Δ-MODULATED FEEDBACK IN DISCRETIZATION OF SLIDING MODE CONTROL

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Abstract:
Modulated feedback control introduces periodicity. The global attracting property of the periodic points is established for a simple scalar discrete-time system under Δ-modulated feedback. Attracting regions of the periodic points are also characterized. When the discretization effects of the equivalent control based sliding mode control systems are studied, we show that the zero-order-hold discretization gives rise to Δ-modulation in the sliding mode direction. The global attracting property of Δ-modulated feedback offers a vivid illustration of the way sliding happens. Interestingly, we find that a ZOH discretization scheme of the equivalent control based sliding mode control system with relative degree one results in only two-periodic orbits.

Keywords: Delta-modulation, periodic points, sliding mode control

1. INTRODUCTION

Discrete sliding mode control arises in two different situations: one associated with sliding mode control of discrete-time systems and the other resulting from discretization of the sliding mode control of continuous-time systems. Studies of both cases have been reported in the literature (see (Wu, Drakuno and Ozguner, 2000; Koshkouei and Zinober, 2000; Yu, 1998; Yu and Chen, 2003) and references therein).

A different line of research is Sigma-Delta (or Δ-) modulation and Δ-modulated feedback of discrete signals and/or systems. Sigma-Delta modulation first appeared in electronic circuits as a method of analog-to-digital conversion(Inose and Yasuda, 1963; Leonov, 1959). An early implementation of Δ-modulated feedback control is the transmitting power control of a mobile unit in the Direct Sequence Code Division Multiple Access (DS-CDMA) cellular network (Ariyavisitakul and Chang, 1991), due to the requirement that only one bit of datum is allowed for the implementation of the power controller. More recent studies, motivated by the renewed interest in hybrid system with hard nonlinearities, include Δ-modulated feedback control systems and the associated complexities (Xia, Gai and Chen, 2004; Gai, Xia and Chen, 2003; Xia and Zinober, 2004; Xia, Chen, Gai and Zinober, 2004).

The link between sliding mode control and Δ-modulated control was first noted in (Zinober and Xia, 2004). This paper explores further the connection. We show that the zero-order-hold discretization of the equivalent control based sliding mode control system gives rise to Δ-modulation in the sliding mode direction. To illustrate vividly how sliding is achieved, we first present a detailed investigation of the global attracting properties of a scalar discrete-time system under Δ-modulated feedback. Global attractiveness of equivalent control based sliding mode control is then realized by the modulation in the sliding direction and followed by the absorption of the stable zero dynamics of the system. Another interesting result is that a ZOH discretization scheme of the equivalent control based sliding mode control system with relative degree one produces only two-periodic orbits, understandably due to sampling. The periodic orbits are regulated by the sampling period to move in the close vicinity of the sliding mode hyperplane.

The layout of the paper is as follows. In Section 2, we investigate the global attracting properties of a scalar discrete-time system under Δ-modulated feedback. The study of the discretization of the equivalent control based sliding mode control system is in Section 3.

2. DELTA-MODULATED CONTROL

In this section we study the periodic orbits of the following scalar, discrete-time linear system:

$$x^+ = ax - \Delta \text{sgn}\ (ax), \tag{1}$$

where $x \in R$ is the state variable, x^+ denotes the system state at the next discrete time, and a is a real number. Δ is a positive real number and sgn (x) is the function defined by

$$\text{sgn}\ (x) = \begin{cases} 1, & \text{when } x \geq 0; \\ -1, & \text{when } x < 0. \end{cases}$$

Here, we will only be concerned with the case $|a| \leq 1$. The existence of periodic points for the case $|a| > 1$ has been discussed in (Gai, Xia and Chen, 2003; Xia, Gai and Chen, 2004; Xia, Chen, Gai and Zinober, 2004).

The case $|a| \leq 1$ provides a relatively thorough investigation of the periodicity and its attractiveness. A detailed analysis gives a vivid illustration of the sliding mode control in the next section.

Theorem 1. 1) when $|a| = 1$, $\Omega = [-\Delta, \Delta]$ is a global attractor on $(-\infty, \infty)$;

2) when $|a| < 1$, the global attractor is the following set of two points:

$$\{-\Delta/(1+|a|), \Delta/(1+|a|)\}; \tag{2}$$

3) when $0 \leq a < 1$, the two points in (2) are 2-periodic points; when $-1 < a < 0$, the two points in (2) are (1-periodic) fixed points.

Proof: Denote the closed-loop system as

$$x^+ = ax - \Delta \text{sgn}\ (ax) \overset{def}{=} f_c(x). \tag{3}$$

In the following, we only give a proof for the case when $0 \leq a \leq 1$. A proof for the case $-1 \leq a < 0$ can be worked out similarly.

Our proof is divided into five parts.

(1) For $0 \leq a \leq 1$, $f_c(\Omega) \subset \Omega$.

This is readily verified by the definition of f_c.

(2) For $0 \leq a \leq 1$, Ω is globally attractive.

(a) If $x > \Delta$, then $x^+ = ax - \Delta \leq x - \Delta$. So $f_c^k(x)$ is decreasing as long as it is positive. We prove, by contradiction, that $\{f_c^k(x)\}$ enters Ω. Suppose this is not the case. Then there are only two situations: Case (i): $f_c^k(x) > \Delta$ for all k. Case (ii): There exists a positive integer l such that $x > f_c(x) > \cdots > f_c^l(x) > \Delta$, but $f_c^{l+1}(x) < -\Delta$.

In Case (i), since $\{f_c^k(x)\}$ is decreasing and bounded from below, we have $f_c^k(x) \to x^* \geq \Delta$ and therefore $x^* = ax^* - \Delta$, or, $(1-a)x^* = -\Delta$, which is impossible, since $(1-a)x^*$ is non-negative when $0 \leq a \leq 1$ and $x^* \geq \Delta$.

In Case (ii), by assumption we have $f_c^{l+1}(x) = af_c^l(x) - \Delta < -\Delta$, hence $af_c^l(x) < 0$, which is impossible since $a \geq 0$ and $f_c^l(x) > \Delta$.

(b) If $x \leq -\Delta$, then

$$x^+ = ax + \Delta \geq x + \Delta.$$

So $f_c^{k+1}(x) \geq f_c^k(x)$, if $f_c^k(x)$ is negative. Similarly, we can prove that $\{f_c^k(x)\}$ enters Ω.

(3) When $a = 1$, Ω is an attractor.

From Part 1 of the proof, $f_c(\Omega) \subset \Omega$. We need only prove that $\Omega \subset f_c(\Omega)$.

To see this, first note that $0 \in f_c(\Omega)$ since $f_c(\Delta) = 0$. For any $0 \neq y \in \Omega$, define $\bar{x} = y - \Delta \text{sgn}\ y$. Then, since $f_c(\Omega) \subset \Omega$, we have $\bar{x} \in \Omega$. Note that $\bar{x}$ and y have opposite signs (*e.g.*, if $y > 0$, then since $0 < y \leq \Delta$, we have $\bar{x} = y - \Delta < 0$), so

$$f_c(\bar{x}) = y.$$

From this last equation we have $f_c^2(y) = y$. We conclude that (when $a = 1$): any point in the half open interval $(-\Delta, \Delta]$ is a 2-periodic point.

It is also straightforward to verify that when $a = -1$: i) all points except for $\pm\Delta/2$ in the closed interval $[-\Delta, \Delta]$ are 2-periodic; ii) $\pm\Delta/2$ are fixed points.

(4) When $0 \le a < 1$, the attractor is $\{-\Delta/(1+|a|), \Delta/(1+|a|)\}$, which is a 2-periodic orbit.

From the above proof, the attractor, if it exists, belongs to $\Omega = [-\Delta, \Delta]$. It is therefore interesting to see how f_c evolves on Ω. Note that f_c transforms Ω into

$$f_c(\Omega) = [-\Delta, -(1-a)\Delta] \cup [(1-a)\Delta, \Delta],$$

therefore $(-(1-a)\Delta, (1-a)\Delta)$ is cut away, and it does not belong to the f_c-invariant set in Ω.

Generally, if we denote $f_c^k(\Omega) = [-a_k, -b_k] \cup [b_k, a_k]$ for $a_k \ge b_k > 0$, then we can derive the following iterative relations of a_k and b_k:

$$\begin{aligned} a_{k+1} &= \Delta - a b_k, \\ b_{k+1} &= \Delta - a a_k. \end{aligned} \tag{4}$$

We can then easily prove, by mathematical induction, that: 1) $a_{2i+1} = a_{2i}$, for $i = 0, 1, 2, \ldots$; 2) $\{a_k\}$ is a decreasing sequence.

Therefore, since $a_k \ge b_k > 0$, there exist $a^* \ge b^* > 0$ such that, as $k \to \infty$,

$$a_k \to a^*, \text{ and } b_k \to b^*.$$

We find that $a^* = b^* = \Delta/(1+a)$. In other words,

$$\bigcap_{k=1}^{\infty} f_c^k[-\Delta, \Delta] = \{-\Delta/(1+a), \Delta/(1+a)\},$$

which, hence, is the global attractor.

It can be easily verified that $\{-\Delta/(1+a), \Delta/(1+a)\}$ is the 2-periodic orbit of the closed-loop system.

(5) When $-1 < a < 0$, it can be similarly verified that $\{-\Delta/(1-a), \Delta/(1-a)\}$ is a global attractor, and these two points are (1-periodic) fixed points of the closed-loop system.

Since the periodic points are globally attractive, it is interesting to find out the attracting region for each of the periodic points.

First, we introduce a new concept. For any real number x and $a \ne 0$ (the case $a = 0$ is trivial), the characteristic index κ is defined as the following non-negative integer:

$$\kappa = \left\lfloor \log_{|a|} \left(\frac{\Delta}{\Delta + (1-|a|)|x|} \right) \right\rfloor,$$

where $\lfloor * \rfloor$ denotes the floor, *i.e.*, the maximal integer bounded by the real number $*$.

Lemma 1. i) For any x, the characteristic index κ is the smallest non-negative integer m such that

$$|f_c^{(m)}| < \frac{\Delta}{|a|}.$$

ii.1) For $-1 < a < 0$, κ is the smallest non-negative integer m such that $f_c^{(m)}$ and $f_c^{(m+1)}$ have the same sign;

ii.2) For $0 < a < 1$, κ is the smallest non-negative integer m such that $f_c^{(m)}$ and $f_c^{(m+1)}$ have opposite signs.

Proof: We prove the result only for the case $0 < a < 1$. Proof for other cases can be worked out along similar lines, and it is therefore omitted.

If $0 < a < 1$, it follows that

$$\begin{aligned} f_c^{(m+1)} &= a f_c^{(m)} - \text{sgn}\,(f_c^{(m)})\Delta \\ &= \begin{cases} a f_c^{(m)} - \Delta, & f_c^{(m)} \ge 0; \\ a f_c^{(m)} + \Delta, & f_c^{(m)} < 0. \end{cases} \end{aligned} \tag{5}$$

It is easy to see that $|f_c^{(m)}| < \Delta/a$ if and only if $f_c^{(m)}$ and $f_c^{(m+1)}$ have different signs.

Note that for $m \le \kappa$:

if $x > 0$, then we can calculate

$$f_c^{(m)}(x) = a^m|x| - \frac{(1-a^m)}{(1-a)}\Delta;$$

if $x \le 0$, then

$$f_c^{(m)}(x) = -a^m|x| + \frac{(1-a^m)}{(1-a)}\Delta.$$

It is then straightforward to verify that the real number $s = \log_a \frac{\Delta}{\Delta+(1-a)|x|}$ satisfies

$$a^s|x| - \frac{(1-a^s)}{(1-a)}\Delta = 0.$$

Therefore, $\kappa = \lfloor s \rfloor$ is the smallest integer such that $f_c^{(m)}$ changes sign.

This completes the proof of the lemma.

The analysis given in the proof can be useful in finding the limiting periodic points. For illustration, we will carry this out separately for $-1 < a < 0$. If $-1 < a < 0$, then we have

$$\begin{aligned} f_c^{(m+1)}(x) &= f_c(f_c^{(m)})(x) \\ &= a f_c^{(m)}(x) + \text{sgn}\,(f_c^{(m)}(x))\Delta. \end{aligned}$$

By ii.1) of Lemma 1, $f_c^{(m)}$ has the same sign as $f_c^{(\kappa)}$, for $m \ge \kappa$. Therefore, we have, for $m \ge \kappa$,

$$f_c^{(m+1)}(x) = a f_c^{(m)}(x) + \text{sgn}\,(f_c^{(\kappa)}(x))\Delta.$$

Hence, by denoting the limit of $f_c^{(m)}$ by x^*, we can solve x^* from $x^* = ax^* + \text{sgn}\,(f_c^{(\kappa)})\Delta$, to obtain $x^* = \frac{\text{sgn}\,(f_c^{(\kappa)})\Delta}{1-a}$.

Theorem 2. For any x, denote its characteristic index as κ.

i) For $-1 < a < 0$, x belongs to the attracting region of $\frac{\Delta}{1-a}$ $(-\frac{\Delta}{1-a})$ if and only if $\text{sgn}\,(x^{(\kappa)}) = 1$ $(\text{sgn}\,(x^{(\kappa)}) = -1)$.

ii) For $0 \le a < 1$, x belongs to the attracting region of $\frac{\Delta}{1+a}$ $(-\frac{\Delta}{1+a})$ if and only if $\text{sgn}\,(x^{(\kappa)}) = (-1)^\kappa$ $(\text{sgn}\,(x^{(\kappa)}) = (-1)^{\kappa+1})$.

3. APPLICATION TO DISCRETIZED SMC SYSTEMS

For a continuous time system

$$\dot{x} = Ax + bu, \tag{6}$$

where $x \in R^n$, A is an $n \times n$ matrix, and b is an n-dimensional vector. A basic Sliding Mode Control (SMC) design (Zinober, 1994) is to seek a sliding mode defined by $s = c^T x$, where c is an n-dimensional vector, such that $c^T x$ has relative degree 1 w.r.t. system (6), *i.e.*,

$$c^T b \neq 0. \tag{7}$$

In this case, a sliding mode controller is obtained:

$$u = -\alpha c^T x - \frac{1}{c^T b} c^T Ax - \frac{\beta}{c^T b}\text{sgn}\,(c^T x), \tag{8}$$

in which $\alpha \ge 0$ and $\beta > 0$ are tuning parameters. There are three parts in the controller

$$u_r = -\alpha c^T x, u_{eq} = -\frac{1}{c^T b} c^T Ax,$$
$$u_s = -\frac{\beta}{c^T b}\text{sgn}\,(c^T x).$$

The *equivalent control* u_{eq} (Zinober, 1994) is derived by solving $\dot{s} = 0$, where $\dot{s} = c^T(Ax + bu)$ is the derivative of s along the dynamics of (6). The *switching control* u_s is designed to satisfy the sliding condition

$$s\dot{s} \le 0.$$

The *reaching control* u_r adds some reaching manipulability to avoid the chattering problem (Gao and Hung, 1993).

The SMC design is applicable to system (6) when it is *minimal phase*, with $c^T x$ as an output (Byrnes and Isidori, 1988).

To study the discretization effects on the sliding mode controller, we assume that the controller u is *digitized* through a zero-order holder (ZOH) at the sampling moments:

$$\begin{aligned} u(t) = u_k &\overset{def}{=} u(kh) \\ &= -\alpha c^T x(kh) - \frac{1}{c^T b} c^T Ax(kh) \\ &\quad -\frac{\beta}{c^T b}\text{sgn}\,(c^T x(kh)) \\ &\triangleq -\alpha c^T x(k) - \frac{1}{c^T b} c^T Ax(k) \\ &\quad -\frac{\beta}{c^T b}\text{sgn}\,(c^T x(k)), \end{aligned} \tag{9}$$

for all $t \in [kh, (k+1)h)$, in which $h > 0$ is the sampling period. A discrete-time conversion of the system (6) under ZOH is obtained:

$$x(k+1) = e^{Ah}x(k) + \int_0^h d^{A\tau}\text{d}\,\tau u_k, \tag{10}$$

where u_k is given in (9).

To reveal the special structure of discretization of the system, let us first make the coordinate transformation on the original (closed-loop) system (6) under feedback (8): $z_1 = c^T x$, and choose $c_2, c_3, \ldots, c_n \in R^n$ satisfying $c_i^T b = 0$, for $i = 2, 3, \ldots, n$, and $\{c, c_2, c_3, \ldots, c_n\}$ is a linearly independent set. This is always possible due to (7).

Hence, let $z_i = c_i^T x$, for $i = 2, 3, \ldots, n$. It is easily seen that system (6) under SMC is written in the new coordinates as

$$\dot{z}_1 = -\alpha z_1 - \beta\text{sgn}\,(z_1),$$
$$\dot{\tilde{z}} = \Psi\tilde{z} + pz_1,$$

in which we denote $\tilde{z} = (z_2, z_3, \ldots, z_n)^T$, $\Psi \in R^{(n-1)\times(n-1)}$ is a stable matrix, due to the assumption that the system is minimal phase, and $p \in R^{n-1}$.

Applying a zero-order hold discretization to the system in coordinates z, we obtain

$$z_1^+ = \kappa z_1 - \Delta\text{sgn}\,(z_1), \tag{11}$$
$$\tilde{z}^+ = \Phi\tilde{z} + \gamma z_1, \tag{12}$$

in which

$$\kappa = e^{-\alpha h}$$
$$\Delta = \begin{cases} \beta(1 - e^{-\alpha h})/\alpha, & \text{when } \alpha \neq 0, \\ \beta h & \text{when } \alpha = 0, \end{cases}$$
$$\Phi = e^{\Psi h},$$
$$\gamma = (-\alpha h I_{n-1} - \Psi h)^{-1}(e^{-\alpha h}I_{n-1} - e^{\Psi h})p,$$

and $h > 0$ is the sampling period. These equations are readily derived by applying the formulae in (10). As a matter of fact, the system (11-12) is the transformed version of (10) under the same coordinate transformation.

We note that the dynamics of z_1 is decoupled from that of $\tilde{z}$. It is exactly in a form that has been considered in the previous section. Since $0 < \kappa \leq 1$, we know from Theorem 1 that when $0 < \kappa < 1$, $\{\pm\Delta/(1+\kappa)\}$ is the only (2-) periodic orbit, and it is globally attracting; when $\kappa = 1$, every point in $(-\Delta, \Delta]$ is 2-periodic, any point is attracted to one pair of these 2-periodic points.

Consider a discrete-time system of order n,

$$x^+ = Ax + bu, \tag{13}$$

where $x \in R^n$ is the state, x^+ denotes the system state at the next discrete time step, $u \in R$ is the scalar input, A is an $n \times n$ matrix of real numbers, and b is a column vector of n real numbers. If A is a stable matrix, *i.e.*, the eigenvalues of A lie within the unit circle.

An input sequence $\{u_i, i = 0, 1, 2, \ldots\}$ is called asymptotically L-periodic, if there are L real numbers $\{u_0^*, u_1^*, \ldots, u_{L-1}^*\}$ such that $\lim_{i\to\infty} u_i = u^*_{(i \bmod L)}$.

Theorem 3. Consider the discrete-time system (13) with a stable A matrix.

(i) For an asymptotically L-periodic input sequence, there is a periodic orbit of period L for system (13).

(ii) This periodic orbit is globally attracting.

Proof: It is easily verified that there is a periodic orbit of period L for the system (13) corresponding to an L-periodic input sequence $\{u_0^*, u_1^*, \ldots, u_{L-1}^*\}$, and it is easily verified that this periodic orbit starts at

$$x^* = (I - A^L)^{-1}(A^{L-1}bu_0^* + \cdots + bu_{L-1}^*).$$

Now we show that this periodic orbit is globally attracting. We need only show that any other orbit $\{y^{(i)}\}$ corresponding to the asymptotically L-periodic input sequence $\{u_i, i = 0, 1, 2, \ldots,\}$ satisfies $y^{(kL)} \to x^*$.

To this end, by definition, we have

$$y^{(L)} = A^L y^{(0)} + A^{L-1}bu_0 + \cdots + bu_{L-1},$$

and iteratively, for $k = 1, 2, 3, \ldots$,

$$\begin{aligned} y^{((k+1)L)} &= A^L y^{(kL)} + A^{L-1}bu_{kL} \\ &\quad + \cdots + bu_{(k+1)L-1}. \end{aligned} \tag{14}$$

We prove that $\{y^{(kL)}\}$ is a Cauchy series. For any $\epsilon > 0$, because $\{u_i, i = 0, 1, 2, \ldots\}$ is an asymptotically L-periodic input sequence, there exists an integer sL such that when $k \geq sL$,

$$|u_{k+L} - u_k| \leq \frac{(1 - ||A||)(1 - ||A||^L)^2}{2||b||}\epsilon, \tag{15}$$

in which we use any norm such that $||A|| < 1$ (which is possible by the stability assumption on A). From (14), we have

$$\begin{aligned} ||y^{((k+1)L)} - y^{(kL)}|| &\leq ||A^L||||y^{(kL)} - y^{((k-1)L)}|| \\ &+ ||A||^{L-1}||b|||u_{kL} - u_{(k-1)L}| \\ &+ \cdots + ||b|||u_{(k+1)L-1} - u_{kL-1}|. \end{aligned}$$

Hence from (15), when $k > s$, we derive

$$\begin{aligned} &||y^{((k+1)L)} - y^{(kL)}|| \leq ||A^L||||y^{(kL)} - y^{((k-1)L)}|| \\ &< ||A^L||||y^{(kL)} - y^{((k-1)L)}|| + \frac{(1 - ||A||^L)^2}{2||b||}\epsilon \end{aligned} \tag{16}$$

Choose an integer $\bar{s} > s$ such that

$$||A^L||^{\bar{s}-s}||y^{(sL)} - y^{((s-1)L)}|| < \frac{(1 - ||A||^L)}{2}\epsilon,$$

then by repeatedly application of (16), we have

$$||y^{(\bar{s}L)} - y^{((\bar{s}-1)L)}|| < (1 - ||A||^L)\epsilon. \tag{17}$$

For any integer $t > 0$, and again by repeatedly application of (16) and (17), we have

$$\begin{aligned} ||y^{((\bar{s}+t)L)} - y^{((\bar{s}-1)L)}|| &\leq ||y^{((\bar{s}+t)L)} - y^{((\bar{s}+t-1)L)}|| \\ &+ \cdots + ||y^{(\bar{s}L)} - y^{((\bar{s}-1)L)}|| \\ &< (||A^L||^t + \cdots + 1)(1 - ||A||^L)\epsilon < \epsilon. \end{aligned}$$

Therefore, $y^{(kL)}$ is converging, and denoting $y^* = \lim_{k\to\infty} y^{(kL)}$, then from (14), we have

$$\begin{aligned} y^* &= A^L y^* + A^{L-1}b \lim_{k\to\infty} u_{kL} + A^{L-2}b \lim_{k\to\infty} u_{kL+1} \\ &\quad + \cdots + b \lim_{k\to\infty} u_{(k+1)L-1} \\ &= A^L y^* + A^{L-1}bu_0^* + A^{L-2}bu_1^* + \cdots + bu_{L-1}^*. \end{aligned}$$

From this last equation we obtain

$$\begin{aligned} y^* &= (I - A^L)^{-1}(A^{L-1}bu_0^* + A^{L-2}bu_1^* \\ &\quad + \cdots + bu_{L-1}^*) = x^*. \end{aligned}$$

To find the periodic points of (11) and (12), first of all we note that z_1 can only be 2-periodic. Therefore, z_1 in (12) can be regarded as a 2-periodic (modulated) orbit, in order to find the periodic orbit for the overall system. Since z_1 is asymptotically 2-periodic and Φ is stable, we can apply Theorem 3 to conclude that there is/are only 2-periodic orbit(s) arising from discretization of SMC. These conclusions are summarized in the following theorem.

Theorem 4. Discretization of SMC results in only 2-periodic orbits. When $\alpha > 0$, there is a unique 2-periodic orbit determined by (in z coordinates): $\{P, -P\}$, in which

$$P = \Delta/(1+\kappa) \begin{bmatrix} -1 \\ (I_{n-1} + \Phi)^{-1}\gamma \end{bmatrix}. \quad (18)$$

This pair is globally attracting.

When $\alpha = 0$, each pair of points of the following form (in z coordinates)

$$\left\{ \varphi \begin{bmatrix} -1 \\ (I_{n-1} + \Phi)^{-1}\gamma \end{bmatrix}, -\varphi \begin{bmatrix} -1 \\ (I_{n-1} + \Phi)^{-1}\gamma \end{bmatrix} \right\}$$

for $\varphi \in [-\Delta, \Delta)$, is a 2-periodic orbit.

Proof: When $\alpha \neq 0$, by Theorem 1, z_1 is globally attracted to two periodic points $\pm\Delta/(1+\kappa)$. It is easily verified that the two points in (18) are the only 2-periodic points for the system. The $\tilde{z}$ part in (12) has exactly the structure discussed in Theorem 3, therefore these two points are globally attracting.

Similar arguments apply to the case when $\alpha = 0$.

This result clearly indicates how sliding mode is achieved: first of all the system is dragged towards the sliding mode (z_1) by a Δ-modulation mechanism, then it is *absorbed* by its stable zero dynamics (the matrix Φ is stable).

From the expressions given in the theorem, the two periodic points are on two different sides of the sliding mode hyperplane (defined by $z_1 = 0$). It is also noticed that the component-wise distance of the two periodic points is ordered at $O(\Delta)$. And from the expression of Δ following (12), in both situations $\alpha = 0$ and $\alpha \neq 0$, $\Delta \sim 2h$. So eventually, we conclude that the distance of any two corresponding components of the two periodic points is ordered at $O(h)$.

Hence, the chattering of SMC still exists in its ZOH sampling implementation, but it is regulated by the sampling period h. When h is very small, chattering becomes "invisible".

Note that a crucial assumption in arriving at Theorem 4 is that system (1) has relative degree 1 taking the sliding surface as an output. If this is not the case, much more complicated situations can arise as illustrated in (Yu, 1998).

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