \rho_k$, then find its interval solution $\mathbf{X_k}$, else go to step 3.
- Step 2: Find the smallest interval vector $\tilde{\mathbf{X}}$ containing $\mathbf{X} \bigcap \mathbf{X_k}$. Put $\mathbf{X} = \tilde{\mathbf{X}}$.
- Step 3: If $k = m-n+1$, then terminate, else set $k = k+1$ and go to step 1.

Algorithm 2 requires the solution of $m-n+1$ interval systems of equations instead of $m/n$ for Algorithm 1, but it provides a more accurate interval estimate.

*Example 3.* Let $n = 2$, $m = 40$ and $x = (1,1)^T$ be the parameter vector to be estimated, i.e. there are two parameters and forty measurements in the model. The data are generated as follows. Take $C$ be a $m \times n$ - matrix with rows $c_i$, which are samples of uniformly dis-

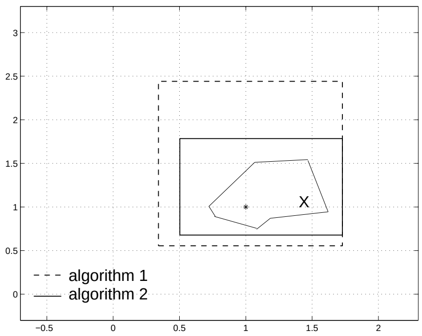


Fig. 4. Interval approximations of $X$.

tributed vectors on the unit sphere. Interval uncertainty is defined by $\varepsilon = 0.2$ and $\delta = 0.5$, and then $\Delta C = 2\varepsilon(\texttt{rand(m,n)} - 0.5)$ and $w = 2\delta(\texttt{rand(m,1)} - 0.5)$. The measurement vector $y \in \mathbb{R}^n$ is taken $y = (C + \Delta C)x + w$. These measurements are compatible with model (2) and given parameter vector $x$. Our purpose is to estimate $x$ under given $C, y$. Algorithm 1 considers $K = m/n = 20$ linear interval systems. Let the initial interval approximation be $\mathbf{X_0} = ([-5,5],[-5,5])^T$. The interval estimator is constructed as an intersection of the optimal interval solutions for the linear interval systems. Algorithm 1 provides $\mathbf{X_1}$ (dashed line box on Fig. 4) while Algorithm 2 computes a more precise interval approximation $\mathbf{X_2}$ of the non-convex feasible parameter set $X$ as the intersection of $m - n + 1 = 39$ optimal interval solutions for linear interval systems (solid line box on Fig. 4).

## 6. CONCLUSIONS

In this paper we considered the parameter estimation problem for linear multi-output models under interval uncertainty. The model uncertainty involves additive measurement noise vector and a bounded uncertain regressor matrix. Outer-bounding interval approximations of the non-convex feasible parameter set for this uncertain model are obtained. The algorithms described are based on the computation of interval solutions for square interval systems of linear equations. This approach allows computational difficulties to be avoided and provides a parameter estimator for models with a large number of measurements.

## REFERENCES


Cerone, V. (1993). Feasible parameter set for linear models with bounded errors in all variables. *Automatica*, **29**, 1551–1555.

Chernousko, F.L. and D.Ya. Rokityanskii (2000). Ellipsoidal bounds on reachable sets of dynamical systems with matrices subjected to uncertain perturbations. *Journ. Optim. Theory and Appl.*, **104**, 1–19.

Cope, J.E. and B.W. Rust (1979). Bounds on solutions of linear systems with inaccurate data. *SIAM J. Numer. Anal.*, **16**, 950–963.

Higham, N.J. (1996). *Accuracy and stability of numerical algorithms*. SIAM, Philadelphia.

Jaulin, L., M. Kieffer, O. Didrit and E. Walter (2001). *Applied interval analysis*. Springer, London.

Kostousova, E.K. (1998). State estimation for dynamic systems via parallelotopes: optimization and parallel computations. *Optim. Methods and Software*, **9**, 269–306.

Kreinovich, V., A.V. Lakeev and S.I. Noskov (1993). Optimal solution of interval linear systems is intractable (NP-hard). *Interval Comput.*, **1**, 6–14.

Kurzhanskii, A.B. and I. Valyi (1997). *Ellipsoidal calculus for estimation and control*. Birkhauser, Basel.

Milanese, M., J.P. Norton, H. Piet-Lahanier and E. Walter (Eds), (1996). *Bounding approaches to system identification*. Plenum, New York.

Neumaier, A. (1990). *Interval methods for systems of equations*. Cambridge University Press, Cambridge.

Norton, J.P. (Ed.) (1994, 1995). Special issues on bounded-error estimation, 1, 2. *Intern. Journ. of Adaptive Control and Signal. Proc.*, **8**, No. 1, **9**, No. 2.

Norton, J.P. (1999). Modal robust state estimation with deterministic specification of uncertainty. In: *Robustness in identification and control* (A. Garulli, A. Tesi, A. Vicino (Eds)), pp. 62–71. Springer, London.

Oettli, W. and W. Prager (1964). Compatibility of approximate solution of linear equations with given error bounds for coefficients and right-hand sides. *Numer. Math.*, **6**, 405–409.

Polyak, B.T. (2003). Robust linear algebra and robust aperiodicity. In: *Directions in mathematical system theory and optimization* (A. Rantzer and C. Byrnes (Eds)), pp. 249–260. Springer, Berlin.

Polyak, B.T. and S.A. Nazin (2004). Interval solution for interval algebraic equations. *Math. and Comput. in Simulation*, **66**, 207–217.

Polyak, B.T., S.A. Nazin, C. Durieu and E. Walter (2004). Ellipsoidal parameter or state estimation under model uncertainty. *Automatica*, **40**, 1171–1179.

Rohn, J. (1989). Systems of linear interval equations. *Linear Algebra and Its Appl.*, **126**, 39–78.

Rust, B.W. and W.R. Burrus, (1972). *Mathematical programming and the numerical solution of linear equations*. Amer. Elsevier, N.Y.

Shary, S.P. (1995). On optimal solution of interval linear equations. *SIAM J. Numer. Anal.*, **32**, 610–630.

Walter, E. and H. Piet-Lahanier (1989). Exact recursive polyhedral description of the feasible parameter set for bounded-error models. *IEEE Trans. Autom. Contr.*, **34**, 911–914.

Walter, E. (Ed.) (1990). Special issue on parameter identification with error bound. *Mathem. and Comput. in Simul.*, **32**, No. 5–6.