OPTIMAL CONTROL PROBLEMS INVOLVING SET MEASURES

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Abstract: Various optimal control problems involving set measures are formulated. An example for such problems is that of optimal motion planning for a mobile observer or robot equipped with cameras to obtain maximum visual coverage of a given terrain. Optimality conditions in the form of variational inequality and maximum principle are presented. The results are applied to the optimal motion planning problem with a simple terrain. *Copyright© 2005 IFAC*

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1. INTRODUCTION

Optimal control problems involving set measures arise in many physical situations. An example is the problem of optimal motion planning for a mobile observer or robot equipped with cameras for planetary exploration or surveillance. It is required to select a path along which complete or maximum visual coverage of a given terrain is attained over the shortest or a specified observation time interval respectively. To fix ideas, we begin with a detailed discussion of this example which provides the motivation for the mathematical formulation of more general optimal control problems involving set measures. Then, optimality conditions in the form of variational inequality and maximum principle for these problems are developed. The paper concludes with the solutions to the optimal motion planning problem with a simple terrain.

2. OPTIMAL MOTION PLANNING PROBLEM

Let $\mathcal{B} = \{e_1, \ldots, e_n\}$ be an orthonormal basis for the n-dimensional real Euclidean space $\mathbb{R}^n$. The representation of a point $x \in \mathbb{R}^n$ with respect to $\mathcal{B}$ is denoted by $[x] = (x_1, \ldots, x_n)^T$, and the usual Euclidean norm of x by $\|x\|$. When ambiguity does not arise, x is also used to denote $[x]$. Let $S = S(x)$ be a specified real-valued C_2-function defined on Ω, a specified simply connected, compact subset of $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$. Let $G_S \stackrel{\text{def}}{=} \{(x, S(x)) \in \mathbb{R}^3 : x \in \Omega\}$ denote the *graph of* S; and $Epi_S \stackrel{\text{def}}{=} \{(x, z) \in \Omega \times \mathbb{R} : z \geq S(x)\}$, the *epigraph of* S. The spatial profile of the terrain under observation corresponds to G_S, The observation platform on which the cameras are attached corresponds to the *elevated surface of* G_S given by G_{S_ϵ}, where $S_\epsilon = S + \epsilon$ with ϵ being a specified positive number. This implies that for any $x \in \Omega$, the cameras are at a fixed vertical height ϵ above the surface G_S.

Definition 2.1 A point $(x, S(x)) \in G_S$ is said to be *visible* from a point $(\tilde{x}, \tilde{z}) \in Epi_S$, if the line segment $\{(x', z) \in \mathbb{R}^3 : (x', z) = \lambda(x, S(x)) + (1 - \lambda)(\tilde{x}, \tilde{z}), 0 \leq \lambda \leq 1\}$ joining the points $(x, S(x))$ and $(\tilde{x}, \tilde{z})$ lies in Epi_S.

Definition 2.2 The *visible set* $\mathcal{V}((x, z))$ of a given point $(x, z) \in Epi_S$ is the set of all points in G_S that are visible from (x, z), i.e. $\mathcal{V}((x, z)) = \{(x', S(x')) \in G_S : (x', S(x'))$ is visible from $(x, z)\}$. If $\mathcal{V}((x, z)) = G_S$, then G_S is said to be *totally visible from* (x, z).

The definitions of visible set and total visibility can be extended to a *set* of observation points. Since for any $(x,z) \in Epi_S$, the point $(x, S(x)) \in G_S$ is always visible from (x,z), hence $\mathcal{V}((x,z))$ is nonempty. Since S is assumed to be a C_2-function defined on a compact set Ω, G_S is compact. Moreover, $\mathcal{V}((x,z))$ and its projection on Ω (denoted by $\Pi_\Omega \mathcal{V}((x,z))$) are also compact. Thus, $(x,z) \rightarrow \mathcal{V}((x,z))$ (resp. $\Pi_\Omega \mathcal{V}((x,z))$) is a set-valued mapping on Epi_S into 2^{G_S} (resp. 2^Ω), the space of all nonempty compact subsets of G_S (resp. Ω). In general, $\Pi_\Omega \mathcal{V}((x,z))$ may be the union of disjoint compact subsets of Ω, and it may consist of isolated points and/or arcs in Ω. It was shown by Wang (2003) that for each point $x \in \Omega$, there exists a *minimal* or *critical height* $h_c(x) \geq S(x)$ such that $\mathcal{V}((x, h_c(x))) = G_S$, or G_S is totally visible from the point $(x, h_c(x))$. Moreover, the set-valued mapping $x \rightarrow \mathcal{V}((x, S_\epsilon(x)))$ on Ω into 2^{G_S} may be discontinuous with respect to the Euclidean metric ρ_E and Hausdorff metric ρ_H when G_S has flat parts (See Wang (2003) for an example).

Consider the nontrivial case where $\epsilon < h_c(x)$ for all $x \in \Omega$ so that the mobile observer must move to achieve total or maximum visibility. Let $I_{t_1} = [0, t_1]$ denote the observation time interval, where t_1 may be a finite fixed or variable terminal time. For simplicity, the mobile observer is represented by a point mass M. Its position in $\mathbb{R}^3$ at any time t is specified by $p(t)$ whose representation with respect to a given orthonormal basis $\mathcal{B}$ is denoted by $[p(t)] = (x_1(t), x_2(t), h(t))^T$, where $h(t)$ corresponds to the vertical height along the z-axis. The motion of the mobile observer can be described by Newton's law:

$$M\ddot{x}(t) + D\dot{x}(t) = u(t); \tag{1}$$

$$M\ddot{h}(t) + \nu_z(x(t), \dot{x}(t), h(t), \dot{h}(t)) = \xi(t) - Mg, \tag{2}$$

where $x(t) = (x_1(t), x_2(t))^T$; (u, ξ) is the external force with $u = (u_1, u_2)$ being the control; $-Mg$ is the gravitational force aligned with the z-axis in the downward direction. D is a 2×2 diagonal matrix with constant diagonal elements ν_{x1} and ν_{x2}; ν_z is a specified real-valued function of its arguments describing the z-component of the friction force. Assuming that the mobile observer is constrained to move on G_S at all times without slipping, the mobile observer motion satisfies a holonomic constraint:

$$h(t) = S(x(t)) \text{ for all } t \in I_{t_1}, \tag{3}$$

and a state variable (position) constraint: $x(t) \in \Omega$ for all $t \in I_{t_1}$. Since S is a C_2-function on Ω, we may differentiate (3) twice with respect to t to obtain

$$\dot{h}(t) = \nabla_x S(x(t))^T \dot{x}(t);$$

$$\ddot{h}(t) = \nabla_x S(x(t))^T \ddot{x}(t) + \dot{x}(t)^T H(x(t)) \dot{x}(t), \tag{4}$$

where ∇_x denotes the gradient operator with respect to x, and $H(x(t))$ the Hessian matrix of S with respect to x evaluated at $x(t)$. Substituting (4) into (2) gives the required vertical component $\xi(t)$ of the external force for keeping the mobile observer on the surface G_S at all times:

$$\begin{aligned}\xi(t) = M(&\nabla_x S(x(t))^T \ddot{x}(t) + \dot{x}(t)^T H(x(t)) \dot{x}(t)\\ &+ \nu_z(x(t), \dot{x}(t), \nabla_x S(x(t))^T \dot{x}(t)) + g).\end{aligned} \tag{5}$$

Assuming that the mobile observer lies on G_S at the starting time $t = 0$, then

$$h(0) = S(x(0)), \quad \dot{h}(0) = \nabla_x S(x(0))^T \dot{x}(0). \tag{6}$$

Let $s_x(t) = (x(t), \dot{x}(t))$ denote the state of system (1) at time t. When necessary, $x_u(t; s_x(0))$ is used to indicate the dependence of a solution of (1) on the control u and $s_x(0)$. A control $u = u(t)$ defined on a given time interval I_{t_1} is said to be *admissible*, if it is a measurable function on I_{t_1}, and takes its values in the control region U_2, where $U_m = \{(u_1, \ldots, u_m) \in \mathbb{R}^m : |u_i| \leq \bar{u}_i, i = 1, \ldots, m\}$, with $\bar{u}_i$'s being given positive constants. The set of all admissible controls defined on I_{t_1} is denoted by $\mathcal{U}_{\rm ad}(I_{t_1})$, where t_1 is a fixed or variable terminal time. In what follows, it is assumed that no constraint is imposed on the magnitude of the vertical force ξ.

Now, a few physically meaningful optimal motion planning problems incorporating the foregoing notion of visibility into the formulation can be stated as follows:

• *P1. Minimum-time Total Visibility Problem.* Let $\mathcal{U}_{\rm ad} = \bigcup_{t_1 \geq 0} \mathcal{U}_{\rm ad}(I_{t_1})$ be the set of all admissible controls. Given $s_x(0) = (x(0), \dot{x}(0))$ or the initial state of the mobile observer with initial position $p(0) = (x(0), S(x(0))) \in G_S$ and initial velocity $v(0) = (\dot{x}(0), \nabla_x S(x(0))^T \dot{x}(0))$, find the smallest time $t_1^* \geq 0$ and an admissible control $u^* = u^*(t)$ defined on $I_{t_1^*}$ such that its corresponding path $\Gamma^* = \{(x_{u^*}(t), S(x_{u^*}(t)) \in \mathbb{R}^3 : t \in I_{t_1^*}\}$ satisfies the total visibility condition at t_1^*:

$$\bigcup_{t \in I_{t_1^*}} \mathcal{V}((x_{u^*}(t), S_\epsilon(x_{u^*}(t))) = G_S$$

or alternatively,

$$\mu_2\{\bigcup_{t \in I_{t_1^*}} \Pi_\Omega \mathcal{V}((x_{u^*}(t), S_\epsilon(x_{u^*}(t)))\} = \mu_2\{\Omega\}, \tag{7}$$

where $\mu_2\{\sigma\}$ denotes the Lebesgue measure of the set $\sigma \subset \mathbb{R}^2$.

In the foregoing problem statement, condition (7) only involves the position $x_{u^*}(t)$, not the velocity $\dot{x}_{u^*}(t)$. In certain physical situations, it is required to move the mobile observer from one rest position to another, i.e. $\dot{x}_{u^*}(0) = 0$ and $\dot{x}_{u^*}(t_1^*) = 0$.

• *P2. Maximum Visibility Problem with Fixed Observation Time-Interval.* Given a finite observation time interval I_{t_1} and $s_x(0) = (x(0), \dot{x}(0))$, or the initial state of the mobile observer with initial position $p(0) = (x(0), S(x(0))) \in G_S$ and initial velocity $v(0) = (\dot{x}(0), \nabla_x S(x(0))^T \dot{x}(0))$, find an admissible control $u^* = u^*(t)$ and its corresponding path

$\Gamma^* = \{(x_{u^*}(t), S(x_{u^*}(t))) \in \mathbb{R}^3 : t \in I_{t_1}\}$ such that the visibility functional given by

$$J_1(u) = \int_0^{t_1} \mu_2\{\Pi_\Omega \mathcal{V}((x_u(t), S_\epsilon(x_u(t))))\}dt \quad (8)$$

is defined, and satisfies $J_1(u^*) \geq J_1(u)$ for all $u(\cdot) \in \mathcal{U}_{\rm ad}(I_{t_1})$.

Another meaningful visibility functional is given by

$$J_2(u) = \mu_2\{\bigcup_{t\in I_{t_1}} \Pi_\Omega \mathcal{V}((x_u(t), S_\epsilon(x_u(t))))\}. \quad (9)$$

The foregoing problem with J_1 replaced by J_2 corresponds to selecting an admissible control u^* such that the area of the union of the projected visibility sets on Ω for all the points along the corresponding path Γ^* is maximized.

3. OPTIMAL CONTROL PROBLEMS

The foregoing example suggests the following optimal control problems involving set measures. As in Sec.2, let I_{t_1} denote the control time interval, and $\mathcal{U}_{\rm ad}(I_{t_1})$ the set of all admissible controls $u(\cdot)$ defined on I_{t_1}. Let the system be described by

$$\dot{x}(t) = Ax(t) + \psi(u(t)), \quad x(0) = x_o \in \mathbb{R}^n, \quad (10)$$

where A is a given $n \times n$ constant matrix, and ψ is a specified C_1-function on U into R^n, where U is a given compact subset of $\mathbb{R}^m$. The set of all admissible controls defined on I_{t_1} is denoted by $\mathcal{U}_{\rm ad}(I_{t_1})$. Let $x \to \tilde{\mathcal{V}}(x)$ be a given set-valued mapping on $\mathbb{R}^n \to 2^{\mathbb{R}^n}$ such that $\tilde{\mathcal{V}}(x)$ is compact, and $\mu_n\{\tilde{\mathcal{V}}(x)\} \leq \hat{\mu} < \infty$ for all $x \in \mathbb{R}^n$, where μ_n denotes the Lebesgue measure for sets in $\mathbb{R}^n$, and $\hat{\mu}$ is a given positive number. Moreover, $\tilde{\mathcal{V}}$ is continuous with respect to metrics ρ_E and ρ_H. Now, optimal control problems similar to *P1* and *P2* can be stated as follows:

• *Problem P1':* Let $\mathcal{U}_{\rm ad} = \bigcup_{t_1\geq 0}\mathcal{U}_{\rm ad}(I_{t_1})$ be the set of all admissible controls. Given x_o the initial state of (10) at $t = 0$, find the smallest time $t_1^* \geq 0$ and an admissible control $u^* = u^*(t)$ defined on $I_{t_1^*}$ such that its corresponding trajectory x_{u^*} satisfies the terminal condition at t_1^*:

$$\mu_n\{\bigcup_{t\in I_{t_1^*}} \tilde{\mathcal{V}}(x_{u^*}(t; x_o))\} = c_o, \quad (11)$$

where c_o is a specified positive constant.

• *Problem P2':* Given a finite control time interval I_{t_1} and x_o, the initial state of (10) at $t = 0$, find an admissible control $u^* = u^*(t)$ and its corresponding trajectory $x_{u^*}(t; x_o), t \in I_{t_1}$ such that the functional given by

$$J_1^{'}(u) = \int_0^{t_1} \mu_n\{\tilde{\mathcal{V}}(x_u(t; x_o))\}dt \quad (12)$$

is defined, and satisfies $J_1^{'}(u^*) \geq J_1^{'}(u)$ for all $u(\cdot) \in \mathcal{U}_{\rm ad}(I_{t_1})$.

4. OPTIMALITY CONDITIONS

In what follows, optimality conditions for Problem *P1'* will be derived under the assumption that a solution exists. Let

$$w_u(t; x_o) = \mu_n\{\bigcup_{\tau\in I_t} \tilde{\mathcal{V}}(x_u(\tau; x_o))\} \quad (13)$$

with $w_u(0; x_o) = \mu_n\{\tilde{\mathcal{V}}(x_o)\}$. It is required to find the smallest time $t_1^* \in \mathbb{R}^+ = [0, \infty[$ and an admissible control $u^* = u^*(t)$ defined on $I_{t_1^*}$ such that $w_{u^*}(t_1^*; x_o) = c_o$.

A necessary condition for optimality can be derived by considering the augmented system:

$$\frac{d}{dt}\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} Ax + \psi(u) \\ g(x) \end{bmatrix}, \quad (14)$$

where

$$g(x(t)) \stackrel{\rm def}{=} \lim_{\delta t\to 0^+} \sup \frac{1}{\delta t}[w(t + \delta t) - w(t)]$$
$$= \lim_{\delta t\to 0^+} \sup \frac{1}{\delta t}\Big[\mu_n\{\bigcup_{\tau\in I_{t+\delta t}} \tilde{\mathcal{V}}(x(\tau))\} - \mu_n\{\bigcup_{\tau\in I_t} \tilde{\mathcal{V}}(x(\tau))\}\Big], \quad (15)$$

and the initial state at $t = 0$ is given by $s_{(x,w)}(0) = (x_o, \mu_n\{\tilde{V}(x_o)\}) \in \mathbb{R}^n \times \mathbb{R}^+$. Since for $\delta t \geq 0$, $\bigcup_{\tau\in I_t} \tilde{\mathcal{V}}(x(\tau)) \subseteq \bigcup_{\tau\in I_{t+\delta t}} \tilde{\mathcal{V}}(x(\tau))$, $g(x(t))$ can be rewritten as

$$g(x(t)) = \lim_{\delta t\to 0^+} \sup \frac{1}{\delta t}\mu_n\{[\bigcup_{\tau\in I_{t+\delta t}} \tilde{\mathcal{V}}(x(\tau))] \cap [\bigcup_{\tau\in I_t} \tilde{\mathcal{V}}(x(\tau))]^c\}, \quad (16)$$

where σ^c denotes the complement of the set σ in $\mathbb{R}^n$.

The target set is a hyperplane specified by $\mathcal{T} = \{(x, w) \in \mathbb{R}^n \times \mathbb{R}^+ : w = c_o\}$. Thus, Problem *P1'* can be restated in the form of a standard time-optimal control problem, i.e. find an admissible control $u^*(\cdot)$ which steers the initial state $s_{(x,w)}(0)$ of system (14) at $t = 0$ to the target set $\mathcal{T}$ in minimum time t_1^*.

Let the Hamiltonian for (14) be defined by:

$$\mathcal{H}(x, \eta, u) = -1 + \tilde{\eta}^T(Ax + \psi(u)) + \eta_{n+1}g(x), \quad (17)$$

where $\eta = (\eta_1, \ldots, \eta_n, \eta_{n+1})^T$ corresponds to the state of the adjoint system:

$$\dot{\tilde{\eta}} = -A^T\tilde{\eta} - \eta_{n+1}\nabla_x g(x), \quad \dot{\eta}_{n+1} = 0, \quad (18)$$

where $\tilde{\eta} = (\eta_1, \ldots, \eta_n)^T$, and ∇_x denotes the gradient operator with respect to x.

If the real-valued function $x \to g(x)$ on $\Omega \to \mathbb{R}^+$ is smooth, then the following necessary condition for optimality follows from the Pontryagin Maximum Principle (Lee and Markus (1967)):

Theorem 4.1 *Suppose that the function* $x \to g(x)$ *on* $\mathbb{R}^n \to \mathbb{R}^+$ *is* C_1. *Let* $u^* = u^*(t)$ *be an optimal* *control for Problem P1 with corresponding response $x^* = x^*(t)$ defined on $I_{t_1^*}$. Then there exists an absolutely continuous function $\tilde{\eta}^* = \tilde{\eta}^*(t)$ satisfying* (18) *for almost all $t \in I_{t_1^*}$ with*

$$\mathcal{H}(x^*(t), \eta^*(t), u^*(t)) \\ = \mathcal{M}(x^*(t), \eta^*(t)) \text{ for almost all } t \in I_{t_1^*}, \quad (19)$$

where $\tilde{\eta} = (\eta_1, \ldots, \eta_n)^T$, and

$$\mathcal{M}(x^*(t), \eta^*(t)) \stackrel{\text{def}}{=} \max_{u \in U} \mathcal{H}(x^*(t), \eta^*(t), u). \quad (20)$$

Moreover,

$$\mathcal{M}(x^*(t), \eta^*(t)) \equiv 0 \text{ on } I_{t_1^*}, \quad (21)$$

and the transversality condition: $-\eta(t_1^*) = (0, \kappa)^T$ for some $\kappa > 0$ (the normal to the target set $\mathcal{T}$ at $(x^*, w^*)(t_1^*)$ in the positive w-direction), or

$$\tilde{\eta}^*(t_1^*) = 0, \quad \eta_{n+1}^*(t_1^*) = -\kappa \quad (22)$$

is satisfied.

From Theorem 4.1, it is evident from (17) that the optimal control u^* is a function of $\tilde{\eta}$ only. For the special case where $\psi(u) = u$ and $U = U_n$ as in system (1), $u^*(t)$ takes on the form:

$$u_i^*(t) = \bar{u}_i \text{sgn}(\eta_i^*(t)), \quad i = 1, \ldots, n. \quad (23)$$

Equations (14),(18)-(21) with initial conditions $x(0) = x_o, w(0) = \mu_n\{\tilde{\mathcal{V}}(x_o)\}$), and terminal conditions $w(t_1) = c_o, \tilde{\eta}(t_1) = 0, \eta_{n+1}(t_1) = -\kappa$ form a family of two-point-boundary-value problems (TPBVP) with the terminal time t_1 as a variable parameter. The optimal trajectory is a solution of the TPBVP with the smallest terminal time t_1^*. It is evident from the second equation in (18) and (22), $\eta_{n+1}(t) \equiv -\kappa$ on $I_{t_1^*}$. Thus, (18) can be rewritten as: $\dot{\tilde{\eta}} = -A^T\tilde{\eta} + \kappa\nabla_x g(x)$.

Now, consider Problem P2' with the assumption that an optimal control $u^* = u^*(t)$ defined on I_{t_1} exists. Let δu be a control perturbation such that $u = u^* + \delta u$ is admissible. Let the solutions of (10) at time t corresponding to u and u^*, and the same initial state x_o be denoted by $x_u(t)$ and $x_{u^*}(t)$ respectively. To derive optimality conditions, consider

$$\Delta J_1' \stackrel{\text{def}}{=} J_1'(u^*) - J_1'(u^* + \delta u) \\ = \int_0^{t_1} (\mu_n\{\tilde{\mathcal{V}}(x_{u^*}(t))\} - \mu_n\{\tilde{\mathcal{V}}(x_{u^*+\delta u}(t))\})dt, \quad (24)$$

Using the identity:

$$\tilde{\mathcal{V}}(x_{u^*}(t)) - \tilde{\mathcal{V}}(x_u(t)) \\ = (\tilde{\mathcal{V}}(x_u(t)))^c \cap \tilde{\mathcal{V}}(x_{u^*}(t)) \cup (\tilde{\mathcal{V}}(x_{u^*}(t))^c \cap \tilde{\mathcal{V}}(x_u(t))), \quad (25)$$

$\Delta J_1'$ can be rewritten as:

$$\Delta J_1' = \int_0^{t_1} (\mu_n\{(\tilde{\mathcal{V}}(x_{u^*+\delta u}(t)))^c \cap \tilde{\mathcal{V}}(x_{u^*}(t))\} \\ -\mu_n\{(\tilde{\mathcal{V}}(x_{u^*}(t)))^c \cap \tilde{\mathcal{V}}(x_{u^*+\delta u}(t))\})dt. \quad (26)$$

Thus, a sufficient but not necessary condition for optimality is given by

$$\mu_n\{(\tilde{\mathcal{V}}(x_{u^*+\delta u}(t)))^c \cap \tilde{\mathcal{V}}(x_{u^*}(t))\} \\ \geq \mu_n\{(\tilde{\mathcal{V}}(x_{u^*}(t)))^c \cap \tilde{\mathcal{V}}(x_{u^*+\delta u}(t))\} \quad (27)$$

for almost all $t \in I_{t_1}$ and all admissible $u^* + \delta u$, where $x_{u^*+\delta u}(t)$ can be written in the form:

$$x_{u^*+\delta u}(t) = x_{u^*}(t) + \delta x(t) + o(\|\delta x(t)\|), \quad (28)$$

with

$$x_{u^*}(t) = e^{At}x_o + \int_0^t e^{A(t-\tau)}\psi(u^*(\tau))d\tau,$$

$$\delta x(t) = \int_0^t e^{A(t-\tau)} J_\psi(u^*(\tau))\delta u(\tau)d\tau, \quad (29)$$

where J_ψ denotes the Jacobian matrix of ψ with respect to u.

Now, consider perturbed admissible controls of the form $u^* + \alpha\delta u$, where δu is a given control perturbation, and $0 \leq \alpha < 1$. If the real-valued function $x \to \mu_n\{\tilde{\mathcal{V}}(x)\}$ on $\mathbb{R}^n \to \mathbb{R}^+$ is C_1, then the Gateaux differential of J_1' at $x_{u^*}(\cdot)$ with increment $\delta x(\cdot)$ exists. Thus, we have the following necessary condition for optimality:

Theorem 4.2 *Suppose that an optimal control $u^* = u^*(t)$ defined on I_{t_1} for Problem P2' exists, and the function $x \to \mu_n\{\tilde{\mathcal{V}}(x)\}$ on $\Omega \to \mathbb{R}^+$ is C_1. Then $u^*(\cdot)$ must satisfy the following variational inequality:*

$$DJ_1'(u^*; \delta u) = \\ \int_0^{t_1} \lim_{\alpha \to 0} \frac{1}{\alpha}(\mu_n\{\tilde{\mathcal{V}}(x_{u^*+\alpha\delta u(t)})^c \cap \tilde{\mathcal{V}}(x_{u^*}(t))\} \\ -\mu_n\{\tilde{\mathcal{V}}(x_{u^*}(t))^c \cap \tilde{\mathcal{V}}(x_{u^*+\alpha\delta u}(t))\})dt \geq 0 \quad (30)$$

for all admissible $u^* + \alpha\delta u$.

Another necessary condition for optimality can be obtained by introducing a new state variable y. The evolution of $y(t)$ with time t is described by

$$\dot{y}(t) = \mu_n\{\tilde{\mathcal{V}}(x_u(t))\}, \quad y(0) = 0. \quad (31)$$

Thus, $J_1'(u) = y(t_1)$. Let the Hamiltonian associated with the augmented system (10) and (31) be defined by:

$$\mathcal{H}(x, \eta, u) = \mu_n\{\tilde{\mathcal{V}}(x)\} + \eta^T(Ax + \psi(u)), \quad (32)$$

where $\eta = (\eta_1, \ldots, \eta_n)^T$ corresponds to the state of the adjoint system:

$$\dot{\eta} = -\nabla_x\mathcal{H}. \quad (33)$$

Again, if the function $x \to \mu_n\{\tilde{\mathcal{V}}(x)\}$ on $\mathbb{R}^n \to \mathbb{R}^+$ is smooth, then a necessary condition for optimality is given by:

Theorem 4.3 *Suppose that the function $x \to \mu_n\{\tilde{\mathcal{V}}(x)\}$ on $\mathbb{R}^n \to \mathbb{R}^+$ is C_1. Let $u^* = u^*(t)$ be an optimal control for Problem P2' with corresponding response $x^* = x^*(t)$. Then there exists an absolutely continuous function $\eta^* = \eta^*(t)$ satisfying* (33) *given explicitly by*

$$\dot{\eta} = -\nabla_x\mu_n\{\tilde{\mathcal{V}}(x)\} - A^T\eta \quad (34)$$

for almost all $t \in I_{t_1}$ *and terminal condition:* $\eta^*(t_1) = 0$ *with*

$$\mathcal{H}(x^*(t), \eta^*(t), u^*(t)) = \mathcal{M}(x^*(t), \eta^*(t)) \qquad (35)$$

for almost all $t \in I_{t_1}$*, where*

$$\mathcal{M}(x^*(t), \eta^*(t)) \stackrel{\text{def}}{=} \max_{u \in U} \mathcal{H}(x^*(t), \eta^*(t), u). \qquad (36)$$

Thus, (10),(31) and (34) with terminal condition $\eta(t_1) = 0$ and initial condition $(x, y)(0) = (x_o, 0)$ along with (35) and (36) constitute a nonlinear TPBVP for which the optimal trajectory $(x^*, y^*, \eta^*)(\cdot)$ must satisfy. For the special case where $\psi(u) = u$ and $U = U_n$, Theorem 4.3 implies that $u^*(t)$ has the form given by (23).

The main difficulty in applying Theorems 4.1-4.3 to concrete problems such as the motion planning problem is that the mapping $\tilde{\mathcal{V}}$ derived from physical situations (e.g. the visible sets $\mathcal{V}((x, S_\epsilon(x)))$ for $x \in \Omega$) cannot be expressed analytically in term of x. Consequently, the Gateaux differential in (30); $g(x)$, $\nabla_x g(x)$ and $\nabla_x \mu_n\{\tilde{\mathcal{V}}(x)\}$ in (14),(18) and (34) respectively cannot be readily computed.

5. EXAMPLE

Consider a simple case of the optimal motion planning problems discussed in Sec.2 in which the visible sets at any point in G_{S_ϵ} can be computed analytically. Let Ω be the normalized spatial domain specified by the unit disk $\{x \in \mathbb{R}^2 : \|x\| \le 1\}$, where x has been normalized with respect to the radius r_o of the actual spatial domain. The surface under observation corresponds to the graph of the real-valued function S given by

$$S(x) = 1 - \|x\|^2, \quad x \in \Omega, \qquad (37)$$

where $x = (x_1, x_2)$, and $\|x\|^2 = x_1^2 + x_2^2$. It can be verified by elementary computations that for any given $\epsilon > 0$, the projection of the visible set from a point $(x, S_\epsilon(x)) \in G_{S_\epsilon}$ onto Ω is simply the intersection of the unit disk with the disk centered at x with radius $\sqrt{\epsilon}$, i.e. $\Pi_\Omega \mathcal{V}((x, S_\epsilon(x))) = \Omega \cap \{x' = (x_1', x_2') \in \mathbb{R}^2 : \|x - x'\| \le \sqrt{\epsilon}\}$. Moreover, for $0 < \epsilon \le 1$,

$$\mu_2\{\Pi_\Omega \mathcal{V}((x, S_\epsilon(x)))\} = \begin{cases} \epsilon\pi, & \text{if } 0 \le r \le 1 - \sqrt{\epsilon}; \\ \beta(r, \tilde{r}), & \text{if } 1 - \sqrt{\epsilon} < r \le 1, \end{cases} \qquad (38)$$

where $r = \|x\|$, $\tilde{r} = (1 - \epsilon + r^2)/2r$, and

$$\beta(r, \tilde{r}) = \cos^{-1}(\tilde{r}) + \epsilon \cos^{-1}\left(\frac{r - \tilde{r}}{\sqrt{\epsilon}}\right) - r\sqrt{1 - \tilde{r}^2}. \qquad (39)$$

Note that for $\epsilon = 1$, the surface G_{μ_2} is not smooth at the origin. Moreover, $\epsilon = 1$ corresponds to the critical height h_c at the origin defined in Sec.2.

Consider the Minimum-Time Total Visibility Problem *P1*. To compute the trajectory and control that satisfy the necessary condition for optimality for Problem *P1* given by Theorem 4.1, it is convenient to introduce a normalized time $\tau = t/t_1$ and solve the TPBVP stated in Theorem 4.1 with a variable parameter t_1. However, even for the simple S given by (37), the computation of $g(x)$ defined by (16) for any $x \in \Omega$ is a tedious task. The characterization of optimal control given by (23) suggests seeking "bang-bang" controls with a finite number of switchings to achieve total visibility in minimum time. Starting with the case with no switchings in both u_1 and u_2, then one switching in u_1 and no switching in u_2 etc, we obtain a trajectory in the (x_1, x_2)-plane with total visibility with the smallest terminal time t_1. Figures 1-3 show the results for the case where $\epsilon = 1/4$. It can be seen from Fig.1 that the computed trajectory in the (x_1, x_2)-plane approaches the circle with radius $\sqrt{\epsilon} = 1/2$ as fast as possible, and then stays in the neighborhood of this circle for the remaining times until total visibility is attained. At any point x on this circle, $\mu_2\{\Pi_\Omega \mathcal{V}((x, S_\epsilon(x)))\}$ takes on its maximum value $\pi\epsilon$.

Next, consider the Maximum Visibility Problem *P2*. Here, the augmented system corresponding to (1) and (31) has the following form:

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} \dot{x} \\ (-D\dot{x} + u)/M \\ \mu_2\{\Pi_\Omega \mathcal{V}((x, S_\epsilon(x)))\} \end{bmatrix}, \qquad (40)$$

where $\mu_2\{\Pi_\Omega \mathcal{V}((x, S_\epsilon(x)))\}$ is given by (38). The adjoint system corresponding to (34) has the following explicit form:

$$\frac{d}{dt}\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} = -\begin{bmatrix} h_1(x) \\ h_2(x) \\ \eta_1 - (\nu_{x1}/M)\eta_3 \\ \eta_2 - (\nu_{x2}/M)\eta_4 \end{bmatrix}, \qquad (41)$$

where $h_i(x) =$

$$\begin{cases} 0, & \text{if } 0 \le r \le 1 - \sqrt{\epsilon}; \\ -2x_i\sqrt{1 - \tilde{r}^2}/r, & \text{if } 1 - \sqrt{\epsilon} < r < 1, \quad i = 1, 2. \end{cases} \qquad (42)$$

Since $x \to \mu_2\{\Pi_\Omega \mathcal{V}((x, S_\epsilon(x)))\}$ given by (37) is C_1, it follows from Theorem 4.3 that the optimal control u^* has the form:

$$u_1^*(t) = \bar{u}_1 \text{sgn}(\eta_3^*(t)), \;\; u_2^*(t) = \bar{u}_2 \text{sgn}(\eta_4^*(t)). \qquad (43)$$

Consider the nonlinear TPBVP corresponding to (40)-(43) with initial condition $(x, \dot{x}, y)(0) = (x(0), \dot{x}(0), \mu_2\{\Pi_\Omega \mathcal{V}((x(0), S_\epsilon(x(0))))\})$ and terminal condition $\eta(t_1) = 0$. Numerical solutions for this problem are obtained for specified values of the system parameters using the MATLAB algorithm "BVP4C" by Shampine et al. Figures 4-6 show the results for the case where $\epsilon = 1/4; x(0) = (-1, 0), \dot{x}(0) = (0.025, -0.05), y(0) = \mu_2\{\Pi_\Omega \mathcal{V}((x(0), S_\epsilon(x(0))))\}$; $M = 10$ kg, $\nu_{x1} = 0.01$ N. sec./m, $\nu_{x2} = 0.02$

N.sec./m, $t_1 = 25$ sec., and $\bar{u} = 0.1$ N/m. In the computation, the signum function in (43) was approximated by $\tanh(50\eta_i^*(t)), i = 3, 4$. It can be seen from Fig.5 that the main portion of the projected trajectory on Ω lies inside the disk D with radius $\sqrt{\epsilon} = 1/2$, where $\mu_2\{\Pi_\Omega \mathcal{V}((x, S_\epsilon(x)))\}$ takes on its maximum value $\pi\epsilon$. This result is consistent with intuition that in order to maximize J_1, the trajectory should enter the disk D as quickly as possible and stay inside D as long as possible for the remaining times.

6. CONCLUSIONS

In this paper, a class of optimal control problems involving set measures motivated from optimal motion planning problems based on visibility has been studied. These optimization problems are generally non-smooth. The derived optimality conditions are not easily applicable to problems derived form real-world situations. Efficient algorithms are needed for the numerical solution of these problems. Some progress has been made in this direction recently for optimal motion planning problems (See Balmes and Wang (2000), Wang (2004)).

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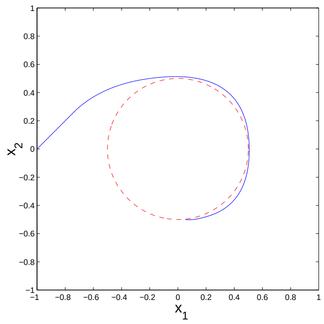

Fig. 1. Projection of the computed trajectories onto the (x_1, x_2)-plane for Problem *P1* with $\epsilon = 1/4$.

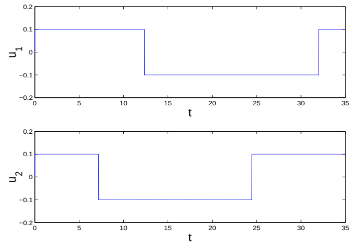

Fig. 2. Computed controls $u_1 = u_1(t)$ and $u_2 = u_2(t)$ for Problem *P1*.

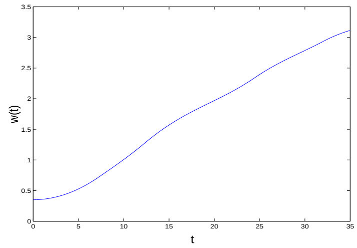

Fig. 3. Time-domain plot of $w = w(t)$ along the computed path for Problem *P1*.

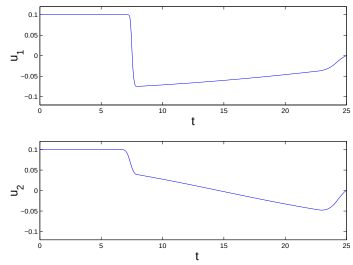

Fig. 4. Computed controls $u_1 = u_1(t)$ and $u_2 = u_2(t)$ for Problem *P2*.

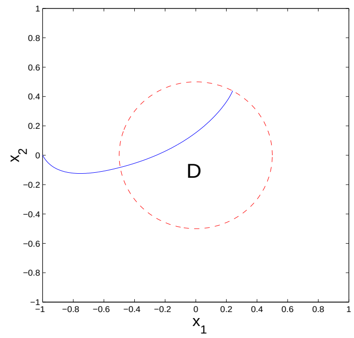

Fig. 5. Computed solution to TPBVP corresponding to Problem *P2* projected onto the (x_1, x_2)-plane.

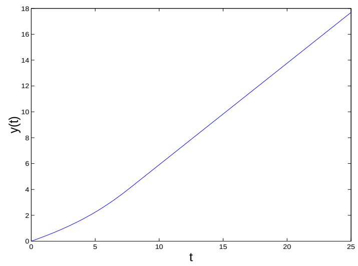

Fig. 6. Computed visibility functional $y = y(t)$ for Problem *P2*.