STABLE REDUCED ORDER MODELLING OF LARGE SCALE SYSTEMS USING PRESCRIBED POLES

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Abstract: By parameterizing all reduced order models matching some of the first characteristic parameters (moments or Markov parameters or both) of the original system, the poles of the reduced system are prescribed as a key to find a stable reduced order system. Two procedures based on Lanczos and Arnoldi algorithms are proposed to approximate the dominant poles of the original system and to choose them as the poles of the reduced order model. If the number of iterations approximating the dominant poles is more than the order of the reduced order model, the new approach leads to a better result than the standard Krylov subspace methods.

Keywords: Model reduction, Large scale systems, Reduced-order models, Canonical form.

1. INTRODUCTION

A widely used approach for order reduction of large scale systems is based on matching some of the characteristic parameters called moments and Markov parameters of the original and reduced systems. Such reduced models can be calculated explicitly in different ways which has extensively been used in the field of circuit simulation (Pillage and Rohrer 1990, Chiprout and Nakhla 1993, Achar and Nakhla 2001). The maximum number of characteristic parameters that can be matched is double the order of the reduced system where the method is called a Padé approximation.

To overcome the numerical problems in explicit moment matching , it is recommended to use the Krylov subspace methods which finds the reduced order model by applying a projection to the original system (Achar and Nakhla 2001, Gallivan *et al.* 1994, Villemagne and Skelton 1987, Freund 2003). Based on the coefficients to be matched, the columns of the projection matrices should form bases of the so called input and output Krylov subspaces. The great advantage of using Krylov subspaces is the existence of some numerical reliable algorithms to find the the required projection matrices like Arnoldi and Lanczos.

In Krylov subspace methods, there is no general guarantee to preserve stability of the original system. There exists a guarantee using a one-sided method only for some types of systems which are related to passive systems; see e.g. (Odabasioglu *et al.* 1998, Freund 2003, Kerns and Yang 1998). In general, if the reduced system is unstable, a stable reduced order model can be found by post processing. By using restarted algorithms, it is possible to remove the unstable poles from the reduced system and recover stability (Grimme *et al.* 1995, Jaimoukha and Kasenally 1997). However, by using the restarted algorithms, the moments do not match after deleting the unstable poles and it is not guaranteed to always find a stable reduced model by a finite number of restarts.

By knowing that a reduced order model by a one-sided method is not unique, in this paper, first all possible reduced systems of order q whose q characteristic parameters match with the original system

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are parameterized. Then, the parameters are chosen to put the poles at some prescribed points in the complex plane. Different ways are proposed to calculate the approximate poles of a large scale system using Arnoldi and Lanczos algorithms. By choosing stable values from the set of approximated poles, the stability of the original system is preserved.

In comparison to the restarted algorithms, the proposed method not only guarantees the stability, but also does not change the characteristic parameters which match with the original model (like moments or Markov parameters). Furthermore, it is suggested to find approximate poles of the original model by doing some iterations more than the order of the reduced order models which improves the results in both one-sided and two-sided methods compared to the standard Krylov subspace methods.

2. KRYLOV SUBSPACE METHODS

We consider the dynamical system of the form

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \\ y(t) = \mathbf{c}^T\mathbf{x}(t), \end{cases} \quad (1)$$

For the system (1), the *moments* around s_0 are,

$$m_i = \mathbf{c}^T \left((\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{E}\right)^i (\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{b}, \quad (2)$$

for $i = 0, 1, 2, \cdots$ which are the negative coefficients of the Taylor series expansion (around s_0) of the system's transfer function (Villemagne and Skelton 1987). Higher values of s_0 reflect the behavior of the system at higher frequencies. For $s_0 = \infty$, the Markov parameters are defined (Kailath 1980),

$$M_i = \mathbf{c}^T \left(\mathbf{E}^{-1}\mathbf{A}\right)^{i-1} \mathbf{E}^{-1}\mathbf{b}, i = 1, 2, \cdots . \quad (3)$$

The *Krylov subspace* is defined as,

$$\mathcal{K}_q(\mathbf{A}_1, \mathbf{b}_1) = span\{\mathbf{b}_1, \mathbf{A}_1\mathbf{b}_1, \cdots, \mathbf{A}_1^{q-1}\mathbf{b}_1\}, \quad (4)$$

where $\mathbf{A}_1 \in \mathbb{R}^{n\times n}$, and $\mathbf{b}_1 \in \mathbb{R}^n$ is called the starting vector. The Krylov subspaces with the help of Arnoldi and Lanczos algorithms can be used for the reduction of large scale systems and match some of the moments and Markov parameters, see e.g. (Salimbahrami and Lohmann 2002, Freund 2003).

Using matrices $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n\times q}$ where $q < n$, the system (1) can be reduced by applying a projection,

$$\begin{cases} \underbrace{\mathbf{W}^T\mathbf{E}\mathbf{V}}_{\mathbf{E}_r}\dot{\mathbf{x}}_r = \underbrace{\mathbf{W}^T\mathbf{A}\mathbf{V}}_{\mathbf{A}_r}\mathbf{x}_r + \underbrace{\mathbf{W}^T\mathbf{b}}_{\mathbf{b}_r}u, \\ y = \underbrace{\mathbf{c}^T\mathbf{V}}_{\mathbf{c}_r^T}\mathbf{x}_r. \end{cases} \quad (5)$$

If the columns of $\mathbf{V}$ form a basis for the Krylov subspace $\mathcal{K}_q\left((\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{b}\right)$ and $\mathbf{W} = \mathbf{V}$ then the first q moments around s_0 match, and the method is called a *one-sided Krylov method*. If the columns of $\mathbf{W}$ also form a basis for the Krylov subspace $\mathcal{K}_q\left((\mathbf{A} - s_0\mathbf{E})^{-T}\mathbf{E}^T, (\mathbf{A} - s_0\mathbf{E})^{-T}\mathbf{c}\right)$, then the first $2q$ moments around s_0 match and the method is called a *two-sided Krylov method*.

To match the Markov parameters the Krylov subspaces $\mathcal{K}_q\left(\mathbf{E}^{-1}\mathbf{A}, \mathbf{E}\mathbf{b}\right)$ and $\mathcal{K}_q\left(\mathbf{E}^{-T}\mathbf{A}^T, \mathbf{E}^T\mathbf{c}\right)$ should be considered. It is also possible to match the moments and Markov parameters simultaneously (Salimbahrami and Lohmann 2002, Villemagne and Skelton 1987) or match the moments around different expansion points (Grimme 1997).

In one sided methods, *Arnoldi algorithm* (Arnoldi 1951, Freund 2003) is commonly used to calculate the projection matrix $\mathbf{V}$ where $\mathbf{V}^T\mathbf{V} = \mathbf{I}$. This algorithm simplifies the transfer function of the reduced order model for moment matching around s_0 to $g_r(s - s_0) = \mathbf{c}^T\mathbf{V}(s\mathbf{H}_q - \mathbf{I})\mathbf{e}_1\|\mathbf{b}\|_2$ where $\mathbf{H}_q \in \mathbb{R}^{q\times q}$ is an upper Hessenberg matrix. In matching the Markov parameters, the transfer function of the reduced order model is simplified to $g_r(s) = \mathbf{c}^T\mathbf{V}(s\mathbf{I} - \mathbf{H}_q)\mathbf{e}_1\|\mathbf{b}\|_2$.

The well known algorithm in two sided methods is *Lanczos algorithm* (Lanczos 1950, Feldmann and Freund 1995) to calculate $\mathbf{V}$ and $\mathbf{W}$ where $\mathbf{W}^T\mathbf{V} = \mathbf{I}$. In this case, the transfer function of the reduced order model for moment matching around s_0 satisfies the equation $g_r(s - s_0) = \mathbf{c}^T\mathbf{b}\mathbf{e}_1^T(s\mathbf{T}_q - \mathbf{I})\mathbf{e}_1$ where $\mathbf{T}_q$ is a tridiagonal matrix. In matching the Markov parameters, the transfer function of the reduced order model is $g_r(s) = \mathbf{c}^T\mathbf{b}\mathbf{e}_1^T(s\mathbf{I} - \mathbf{T}_q)\mathbf{e}_1$.

3. MOMENT MATCHING BASED ON PRESCRIBED POLES

In this section, a method to find a reduced order model is proposed such that the first q moments match and its poles are located at some desired points. To apply the new approach, the moments and the poles of the reduced order system should be known a priory or be calculated as discussed here.

3.1 Moments and approximation of dominant poles

A Krylov subspace method, depending on the starting vector and the expansion point, approximates some of the dominant eigenvalues of a large matrix (Saad 1992). In order reduction, the poles of a reduced system by a Krylov subspace method are approximations of some of the poles of the original system. In other words, the eigenvalues of $\mathbf{A}^{-1}\mathbf{E}$ are approximated by the eigenvalues of $\mathbf{W}^T\mathbf{A}_r^{-1}\mathbf{E}_r\mathbf{V}$ which is Hessenberg using Arnoldi or tridiagonal using Lanczos algorithm.

The Krylov subspace method may find some unstable poles for a stable model or some poles out of the frequency range of interest. As observed in (Feldmann and Freund 1995), the residues of such poles are

negligible and can simply be deleted. In algorithm 1, the Arnoldi algorithm is run and the stable poles with larger residues or larger dominant measures are chosen which is a good choice for non stiff systems.

Algorithm 1. Dominant poles and moment generation using Arnoldi algorithm

(1) Run Arnoldi algorithm on $\mathcal{K}_{q_1}((\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{b})$ where $q_1 > q$ to find the Hessenberg matrix $\mathbf{H}_{q_1}$.
(2) Calculate the eigenvalue decomposition $\mathbf{H}_{q_1} = \mathbf{U}^{-1}\mathbf{\Lambda}\mathbf{U}$ where $\mathbf{\Lambda} = diag(\lambda_1, \cdots, \lambda_{q_1})$.
(3) Calculate the residues using $k_i = \frac{c_i u_{i1} \|\mathbf{b}\|_2}{\lambda_i}$ where c_i and u_{i1} are the i-th and $(i,1)$-th entries of $\mathbf{c}^T\mathbf{V}\mathbf{U}^{-1}$ and $\mathbf{U}$, respectively.
(4) Choose q eigenvalues of $\mathbf{H}_{q_1}$ with largest residues such that $Re(\frac{1}{\lambda_i}) < s_0$. It is recommended to choose the slower poles with larger residues by using $\frac{k_i}{Re(s_0+\frac{1}{\lambda_i})}$ as a dominant measure for λ_i. The poles of the system are $s_0 + \frac{1}{\lambda_i}$.
(5) Calculate the first q moments around s_0, iteratively, using $m_i = \mathbf{c}^T\mathbf{V}\mathbf{H}_{q_1}^{i-1}\mathbf{e}_1\|\mathbf{b}\|_2$.

To have a better approximation in reducing the systems with a defined output equation, the Lanczos algorithm can be used as in algorithm 2.

Algorithm 2. Dominant poles and moment generation using Lanczos algorithm

(1) Run Lanczos algorithm on $\mathcal{K}_{q_1}((\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{b})$ and $\mathcal{K}_{q_1}((\mathbf{A} - s_0\mathbf{E})^{-T}\mathbf{E}^T, \mathbf{c})$ where $q_1 > q$ to find the tridiagonal matrix $\mathbf{T}_{q_1}$.
(2) Calculate the eigenvalue decomposition $\mathbf{T}_{q_1} = \mathbf{U}^{-1}\mathbf{\Lambda}\mathbf{U}$ where $\mathbf{\Lambda} = diag(\lambda_1, \cdots, \lambda_q)$.
(3) Calculate the residues using $k_i = \frac{\bar{u}_{1i} u_{i1} \alpha}{\lambda_i}$ where $\bar{u}_{1i}$ and u_{i1} are the $(1,i)$ and $(i,1)$ entries of $\mathbf{U}^{-1}$ and $\mathbf{U}$, respectively and $\alpha = \mathbf{c}^T(\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{b}$.
(4) Choose q eigenvalues of $\mathbf{T}_{q_1}$ with largest dominant measures such that $Re(\frac{1}{\lambda_i}) < s_0$. $\frac{k_i}{Re(s_0+\frac{1}{\lambda_i})}$ can be used as a dominant measure for λ_i and the poles of the system are $s_0 + \frac{1}{\lambda_i}$.
(5) Calculate the first q moments around s_0 using $m_i = \alpha\mathbf{e}_1^T\mathbf{T}_{q_1}^{i-1}\mathbf{e}_1$.

By using the Lanczos algorithm, not only the output equation affects the procedure to produce the poles, but also because of a simpler structure of the matrices produced by Lanczos (a tridiagonal matrix $\mathbf{T}_q$ and two vectors $\mathbf{b}_r$ and $\mathbf{c}_r$ as multiple of the first unit vector), calculating the moments and dominant poles is faster.

3.2 Matching the moments

Consider that the q poles and moments are found. We construct a reduced order model with the prescribed poles such that its first q moments match with the original one. Consider the reduced system of the form,

$$\begin{cases} \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{q-1} \end{bmatrix}}_{\mathbf{E}_r} \dot{\mathbf{x}}_r = \mathbf{A}_r\mathbf{x}_r + \mathbf{b}_r u, \\ y = \underbrace{\begin{bmatrix} m_0 & m_2 & \cdots & m_{q-1} \end{bmatrix}}_{\mathbf{c}_r^T} \mathbf{x}_r. \end{cases} \tag{6}$$

where $\mathbf{A}_r = s_0\mathbf{E}_r + \mathbf{I}$ and $\mathbf{b}_r = \mathbf{e}_1$ is the first unit vector. The moments of the system (6) around s_0 are,

$$\begin{aligned} \mathbf{m}_{ri} &= \mathbf{c}_r^T \left((\mathbf{A}_r - s_0\mathbf{E}_r)^{-1}\mathbf{E}_r\right)^i (\mathbf{A}_r - s_0\mathbf{E}_r)^{-1}\mathbf{b}_r \\ &= \mathbf{c}_r^T\mathbf{E}_r^i\mathbf{b}_r, \quad i = 0, 1, \cdots . \end{aligned}$$

Because of structure of the matrix $\mathbf{E}_r$ and $\mathbf{b}_r$, the first q moments of the system (6) around s_0 are equal to the elements of the matrix $\mathbf{c}_r$.

Theorem 1. The reduced system (6) parameterizes all reduced system $g_r(s)$ of order q matching the first q moments around s_0 of the higher order system $g(s)$ in terms of the parameters in the last column of $\mathbf{E}_r$.

PROOF. From the discussion above, for any value in the last column of $\mathbf{E}_r$ the first q moments of system (6) match with the original system if the entries of $\mathbf{c}_r$ are the moments of the original system around the specified point. Now, consider the first q moments of $g_r(s)$ and the original system around s_0 are equal. We find a state space equation for $g_r(s - s_0)$,

$$\bar{\Sigma} : \begin{cases} \bar{\mathbf{E}}\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{b}}u, \\ y = \bar{\mathbf{c}}^T\bar{\mathbf{x}}. \end{cases} \tag{7}$$

Because the first q moments of this system around zero are equal to the moments of $g_r(s)$ around s_0, the matrix $\bar{\mathbf{A}}$ is nonsingular. Now, we multiply the state equation by inverse of $\bar{\mathbf{A}}$ and apply the state space transformation, $\hat{\mathbf{x}} = \mathbf{T}\bar{\mathbf{x}}$ where,

$$\mathbf{T} = \left[\bar{\mathbf{A}}^{-1}\bar{\mathbf{b}},\ \bar{\mathbf{A}}^{-1}\bar{\mathbf{E}}\bar{\mathbf{A}}^{-1}\bar{\mathbf{b}},\ \cdots,\ \left(\bar{\mathbf{A}}^{-1}\bar{\mathbf{E}}\right)^{q-1}\bar{\mathbf{A}}^{-1}\bar{\mathbf{b}} \right].$$

The state space equation after transformation is,

$$\hat{\Sigma} : \begin{cases} \underbrace{\mathbf{T}\bar{\mathbf{A}}^{-1}\bar{\mathbf{E}}\mathbf{T}^{-1}}_{\hat{\mathbf{E}}}\dot{\hat{\mathbf{x}}} = \hat{\mathbf{x}} + \underbrace{\mathbf{T}\bar{\mathbf{A}}^{-1}\bar{\mathbf{b}}}_{\hat{\mathbf{b}}}u, \\ y = \underbrace{\bar{\mathbf{c}}^T\mathbf{T}^{-1}}_{\hat{\mathbf{c}}^T}\hat{\mathbf{x}}. \end{cases}$$

With the same proof as in controllability canonical forms (Kailath 1980), the vector $\mathbf{T}\bar{\mathbf{A}}^{-1}\bar{\mathbf{b}}$ and the matrix $\mathbf{T}\bar{\mathbf{A}}^{-1}\bar{\mathbf{E}}\mathbf{T}^{-1}$ have the same structure as $\mathbf{b}_r$ and $\mathbf{E}_r$ in (6). The moments of the system $\hat{\Sigma}$ around zero are now the entries of the vector $\hat{\mathbf{c}}$. Now, we shift the state space equation back to have the entries of $\hat{\mathbf{c}}$ as the moments around s_0 by defining $\hat{\mathbf{A}} = s_0\hat{\mathbf{E}} + \mathbf{I}$ and the proof is completed. □

The characteristic equation of the system (6) is $\det(s\mathbf{E}_r - \mathbf{A}_r)$. However, the poles of the system can be calculated from,

$$\begin{aligned} a(s) &= \det\left(\mathbf{E}_r^{-1}\left(s\mathbf{E}_r - \mathbf{A}_r\right)\right) \\ &= \det\left((s-s_0)\mathbf{I} - \mathbf{E}_r^{-1}\right). \end{aligned}$$

Because the matrix $\mathbf{E}_r$ is in companion form, we have,

$$a(s) = (s-s_0)^q + \frac{a_1}{a_0}(s-s_0)^{q-1} + \cdots + \frac{a_{q-1}}{a_0}(s-s_0) + \frac{1}{a_0}. \quad (8)$$

Therefore, the parameters a_i-s are related to the coefficients of the shifted poles of the system (6). If the poles of the reduced order model are prescribed without using the algorithm 1, then a_i-s are found by matching the coefficients of (8) and the polynomial,

$$a(s-s_0) = \prod_{i=1}^{q}(s-(p_i+s_0)).$$

where p_i-s are the prescribed poles. If algorithm 1 is used to find the prescribed poles, the whole reduction procedure is simplified to the steps in algorithm 3.

Algorithm 3. Moment matching using dominant poles

(1) Choose the values of s_0, q_1 and q and run algorithm 1 or 2 to find the eigenvalues $\lambda_{r1}, \cdots, \lambda_{rq}$ and the first q moments of the system. The prescribed poles of the reduced system are $\frac{1}{\lambda_{ri}} + s_0$.
(2) Calculate the coefficients of the polynomial $\prod_{i=1}^{q}(s-\lambda_i) = s^q + a_{q-1}s^{q-1} + \cdots + a_1 s + a_0$ to produce the last column of $\mathbf{E}_r$.
(3) Put the calculated q moments around s_0 in $\mathbf{c}_r$.
(4) Construct the reduced order model (6) to match q moments at s_0 and have the prescribed poles.

Based on theorem 1, all reduced order model matching the first q moments can be expressed as (6). Here, the remaining parameter are chosen to put the poles at some desired points. Some other criteria can be used to find the best parameters. One option is to match as many number of moments as possible. This can be done directly using any type of Padé approximation like Padé via Lanczos (PVL) (Feldmann and Freund 1995) or two-sided Arnoldi (Salimbahrami *et al.* 2003) where $2q$ moments match and the result is unique and independent of the algorithm used to find the reduced order model (Salimbahrami and Lohmann 2002). If the algorithm 3 is used with Lanczos and $q_1 = q$, then the reduced order model is a Padé type approximation leading to the same transfer function as the standard Lanczos algorithm.

4. MATCHING THE MARKOV PARAMETERS

If the value of s_0 tends to infinity, then the method of section 3.2 can be changed to match the Markov parameters. Consider the reduced system of the form,

$$\begin{cases} \dot{\mathbf{x}}_r = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{q-1} \end{bmatrix}}_{\mathbf{A}_r} \mathbf{x}_r + \mathbf{b}_r u, \\ y = \underbrace{\begin{bmatrix} M_1 & M_2 & \cdots & M_q \end{bmatrix}}_{\mathbf{c}_r^T} \mathbf{x}_r. \end{cases} \quad (9)$$

where $\mathbf{b}_r = \mathbf{e}_1$ is the first unit vector. The i-th Markov parameter of system (9) is $\mathbf{M}_{ri} = \mathbf{c}_r^T \mathbf{A}_r^{i-1} \mathbf{b}_r$ and because of the structure of $\mathbf{A}_r$ and $\mathbf{b}_r$, the first q Markov parameters of the system (9) are equal to the elements of the matrix $\mathbf{c}_r$. A result similar to theorem 1 can be proved in this case.

Theorem 2. All order q reduced system $g_r(s)$ matching the first q Markov parameters of the original system can be parameterized as the reduced system (9) in terms of the parameters in the last column of $\mathbf{A}_r$.

The proof of theorem 2 is quite similar to theorem 1 but the transformation $\hat{\mathbf{x}} = \mathbf{T}\bar{\mathbf{x}}$ is applied to the reduced model (7) after multiplying the state equation with $\bar{\mathbf{E}}^{-1}$ where,

$$\mathbf{T} = \left[\, \bar{\mathbf{E}}^{-1}\bar{\mathbf{b}},\ \bar{\mathbf{E}}^{-1}\bar{\mathbf{A}}\bar{\mathbf{E}}^{-1}\bar{\mathbf{b}},\ \cdots,\ (\bar{\mathbf{E}}^{-1}\bar{\mathbf{A}})^{q-1}\bar{\mathbf{E}}^{-1}\bar{\mathbf{b}} \,\right],$$

which transforms the system into the controllability canonical form (9).

The poles of the system (9) are the eigenvalues of the companion matrix $\mathbf{A}_r$ with the characteristic equation,

$$a(s) = \det(s\mathbf{I} - \mathbf{A}_r) = s^q + a_{q-1}s^{q-1} + \cdots + a_0.$$

The parameters a_i-s can be chosen based on a set of prescribed poles. It is possible to find a set of low frequency dominant poles as described in section 3.1 and then match the Markov parameters using reduced system (9). To find an approximation of the high frequency dominant poles and to match the Markov parameters, the algorithm 4 can be used.

4.1 Matching the moments and Markov parameters

To achieve a good accuracy at both high and low frequencies, the Markov parameters can be matched while the low frequency poles are prescribed or vice versa. It is also possible to match some of the moments and Markov parameters simultaneously.

To match $q - l$ moments around 0 and l Markov parameters, then a reduced order model of the form,

$$\begin{cases} \mathbf{E}_r\dot{\mathbf{x}}_r = \mathbf{x} + \mathbf{b}_r u, \\ y = \underbrace{\begin{bmatrix} M_l & \cdots & M_1 & m_0 & \cdots & m_{q-l-1} \end{bmatrix}}_{\mathbf{c}_r^T} \mathbf{x}_r, \end{cases} \quad (10)$$

Algorithm 4. Matching the Markov parameters using dominant poles

(1) Run Lanczos algorithm on $\mathcal{K}_{q_1}(\mathbf{E}^{-1}\mathbf{A}, \mathbf{E}^{-1}\mathbf{b})$ and $\mathcal{K}_{q_1}(\mathbf{E}^{-T}\mathbf{A}^T, \mathbf{E}^{-T}\mathbf{c})$ where $q_1 > q$ to find the tridiagonal matrix $\mathbf{T}_{q_1}$.
(2) Calculate the eigenvalue decomposition $\mathbf{T}_{q_1} = \mathbf{U}^{-1}\mathbf{\Lambda}\mathbf{U}$ where $\mathbf{\Lambda} = diag(\lambda_1, \cdots, \lambda_q)$.
(3) Calculate the residues using $k_i = \frac{\bar{u}_{1i}u_{i1}\alpha}{\lambda_i}$ where $\bar{u}_{1i}$ and u_{i1} are the $(1,i)$ and $(i,1)$ entries of $\mathbf{U}^{-1}$ and $\mathbf{U}$, respectively and $\alpha = \mathbf{c}^T\mathbf{E}^{-1}\mathbf{b}$.
(4) Choose q eigenvalues of $\mathbf{T}$ with largest dominant measures and negative real parts. $\frac{k_i}{Re(\lambda_i)}$ can be used as a dominant measure for λ_i.
(5) Calculate the coefficients of the polynomial $\prod_{i=1}^{q}(s-\lambda_{ri}) = s^q + a_{q-1}s^{q-1} + \cdots + a_1 s + a_0$ to produce the last column of $\mathbf{A}_r$ where λ_{ri}-s are the chosen eigenvalues from the previous step.
(6) Calculate the Markov parameters using $M_i = \alpha\mathbf{e}_1^T\mathbf{T}_{q_1}^i\mathbf{e}_1$ for $i = 1, \cdots q$.
(7) Construct the reduced order model (9) to match q Markov parameters have the prescribed poles.

is considered where $\mathbf{E}_r$ is defined in (6) and $\mathbf{b}_r = \mathbf{e}_{l+1}$ is the $l+1$-st unit vector. It is not difficult to show that the first l Markov parameters of system (10) are the first l entries of $\mathbf{c}_r$ and the first $q-l$ moments are the last $q-l$ entries of $\mathbf{c}_r$. By prescribing the poles and calculating the moments and Markov parameters of the original system the reduced order model is found.

Theorem 3. The system (10) parameterizes all order q reduced system $g_r(s)$ matching the first $q-l$ moments ($l \leq q$) around 0 and l Markov parameters with $g(s)$ in terms of the parameters in the last column of $\mathbf{E}_r$.

PROOF. From the discussion above, for any value in the last column of $\mathbf{E}_r$ the first $q-l$ moments and l Markov parameters match with the original system if the entries of $\mathbf{c}_r$ are the moments and Markov parameters of the original system as in (10). Now, consider the system $g_r(s)$, with the assumption of the theorem with the state space equation,

$$\bar{\Sigma} : \begin{cases} \bar{\mathbf{E}}\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{b}}u, \\ y = \bar{\mathbf{c}}^T\bar{\mathbf{x}}. \end{cases}$$

where $\bar{\mathbf{A}}$ and $\bar{\mathbf{E}}$ are nonsingular. Now, we multiply the state equation by inverse of $\bar{\mathbf{A}}$ and apply the state space transformation, $\hat{\mathbf{x}} = \mathbf{T}\bar{\mathbf{x}}$ where,

$$\mathbf{T} = \Big[\left(\bar{\mathbf{A}}^{-1}\bar{\mathbf{E}}\right)^{q-l-1}\bar{\mathbf{A}}^{-1}\bar{\mathbf{b}} \ \cdots \ \bar{\mathbf{A}}^{-1}\bar{\mathbf{b}} \ \bar{\mathbf{E}}^{-1}\bar{\mathbf{b}} \ \cdots \ \left(\bar{\mathbf{E}}^{-1}\bar{\mathbf{A}}\right)^{l-1}\bar{\mathbf{E}}^{-1}\bar{\mathbf{b}} \Big].$$

The state space equation after transformation is,

$$\hat{\Sigma} : \begin{cases} \underbrace{\mathbf{T}\bar{\mathbf{A}}^{-1}\bar{\mathbf{E}}\mathbf{T}^{-1}}_{\hat{\mathbf{E}}}\dot{\hat{\mathbf{x}}} = \hat{\mathbf{x}} + \underbrace{\mathbf{T}\bar{\mathbf{A}}^{-1}\bar{\mathbf{b}}}_{\hat{\mathbf{b}}}u, \\ y = \underbrace{\bar{\mathbf{c}}^T\mathbf{T}^{-1}}_{\hat{\mathbf{c}}^T}\hat{\mathbf{x}}. \end{cases}$$

The vector $\mathbf{T}\bar{\mathbf{A}}^{-1}\bar{\mathbf{b}}$ and the matrix $\mathbf{T}\bar{\mathbf{A}}^{-1}\bar{\mathbf{E}}\mathbf{T}^{-1}$ have the same structure as $\mathbf{b}_r$ and $\mathbf{E}_r$ in (6). The moments and Markov parameter of the system $\hat{\Sigma}$ are now the entries of the vector $\hat{\mathbf{c}}$ and the proof is completed. □

If it is desired to have $\mathbf{E}_r = \mathbf{I}$, the reduced order model (10) can be written as follows,

$$\begin{cases} \dot{\mathbf{x}}_r = \mathbf{A}_r\mathbf{x} + \mathbf{b}_r u, \\ y = \underbrace{\begin{bmatrix} m_{q-l-1} & \cdots & m_0 & M_1 & \cdots & M_l \end{bmatrix}}_{\mathbf{c}_r^T}\mathbf{x}_r, \end{cases} \quad (11)$$

where $\mathbf{A}_r$ is defined in (9) and $\mathbf{b}_r = \mathbf{e}_{l+1}$. The result of theorem 3 can easily be proved for system (11).

5. TECHNICAL EXAMPLES

We consider a clamped beam model (Antoulas *et al.* 2001, Chahlaoui and Van Dooren 2002) of order 348 with one input and one output [2]. The model is obtained by spatial discretization of an appropriate partial differential equation. The input represents the force applied to the structure at the free end, and the output is the resulting displacement.

Table 1. Relative error norm of different reduced models (multiple of 10^{-3}).

Method	q_1	q	s_0	$\mathcal{H}_2$norm	$\mathcal{H}_\infty$ norm
New method with Lanczos	14	14	2	10.5	6.7
	16	14	2	29.0	0.9099
	19	14	2	5.1442	6.854
	21	14	2	3.7538	0.3149
	23	14	2	3.7508	0.3149
New method with Arnoldi	14	12	2	87.0	103.1
	16	16	2	11.0	0.0744
	19	14	2	66.8	25.0
	20	14	2	42.5	2.9
	23	14	2	5.0150	0.4472
Padé via Lanczos	-	14	0	47.4	3.3398
	-	16	0	32.9	2.3398
	-	16	2	unstable	unstable
	-	14	2	10.5	6.7

The proposed methods and the standard Lanczos algorithm are applied to reduce the order of the system. In table 1, the results are compared to each other. q_1 is the number of iterations to approximate the poles. Although matching the moments around $s_0 \neq 0$ leads to a better result, the standard Lanczos algorithm and even one-sided Arnoldi produce unstable models for so many values of q when $s_0 \neq 0$.

As expected, the new method using the Lanczos algorithm leads to a better result. By increasing the number of iterations to approximate the poles, it is possible to find a better result but the error is not necessarily decreasing. However, if the number of iterations is

[2] The models are available at `http://www.win.tue.nl/niconet/NIC2/benchmodred.html`. We would like to thank the authors of this site for providing us the model.

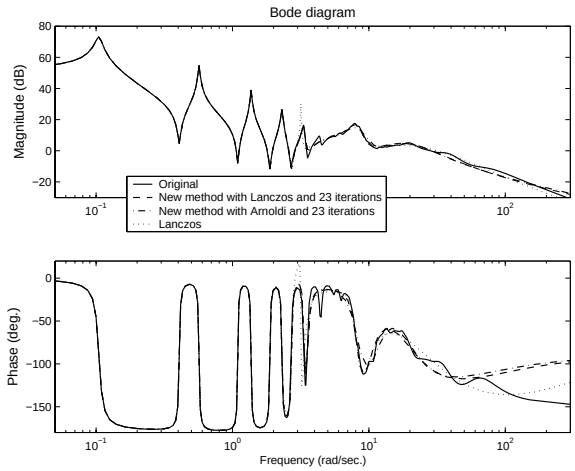

Fig. 1. Bode plot of the original and reduced order systems of order 14 with $s_0 = 2$.

much more than the order of the reduced order model, then choosing the best poles is difficult and is not recommended. For this particular example, the results are improved by increasing the number of iterations up to 23. In figure 1, the bode diagram of the reduced and original models are compared. The best result is for the new method with Lanczos and the worst reduced system is the one with standard Lanczos.

6. CONCLUSION

All reduced models of order q, matching q characteristic parameters (moments or Markov parameters) of the original model have been parameterized. The proposed approach:

- improves the results compared to the methods of moment matching like Arnoldi and Lanczos.
- guarantees the stability of the reduced system.
- compared to the restarted algorithms, matches the characteristic parameters without any error although the undesired poles are deleted.

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