ROBUST $\mathcal{H}_\infty$ OUTPUT FEEDBACK CONTROL DESIGN FOR UNCERTAIN FUZZY SYSTEMS WITH $\mathcal{D}$ STABILITY CONSTRAINTS: AN LMI APPROACH

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Abstract: This paper addresses the problem of designing a robust output feedback controller for a class of fuzzy uncertain dynamic systems that guarantees i) the $\mathcal{L}_2$-gain from an exogenous input to a regulated output is less or equal to a prescribed value and ii) the closed-loop fuzzy system to be quadratically stable within a pre-specified LMI stability region. Based on an LMI approach, solutions to the problem are derived in terms of a family of linear matrix inequalities. The chaotic Lorenz system is used to illustrate the effectiveness of the proposed design techniques.

Keywords: H-infinity Control; Stability robustness; Stabilization methods; Fuzzy models; Lyapunov function.

1. INTRODUCTION

Nonlinear $\mathcal{H}_\infty$ control design has been extensively studied by a number of researchers over the past two decades; see, for instance, (Ball et al. , 1993; Ball and Helton, 1989; Basar and Olsder, 1982; Isidori and Astolfi, 1992; Isidori, 1991; Nguang and Shi , 2000a,b; Nguang and Fu , 1996; Nguang , 1996; van der Schaft, 1992; van der Schaft , 1991). This problem can be stated as follows: given a dynamic system with the exogenous input and measured output, design a control law such that the $\mathcal{L}_2$ gain of the mapping from the exogenous input to the regulated output is minimised or no larger than some prescribed level. Solutions to this problem can be characterized in terms of the so-called Hamilton-Jacobi equation (HJE). However, until now, the problem of finding a global solution to the HJE either analytically or numerically is an open research area.

Over the past two decades, there has been rapidly growing interest in application of fuzzy logic to control problem. Researches have been focused on its application to industrial processes and a number of successful results have been reported in the literature. In spite of these successes, there are many basic issues remain to be addressed. One of them is how to achieve a systematic design that guarantees closed-loop stability and performance. Recently, a great amount of effort has been devoted to describing a nonlinear system using a Takagi-Sugeno fuzzy model; see (Assawinchaichote and Nguang, 2004a,b; Cao et al., 1996; Chen et al., 2000; Lee et al., 2001; Ma et al., 1998; Nguang and Shi, 2003; Nguang and Shi , 2001a,b; Tanaka et. al., 1996; Tanaka , 1995; Tanaka and

Sugeno , 1992; Takagi and Sugeno, 1985). The Takagi-sugeno fuzzy model represents a nonlinear system by a family of local linear models which smoothly blended together through fuzzy membership functions. Unlike conventional modelling techniques which uses a single model to describe the global behavior of a nonlinear system, fuzzy modelling is essentially a multi-model approach in which simple sub-models (typically linear models) are fuzzily combined to described the global behavior of a nonlinear system. Based on this fuzzy model, a number of systematic model-based fuzzy control design methodologies have been developed.

The aim of this paper is to study the problem of designing an output feedback controller for a class of fuzzy uncertain systems that guarantees i) the $\mathcal{L}_2$-gain from an exogenous input to a regulated output is less or equal to a prescribed value, and ii) the closed-loop system to be quadratically stable in a pre-specified LMI stability region. Based on an LMI approach, sufficient conditions for quadratic $\mathcal{D}$-stabilization of the uncertain Takagi-Sugeno fuzzy model with an $\mathcal{H}_\infty$ performance are derived in terms of a family of linear matrix inequalities.

This paper is organized as follows. In Section II, system descriptions and definitions are presented. Based on an LMI approach, we develop a technique in Section III for designing a fuzzy $\mathcal{H}_\infty$ controller that guarantees the $\mathcal{L}_2$-gain of the mapping from the exogenous input noise to the regulated output is less than a prescribed value and the closed-loop system to be quadratically stable within a pre-specified LMI stability region, $\mathcal{D}$. The validity of this approach is demonstrated by using the chaotic Lorenz system from the literature in Section IV. Finally, in Section V, the conclusion is drawn.

2. SYSTEM DESCRIPTIONS AND DEFINITIONS

The class of nonlinear uncertain systems under consideration is described by the following fuzzy system model:

Plant Rule *i:*
IF $\nu_1(t)$ is M_{i1} and $\cdots$ and $\nu_\vartheta(t)$ is $M_{i\vartheta}$ THEN

$$\begin{aligned} \dot{x}(t) &= [A_i + \Delta A_i]x(t) + [B_{1_i} + \Delta B_{1_i}]w(t) \\ &\quad + [B_{2_i} + \Delta B_{2_i}]u(t), \;\; x(0) = 0 \\ z(t) &= [C_{1_i} + \Delta C_{1_i}]x(t) + [D_{12_i} + \Delta D_{12_i}]u(t) \\ y(t) &= [C_{2_i} + \Delta C_{2_i}]x(t) + [D_{21_i} + \Delta D_{21_i}]w(t) \end{aligned} \tag{1}$$

where $i = 1, 2, \cdots, r$, $M_{ij}(j = 1, 2, \cdots, \vartheta)$ are fuzzy sets, $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the input, $w(t) \in \Re^p$ is the disturbance which belongs to $\mathcal{L}_2[0, T_f]$ with $T_f > 0$, $y(t) \in \Re^\ell$ is the measurement, $z(t) \in \Re^s$ is the controlled output, the matrices $A_i, B_{1_i}, B_{2_i}, C_{1_i}, C_{2_i}, D_{12_i}$ and D_{21_i} are of appropriate dimensions, r is the number of IF-THEN rules. The matrices $\Delta A_i, \Delta B_{1_i}, \Delta B_{2_i}, \Delta C_{1_i}, \Delta C_{2_i}, \Delta D_{12_i}$ and ΔD_{21_i} represent the time-varying uncertainties in the system and satisfy the following assumption.

Assumption 2.1.

$$\begin{aligned} \Delta A_i = F(x(t), t)E_{1_i},\ \Delta B_{1_i} = F(x(t), t)E_{2_i}, \\ \Delta B_{2_i} = F(x(t), t)E_{3_i},\ \Delta C_{1_i} = F(x(t), t)E_{4_i}, \\ \Delta C_{2_i} = F(x(t), t)E_{5_i},\ \Delta D_{12_i} = F(x(t), t)E_{6_i}, \\ \Delta D_{21_i} = F(x(t), t)E_{7_i} \end{aligned}$$

where $E_{j_i},\ j = 1, 2, \cdots, 7$ are known matrices which characterize the structure of the uncertainties. Furthermore, there exists a positive constant ρ such that the following inequality holds:

$$\|F(x(t), t)\| \le \rho. \tag{2}$$

Let $\varpi_i(\nu(t)) = \prod_{k=1}^{\vartheta} M_{ik}(\nu_k(t))$ and $\mu_i(x(t)) = \frac{\varpi_i(\nu(t))}{\sum_{i=1}^{r} \varpi_i(\nu(t))}$ where $M_{ik}(\nu_k(t))$ is the grade of membership of $\nu_k(t)$ in M_{ik}. It is assumed in this paper that $\varpi_i(\nu(t)) \ge 0$ and $\sum_{i=1}^{r} \varpi_i(\nu(t)) > 0$ for all t. Hence, $\mu_i(\nu(t)) \ge 0$ and $\sum_{i=1}^{r} \mu_i(\nu(t)) = 1$ for all t. For the convenience of notations, we let $\varpi_i = \varpi_i(\nu(t))$ and $\mu_i = \mu_i(\nu(t))$.

The resulting fuzzy system model is inferred as the weighted average of the local models of the form:

$$\begin{aligned} \dot{x} &= [A(\mu) + \Delta A(\mu)]x + [B_1(\mu) + \Delta B_1(\mu)]w \\ &\quad + [B_2(\mu) + \Delta B_2(\mu)]u \\ z &= [C_1(\mu) + \Delta C_1(\mu)]x + [D_{12}(\mu) + \Delta D_{12}(\mu)]u \\ y &= [C_2(\mu) + \Delta C_2(\mu)]x + [D_{21}(\mu) + \Delta D_{21}(\mu)]w \end{aligned} \tag{3}$$

where $A(\mu) = \sum_{i=1}^{r} \mu_i A_i$, $B_1(\mu) = \sum_{i=1}^{r} \mu_i B_{1_i}$, $B_2(\mu) = \sum_{i=1}^{r} \mu_i B_{2_i}$, $C_1(\mu) = \sum_{i=1}^{r} \mu_i C_{1_i}$, $C_2(\mu) = \sum_{i=1}^{r} \mu_i C_{2_i}$, $D_{12}(\mu) = \sum_{i=1}^{r} \mu_i D_{12_i}$, $D_{21}(\mu) = \sum_{i=1}^{r} \mu_i D_{21_i}$, $\Delta A(\mu) = F(x, t)E_1(\mu)$, $\Delta B_1(\mu) = F(x, t)E_2(\mu)$, $\Delta B_2(\mu) = F(x, t)E_3(\mu)$, $\Delta C_1(\mu) = F(x, t)E_4(\mu)$, $\Delta C_2(\mu) = F(x, t)E_5(\mu)$, $\Delta D_{12}(\mu) = F(x, t)E_6(\mu)$, $\Delta D_{21}(\mu) = F(x)E_7(\mu)$ with $E_k(\mu) = \sum_{i=1}^{r} \mu_i E_{k_i}$, $k = 1, 2, \cdots, 7$.

Definition 2.1. (van der Schaft, 1992) Suppose γ is a given positive number. A system of the form (3) is said to have $\mathcal{L}_2[0, T_f]$ gain less than or equal to γ if

$$\int_0^{T_f} z^T(t)z(t)\, dt \le \gamma^2 \left[\int_0^{T_f} w^T(t)w(t)\, dt \right]. \tag{4}$$

Definition 2.2. (Chilali and Gahinet , 1996; Chilali et al., 1994) A subset $\mathcal{D}$ of the complex plane is called an LMI region if there exist a symmetric matrix $\Gamma \in \Re^{g\times g}$ and a matrix $\Pi \in \Re^{g\times g}$ such that

$$\mathcal{D} = \{z = x + jy \in \mathcal{C} : \quad f_{\mathcal{D}}(z) < 0\} \tag{5}$$

where the characteristic function $f_{\mathcal{D}}(z)$ is given as follows:

$$f_{\mathcal{D}}(z) = \Gamma + \Pi z + \Pi^T \bar{z}. \tag{6}$$

Definition 2.3. (Quadratic $\mathcal{D}$-stability) Given an LMI stability region $\mathcal{D}$-stable defined by (5), the nonlinear system $\dot{x}(t) = f(x(t))x(t)$ is said to be quadratically $\mathcal{D}$-stable if there exists a positive definite symmetric matrix $X \in \Re^{n\times n}$ such that

$$\Gamma \otimes X + \Pi \otimes (Xf(x)) + \Pi^T \otimes (Xf(x))^T < 0 \tag{7}$$

where $\otimes$ denotes the Kronecker product of the matrices.

In this paper, we seek for an n^{th}-order $\mathcal{H}_\infty$ fuzzy output feedback which is inferred as the weighted average of the local models of the form:

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{A}(\mu)\hat{x}(t) + \hat{B}(\mu)y(t) \\ u(t) &= \hat{C}(\mu)\hat{x}(t) \end{aligned} \tag{8}$$

Before ending this section, we describe the problem under our study as follows.

Problem Formulation: Given a prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$ and an LMI stability region $\mathcal{D}$ with the characteristic function (6), design a robust fuzzy $\mathcal{H}_\infty$ output feedback controller of the form (8) such that i) the inequality (4) holds, and ii) the closed-loop fuzzy system is quadratically stable in the given LMI stability region $\mathcal{D}$.

Note that for the symmetric block matrices, we use $(*)$ as an ellipsis for terms that are induced by symmetry.

3. MAIN RESULTS

This section provides sufficient conditions for the system (3) with (8) to be quadratically $\mathcal{D}$ stable and has a prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$.

Theorem 3.1. Consider the system (3) satisfies Assumption 2.1. Given a prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$, an LMI stability region $\mathcal{D}$ and a positive constant δ, if there exists a matrix $P > 0$ satisfying the following matrix inequalities:

$$\begin{bmatrix} \Pi_{\mathcal{D}}(A_{cl}(\mu), P) & (*)^T & (*)^T \\ \Pi_1 \otimes (\check{B}_{cl}^T(\mu)P) & -\frac{1}{\rho}I & (*)^T \\ \Pi_2 \otimes \check{C}_{cl}(\mu) & 0 & -\frac{1}{\rho}I \end{bmatrix} < 0 \tag{9}$$

and

$$\begin{pmatrix} A_{cl}(\mu)P + PA_{cl}^T(\mu) & (*)^T & (*)^T \\ B_{cl}^T(\mu) & -\gamma I & (*)^T \\ C_{cl}(\mu)P & 0 & -\gamma I \end{pmatrix} < 0, \tag{10}$$

where $\Pi_{\mathcal{D}}(A_{cl}(\mu), P) = \Gamma \otimes P + \Pi \otimes (PA_{cl}(\mu))$ $+\Pi^T \otimes (A_{cl}(\mu)^T P)$, $\check{B}_{cl}(\mu) = \begin{bmatrix} I & I & 0 \\ 0 & 0 & \hat{B}(\mu) \end{bmatrix}$,

$A_{cl}(\mu) = \begin{bmatrix} A(\mu) & B_2(\mu)\hat{C}(\mu) \\ \hat{B}(\mu)C_2(\mu) & \hat{A}(\mu) \end{bmatrix}$, $\check{C}_{cl}(\mu) = \begin{bmatrix} E_1(\mu) & 0 \\ 0 & E_3(\mu)\hat{C}(\mu) \\ E_5(\mu) & 0 \end{bmatrix}$, $B_{cl}(\mu) = \begin{bmatrix} \tilde{B}_1(\mu) \\ \hat{B}(\mu)\tilde{D}_{21}(\mu) \end{bmatrix}$ and $C_{cl}(\mu) = [\tilde{C}_1(\mu) \quad \tilde{D}_{12}(\mu)\hat{C}(\mu)]$ with $\tilde{B}_1(\mu) = [\delta I \quad I \quad \delta I \quad 0 \quad B_1(\mu) \quad 0]$, $\tilde{C}_1(\mu) = \left[\frac{\rho}{\delta}E_1^T(\mu) \quad 0 \quad \frac{\rho}{\delta}E_5^T(\mu)\sqrt{2}\lambda\rho E_4^T(\mu) \quad \sqrt{2}\lambda C_1^T(\mu)\right]^T$, $\tilde{D}_{12}(\mu) = \left[0 \ \frac{\rho}{\delta}E_3^T(\mu) \ 0 \ \sqrt{2}\lambda\rho E_6^T(\mu) \ \sqrt{2}\lambda D_{12}^T(\mu)\right]^T$, $\tilde{D}_{21}(\mu) = [0 \ \ 0 \ \ 0 \ \delta I \ \ D_{21}(\mu) \ \ I]$ and $\lambda^2 = \Big[1 + \rho^2 \sum_{i=1}^r \sum_{j=1}^r \left\{\|E_{2_i}^T E_{2_j}\| + \|E_{7_i}^T E_{7_j}\|\right\}\Big]$. Then the inequality (4) is guaranteed and the closed-loop system is quadratically stable in the given LMI stability region $\mathcal{D}$.

Proof: This theorem can be proved by employing the approaches given in (Chilali et al., 1994; Nguang and Shi, 2003; Assawinchaichote and Nguang, 2004a,b). Due to the page limit, the detail of the proof has been omitted. ■

In general, (9) and (10) are nonconvex nonlinear matrix inequalities. Fortunately, (9) and (10) can be transformed to some convex linear matrix inequalities by the following procedures. Partition P and its inverse as

$$P = \begin{bmatrix} X & M \\ M^T & U \end{bmatrix}, \ P^{-1} = \begin{bmatrix} Y & N \\ N^T & V \end{bmatrix}, \quad X \in \Re^{n\times n}, Y \in \Re^{n\times n}. \tag{11}$$

Define the new controller variables as

$$\begin{aligned} \mathcal{B}(\mu) &:= N\hat{B}(\mu) \\ \mathcal{C}(\mu) &:= \hat{C}(\mu)M^T \\ \mathcal{A}(\mu) &:= N\hat{A}(\mu)M^T + N\hat{B}(\mu)C_2(\mu)X \\ &\quad +YB_2(\mu)\hat{C}(\mu)M^T + XA(\mu)Y. \end{aligned} \tag{12}$$

The identity $PP^{-1} = I$ together with (11) gives

$$MN^T = I - XY \tag{13}$$

Thus, $M \in \Re^{n\times n}$ and $N \in \Re^{n\times n}$ are invertible when $I - XY$ is invertible.

Using this change of variable, and (9) and (10), we have the following LMI-based sufficient conditions for the system (3):

Theorem 3.2. Consider the system (3) satisfies Assumption 2.1. Given a prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$, an LMI stability region, $\mathcal{D}$, and a positive constant δ. If there exist matrices X, Y, $\mathcal{A}_{ij}$, $\mathcal{B}_i$ and $\mathcal{C}_i$ satisfying the following matrix inequalities:

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \tag{14}$$

$$\Omega_{ii} < 0, \quad i = 1, \cdots, r \tag{15}$$

$$\Omega_{ij} + \Omega_{ji} < 0, \quad i < j < r \tag{16}$$

$$\Sigma_{ii} < 0, \quad i = 1, 2, \cdots, r \tag{17}$$

$$\Sigma_{ij} + \Sigma_{ji} < 0, \quad i < j < r \tag{18}$$

where

$$\Sigma_{ij} = \begin{bmatrix} \Psi_{11_{ij}} & \Psi_{21_{ij}}^T \\ \Psi_{21_{ij}} & \Psi_{22_{ij}} \end{bmatrix} \tag{19}$$

$$\Omega_{ij} = \begin{bmatrix} \begin{pmatrix} \Gamma \otimes \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \\ +\Pi \otimes \Phi_{A_{ij}} \\ +\Pi^T \otimes \Phi_{A_{ij}}^T \end{pmatrix} & (*)^T & (*)^T \\ \Pi_1 \otimes \Phi_{B_i}^T & -\dfrac{I}{\rho} & (*)^T \\ \Pi_2 \otimes \Phi_{C_i} & 0 & -\dfrac{I}{\rho} \end{bmatrix} \tag{20}$$

$$\Phi_{A_{ij}} = \begin{bmatrix} A_i X + B_{2_i} \mathcal{C}_j & A_i + B_{2_i} C_{2_j} \\ \mathcal{A}_{ij} & Y A_i + \mathcal{B}_i C_{2_j} \end{bmatrix}$$

$$\Phi_{B_i} = \begin{bmatrix} B_\Delta \\ Y B_\Delta + \mathcal{B}_i D_\Delta \end{bmatrix}$$

$$\Phi_{C_i} = \begin{bmatrix} C_{\Delta_i} X + D_{\Delta\Delta_i} & C_{\Delta_i} \end{bmatrix}$$

$$\Psi_{11_{ij}} = \begin{bmatrix} A_i X + X A_i^T + B_{2_i} \mathcal{C}_j + \mathcal{C}_i^T B_{2_j}^T & \tilde{B}_{1_i} \\ \tilde{B}_{1_i}^T & -\gamma I \end{bmatrix}$$

$$\Psi_{21_{ij}} = \begin{bmatrix} \mathcal{A}_{ij} + A_i^T & Y \tilde{B}_{1_i} + \mathcal{B}_i \tilde{D}_{21_j} \\ \tilde{C}_{1_i} X + \tilde{D}_{12_i} \mathcal{C}_j & 0 \end{bmatrix}$$

$$\Psi_{22_{ij}} = \begin{bmatrix} A_i^T Y + Y A_i + \mathcal{B}_i C_{2_j} + C_{2_i}^T \mathcal{B}_j^T & \tilde{C}_{1_i}^T \\ \tilde{C}_{1_i} & -\gamma I \end{bmatrix}$$

with $B_\Delta = [I \quad I \quad 0]$, $C_{\Delta_i} = \begin{bmatrix} E_{1_i}^T & 0 & E_{5_i}^T \end{bmatrix}^T$, $D_\Delta = [0 \ 0 \ I]$, $D_{\Delta\Delta_i} = \begin{bmatrix} 0 & E_{3_i}^T & 0 \end{bmatrix}^T$. Then the inequality (4) holds and the closed-loop system (3) with (8) is quadratically stable in the given LMI stability region $\mathcal{D}$. Furthermore, a suitable controller ($\hat{A}_{ij}, \hat{B}_i$ and $\hat{C}_i$) is given as follows:

$$\begin{aligned} \hat{B}_i &= N^{-1} \mathcal{B}_i \\ \hat{C}_i &= \mathcal{C}_i (M^T)^{-1} \\ \hat{A}_{ij} &= N^{-1} [\mathcal{A}_{ij} - Y \tilde{A}_i X \\ &\quad - \mathcal{B}_i \tilde{C}_{2_j} X - Y \tilde{B}_{2_i} \mathcal{C}_j] (M^T)^{-1} \end{aligned} \tag{21}$$

where $MN^T = I - XY$, and M and N are invertible.

Proof: Using the change of variable defined in (12), (9) and (10) can be, respectively, re-written as follows:

$$\Omega(\mu) < 0 \tag{22}$$

$$\begin{bmatrix} \Psi_{11}(\mu) & \Psi_{21}^T(\mu) \\ \Psi_{21}(\mu) & \Psi_{22}(\mu) \end{bmatrix} < 0 \tag{23}$$

where

$$\Omega(\mu) = \begin{bmatrix} \begin{pmatrix} \Gamma \otimes \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \\ +\Pi \otimes \Phi_A(\mu) \\ +\Pi^T \otimes \Phi_A^T(\mu) \end{pmatrix} & (*)^T & (*)^T \\ \Pi_1 \otimes \Phi_B^T(\mu) & -\dfrac{I}{\rho} & (*)^T \\ \Pi_2 \otimes \Phi_C(\mu) & 0 & -\dfrac{I}{\rho} \end{bmatrix} \tag{24}$$

$$\Phi_A(\mu) = \begin{bmatrix} \begin{pmatrix} A(\mu) X \\ +B_2(\mu) \mathcal{C}(\mu) \end{pmatrix} & \begin{pmatrix} A(\mu) \\ +B_2(\mu) C_2(\mu) \end{pmatrix} \\ \mathcal{A}(\mu) & \begin{pmatrix} Y A(\mu) \\ +\mathcal{B}(\mu) C_2(\mu) \end{pmatrix} \end{bmatrix}$$

$$\Phi_B(\mu) = \begin{bmatrix} B_\Delta \\ Y B_\Delta + \mathcal{B}(\mu) D_\Delta \end{bmatrix}$$

$$\Phi_C(\mu) = \begin{bmatrix} \begin{pmatrix} C_\Delta(\mu) X \\ +D_{\Delta\Delta}(\mu) \end{pmatrix} & C_\Delta(\mu) \end{bmatrix}$$

$$\Psi_{11}(\mu) = \begin{bmatrix} \begin{pmatrix} A(\mu) X + X A^T(\mu) \\ +B_2(\mu) \mathcal{C}(\mu) \\ +\mathcal{C}^T(\mu) B_2^T(\mu) \end{pmatrix} & \tilde{B}_1(\mu) \\ \tilde{B}_1^T(\mu) & -\gamma I \end{bmatrix}$$

$$\Psi_{21}(\mu) = \begin{bmatrix} \mathcal{A}(\mu) + A^T(\mu) & \begin{pmatrix} Y \tilde{B}_1(\mu) \\ +\mathcal{B}(\mu) \tilde{D}_{21}(\mu) \end{pmatrix} \\ \begin{pmatrix} \tilde{C}_1(\mu) X \\ +\tilde{D}_{12}(\mu) \mathcal{C}(\mu) \end{pmatrix} & 0 \end{bmatrix}$$

$$\Psi_{22}(\mu) = \begin{bmatrix} \begin{pmatrix} A^T(\mu) Y + Y A(\mu) \\ +\mathcal{B}(\mu) C_2(\mu) \\ +C_2^T(\mu) \mathcal{B}^T(\mu) \end{pmatrix} & \tilde{C}_1^T(\mu) \\ \tilde{C}_1(\mu) & -\gamma I \end{bmatrix}$$

with $C_\Delta(\mu) = \begin{bmatrix} E_1^T(\mu) & 0 & E_5^T(\mu) \end{bmatrix}^T$, $D_{\Delta\Delta}(\mu) = \begin{bmatrix} 0 & E_3^T(\mu) & 0 \end{bmatrix}^T$. Expanding (22) and (23), we, respectively, get

$$\Omega(\mu) = \sum_{i=1}^{r} \mu_i \mu_i \Omega_{ii} + \sum_{i=1}^{r} \sum_{i<j}^{r} \mu_i \mu_j \left[\Omega_{ij} + \Omega_{ji} \right] < 0 \tag{25}$$

$$\begin{aligned} \begin{bmatrix} \Psi_{11}(\mu) & \Psi_{21}^T(\mu) \\ \Psi_{21}(\mu) & \Psi_{22}(\mu) \end{bmatrix} &= \sum_{i=1}^{r} \mu_i \mu_i \begin{bmatrix} \Psi_{11_{ii}} & \Psi_{21_{ii}}^T \\ \Psi_{21_{ii}} & \Psi_{22_{ii}} \end{bmatrix} \\ &+ \sum_{i=1}^{r} \sum_{i<j}^{r} \mu_i \mu_j \left\{ \begin{bmatrix} \Psi_{11_{ij}} & \Psi_{21_{ij}}^T \\ \Psi_{21_{ij}} & \Psi_{22_{ij}} \end{bmatrix} \right. \\ &\left. + \begin{bmatrix} \Psi_{11_{ji}} & \Psi_{21_{ji}}^T \\ \Psi_{21_{ji}} & \Psi_{22_{ji}} \end{bmatrix} \right\} < 0. \end{aligned} \tag{26}$$

From (25) and (26), we get (15)-(18). $P > 0$ in Theorem 3.1 implies that $\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0$. ■

4. ILLUSTRATIVE EXAMPLE

Consider the following chaotic Lorenz system (Lee et al., 2001) which is described by

$$\begin{aligned} \dot{x}_1 &= -(\sigma_1 + \Delta\sigma_1) x_1 + (\sigma_1 + \Delta\sigma_1) x_2 + 0.1 w_1 + u \\ \dot{x}_2 &= (\sigma_2 + \Delta\sigma_2) x_1 - x_2 - x_1 x_3 + 0.1 w_2 \\ \dot{x}_3 &= x_1(t) x_2 - (\sigma_3 + \Delta\sigma_3) x_3 + 0.1 w_3 \end{aligned}$$

$$z = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} u(t) \qquad (27)$$
$$y = x_1 + 0.1w_1$$

where x_1, x_2 and x_3 are the state variables, u is the control input, w_1, w_2, w_3 are the disturbance noise inputs, y is the measurement output, z is the controlled output. $\sigma_1 = 10$, $\sigma_2 = 28$ and $\sigma_3 = 8/3$ are the system's parameters. The uncertain time-varying system's parameters are $\Delta\sigma_1 = f(x,t)\sigma_1$, $\Delta\sigma_2 = f(x,t)\sigma_2$ and $\Delta\sigma_3 = f(x,t)\sigma_3$ where $|f(x,t)| \leq 0.3$. Through some simulations, $x_1(t)$ seems to be bounded within $[-20, 30]$.

For the sake of simplicity, we use as few rules as possible. The nonlinear system (27) can be approximated by the following two rules TS model:

Plant Rule 1: IF x_1 is $M_1(x_1)$ THEN

$$\dot{x} = [A_1 + \Delta A_1]x + B_{1_1}w + B_{2_1}u,$$
$$z = C_{1_1}x + D_{12_2}u$$
$$y = C_{2_1}x + D_{21_1}w$$

Plant Rule 2: IF x_1 is $M_2(x_1)$ THEN

$$\dot{x} = [A_2 + \Delta A_2]x + B_{1_2}w + B_{2_2}u,$$
$$z = C_{1_2}x + D_{12_2}u$$
$$y = C_{2_2}x + D_{21_2}w$$

where the membership functions $M_1(x_1) = \frac{-x_1+30}{50}$ are $M_2(x_1) = \frac{x_1+20}{50}$.

$$A_1 = \begin{bmatrix} -\sigma_1 & \sigma_1 & 0 \\ \sigma_2 & -1 & 20 \\ 0 & -20 & -\sigma_3 \end{bmatrix}, A_2 = \begin{bmatrix} -\sigma_1 & \sigma_1 & 0 \\ \sigma_2 & -1 & -30 \\ 0 & 30 & -\sigma_3 \end{bmatrix},$$

$$B_{1_1} = B_{1_2} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad B_{2_1} = B_{2_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$C_{1_1} = C_{1_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, D_{12_1} = D_{12_2} = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix},$$

$$C_{2_1} = C_{2_2} = [1\ 0\ 0] D_{21_1} = D_{21_2} = \begin{bmatrix} 0.1 & 0 & 0 \end{bmatrix},$$

$x(t) = [x_1^T(t)\ x_2^T(t)\ x_3^T(t)]^T, w(t) = [w_1^T\ w_2^T\ w_3^T]^T$. The uncertain time-varying matrices are given as follows:

$$\Delta A_1 = F(x,t)E_{1_1} \text{ and } \Delta A_2 = F(x,t)E_{1_2}$$

where $E_{1_1} = E_{1_2} = \begin{bmatrix} -\sigma_1 & \sigma_2 & 0 \\ \sigma_2 & 0 & 0 \\ 0 & 0 & -\sigma_3 \end{bmatrix}$ and $F(x,t) = \begin{bmatrix} f(x,t) & 0 & 0 \\ 0 & f(x,t) & 0 \\ 0 & 0 & f(x,t) \end{bmatrix}$. We would like the closed-loop system to be quadratically stable in the LMI conic sector region with $\theta = 80°$. Using an LMI optimization algorithm and Theorem 3.2 with $\gamma = 1$, we obtain

$$\hat{A}_{11} = \begin{bmatrix} -52.6459 & 913.0329 & 11.1683 \\ 0.4211 & -93.8119 & -1.1292 \\ 2.3239 & -0.4233 & 0.0865 \end{bmatrix},$$

$$\hat{A}_{12} = \begin{bmatrix} -52.9740 & 909.6351 & 0.8313 \\ 0.5070 & -93.0535 & -0.2157 \\ 2.3414 & -0.2540 & 0.1024 \end{bmatrix},$$

$$\hat{A}_{21} = \begin{bmatrix} -54.8390 & 912.4579 & -6.7553 \\ 1.4467 & -93.6196 & 0.6829 \\ -3.5367 & -0.1599 & 0.2080 \end{bmatrix},$$

$$\hat{A}_{22} = \begin{bmatrix} -54.7676 & 913.4610 & -17.1638 \\ 1.3897 & -94.0748 & 1.5985 \\ -3.5229 & -0.0374 & 0.1865 \end{bmatrix},$$

$$\hat{B}_1 = \begin{bmatrix} -110.4306 \\ 4.8589 \\ 2.9909 \end{bmatrix}, \hat{B}_2 = \begin{bmatrix} 113.2188 \\ 6.1387 \\ -4.5464 \end{bmatrix},$$

$$\hat{C}_1 = \begin{bmatrix} -36.1488 & -710.9845 & -3.2817 \end{bmatrix},$$

$$\hat{C}_2 = \begin{bmatrix} -35.9847 & -709.7215 & 5.1803 \end{bmatrix}.$$

Remark 4.1. The fuzzy controller ensures that the closed-loop system is quadratically stable within the pre-specified LMI conic sector region, and the inequality (4) holds. The ratio of the regulated output energy to the disturbance input noise energy obtained is depicted in Figure 1. A square wave with amplitude=0.8 and frequency=1Hz has been used to simulate the disturbance input signals, $w_1(t)$, $w_2(t)$ and $w_3(t)$. The time-vary uncertain function, $f(x,t) = \sin(x_1(t)x_2(t))$ was chosen in the simulation. After 3 seconds, the ratio of the regulated output energy to the disturbance input noise energy tends to a constant value which is about 0.21. Thus, $\gamma = \sqrt{0.21} = 0.458$ which are less than the prescribed value 1.

5. CONCLUSION

This paper has proposed a technique for designing an $\mathcal{H}_\infty$ output feedback controller for a class of fuzzy dynamic systems that guarantees i) the $\mathcal{L}_2$-gain from an exogenous input to a regulated output is less or equal to a prescribed value and ii) the closed-loop system to be quadratically stable within a pre-specified region. Based on an LMI approach, LMI-based sufficient conditions for quadratic $\mathcal{D}$-stabilization of the uncertain Takagi-Sugeno fuzzy model with an $\mathcal{H}_\infty$ performance are derived. The effectiveness of the proposed design methodology is demonstrated through numerical simulation of the chaotic Lorenz system.

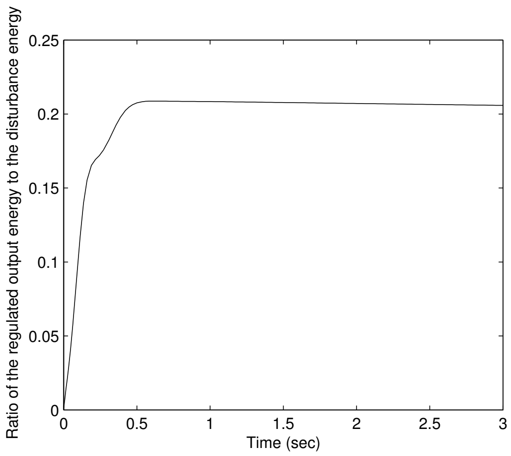

Fig. 1. The ratio of the regulated output energy to the disturbance noise energy: $\frac{\int_0^{T_f} z^T(t)z(t)dt}{\int_0^{T_f} w^T(t)w(t)dt}$.

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