SET-MEMBERSHIP NONLINEAR FILTERING WITH SECOND-ORDER INFORMATION

Giuseppe Calafiore [*,1] **Basilio Bona** [*]

[*] *Dipartimento di Automatica e Informatica*
Politecnico di Torino
Corso Duca degli Abruzzi, 24 – 10129 Torino, Italy.
`giuseppe.calafiore, basilio.bona@polito.it`

Abstract: In this paper, we develop a numerically efficient scheme for set-membership prediction and filtering for discrete-time nonlinear systems, that takes into explicit account the effects of nonlinearities via local second-order information. The filtering scheme is based on a classical prediction/update recursion that requires at each step the solution of a convex semidefinite optimization problem. The technical results discussed in the paper build upon the recently developed paradigm of uncertain linear equations (ULE) and semidefinite relaxations.

Keywords: Set-membership filtering, Ellipsoidal bounds, Nonlinear filters, Semidefinite programming.

1. INTRODUCTION

Deterministic counterparts of discrete-time Kalman filters for linear systems are known in the literature since the early works of Bertsekas and Rhodes in the seventies. The deterministic filter introduced in (Bertsekas and Rhodes, 1971) provided a state estimate in the form of an ellipsoidal set of all possible states that are consistent with the given measurements and a deterministic additive description of the process noise. Further early contributions in this field were due to (Schweppe, 1973), whose ideas were later developed by (Chernousko, 1993; Durieu *et al.*, 2001; Kurzhanski and Valyi, 1996; Maskarov and Norton, 1996), among many others. However, all these contributions deal essentially with the *linear* filtering problem, or with a nonlinear version of it which is based on local linearization, in the same spirit of the so-called Extended Kalman Filter (EKF).

It is well known (see for instance the discussion in (Einicke and White, 1999)) that for nonlinear systems an approach completely based on local linearization, such as the EKF, may lead to poor estimation performances. In this paper, we develop an efficient computational technique for set-membership nonlinear filtering which takes into explicit account the second order local information of the system. This additional structure is expected to improve the filter performance in all applications where strong nonlinearities are present, such as in localization problems arising in mobile robotics, see for instance (Fabrizi *et al.*, 1998; Jetto *et al.*, 1999) and (Di Marco *et al.*, 2004).

The filter recursions developed here are built upon the results on uncertain linear equations (ULE) that appeared recently in (Calafiore and El Ghaoui, 2004). According to this theory, the numerical complexity of each filter step is of the order of $n^{3.5}$, and it is thus comparable to the complexity of the standard Kalman filter iterates.

In the next sections, we develop all the technical details of the second-order filter recursions, whereas we leave the numerical experiments to a subsequent work. For space reasons, the presentation is kept "dry" and rather technical. The paper

[1] This work has been supported by FIRB and PRIN funds from the Italian Ministry of University and Research.

is organized as follows: Section 2 contains some preliminaries on the representation of quadratic functions, and reports a key result on uncertain linear equations; Section 3 sets up the filtering problem, and Sections 4 and 5 provide the key results on the prediction and update steps of the filter, respectively. Section 6 finally draws some conclusions.

2. PRELIMINARIES

2.1 Quadratic functions in linear fractional form

Consider a quadratic function of the form

$$\begin{aligned}&\varphi(\delta_x,\delta_u,\delta_\nu)=\\&\quad\bar{\varphi}+A_x\delta_x+A_u\delta_u+h(\delta_x,\delta_u)+A_\nu\delta_\nu\end{aligned}$$

where $\delta_x\in\mathbb{R}^{n_x}$, $\delta_u\in\mathbb{R}^{n_u}$, $\delta_\nu\in\mathbb{R}^{n_\nu}$, and the i-th component of $h(\delta_x,\delta_u)$ is

$$h_i(\delta_x,\delta_u)\doteq\begin{bmatrix}\delta_x\\\delta_u\end{bmatrix}^T\begin{bmatrix}H_{xx}^{(i)}&H_{xu}^{(i)}\\H_{ux}^{(i)}&H_{uu}^{(i)}\end{bmatrix}\begin{bmatrix}\delta_x\\\delta_u\end{bmatrix}$$

and where the inner matrix in the above expression is symmetric. When useful for the sake of compactness of the formulas, we shall use the notation $XY\sharp$ to denote the product XYX^T. The previous expression is written equivalently as

$$\begin{aligned}&h_i(\delta_x,\delta_u)=\\&\quad{\delta_x}^T(H_{xx}^{(i)}\delta_x+H_{xu}^{(i)}\delta_u)+{\delta_u}^T(H_{xu}^{(i)T}\delta_x+H_{uu}^{(i)}\delta_u)\end{aligned}$$

and hence

$$\begin{aligned}&h(\delta_x,\delta_u)=\begin{bmatrix}h_1(\delta_x,\delta_u)\\\vdots\\h_n(\delta_x,\delta_u)\end{bmatrix}=\\&=\mathrm{diag}\,({\delta_x}^T,\ldots,{\delta_x}^T)\left(\begin{bmatrix}H_{xx}^{(1)}\\\vdots\\H_{xx}^{(n_x)}\end{bmatrix}\delta_x+\begin{bmatrix}H_{xu}^{(1)}\\\vdots\\H_{xu}^{(n_x)}\end{bmatrix}\delta_u\right)\\&+\mathrm{diag}\,({\delta_u}^T,\ldots,{\delta_u}^T)\left(\begin{bmatrix}H_{xu}^{(1)T}\\\vdots\\H_{xu}^{(n_x)T}\end{bmatrix}\delta_x+\begin{bmatrix}H_{uu}^{(1)}\\\vdots\\H_{uu}^{(n_x)}\end{bmatrix}\delta_u\right)\\&=\Delta_x(H_{xx}\delta_x+H_{xu}\delta_u)+\Delta_u(H_{ux}\delta_x+H_{uu}\delta_u)\end{aligned}$$

where we defined

$$\begin{aligned}\Delta_x&\doteq\mathrm{diag}\,({\delta_x}^T,\ldots,{\delta_x}^T)\in\mathbb{R}^{n_x,n_x^2}\\\Delta_u&\doteq\mathrm{diag}\,({\delta_u}^T,\ldots,{\delta_u}^T)\in\mathbb{R}^{n_x,n_xn_u}\end{aligned}$$

$$H_{xx}\doteq\begin{bmatrix}H_{xx}^{(1)}\\\vdots\\H_{xx}^{(n_x)}\end{bmatrix}\in\mathbb{R}^{n_x^2,n_x};\;H_{xu}\doteq\begin{bmatrix}H_{xu}^{(1)}\\\vdots\\H_{xu}^{(n_x)}\end{bmatrix};$$

$$H_{ux}\doteq\begin{bmatrix}H_{xu}^{(1)T}\\\vdots\\H_{xu}^{(n_x)T}\end{bmatrix}\in\mathbb{R}^{n_xn_u,n_x};\;H_{uu}\doteq\begin{bmatrix}H_{uu}^{(1)}\\\vdots\\H_{uu}^{(n_x)}\end{bmatrix}$$

with $H_{xu}\in\mathbb{R}^{n_x^2,n_u}$, $H_{uu}\in\mathbb{R}^{n_xn_u,n_u}$. Summarizing, we have

$$\begin{aligned}&\varphi(\delta_x,\delta_u,\delta_\nu)=\bar{\varphi}+A_x\delta_x+A_u\delta_u+A_\nu\delta_\nu\\&\quad+\Delta_x(H_{xx}\delta_x+H_{xu}\delta_u)+\Delta_u(H_{ux}\delta_x+H_{uu}\delta_u).\end{aligned}\tag{1}$$

The following lemma holds.

Lemma 1. (LFT representation). The function φ $(\delta_x,\delta_u,\delta_\nu)$ is represented by the feedback connection

$$\begin{bmatrix}\varphi\\\hline z\end{bmatrix}=\left[\begin{array}{c|ccccc}\bar{\varphi}&A_x&A_u&I_{n_x}&I_{n_x}&A_\nu\\\hline 1&0&0&0&0&0\\1&0&0&0&0&0\\0&H_{xx}&H_{xu}&0&0&0\\0&H_{ux}&H_{uu}&0&0&0\\1&0&0&0&0&0\end{array}\right]\begin{bmatrix}1\\\hline w\end{bmatrix}\tag{2}$$

$$w=\Delta z\tag{3}$$

where

$$\Delta\doteq\mathrm{diag}\,(\delta_x,\delta_u,\Delta_x,\Delta_u,\delta_\nu).\tag{4}$$

Equivalently,

$$\varphi(\delta_x,\delta_u,\delta_\nu)=\bar{\varphi}+L\Delta(I-H\Delta)^{-1}R\tag{5}$$

where $\left[\begin{array}{c|c}\bar{\varphi}&L\\\hline R&H\end{array}\right]$ can be deduced from the partition in (2). Notice that $(I-H\Delta)$ is always invertible, i.e. the LFT is always well-posed.

Proof. The representation is easily constructed as follows. First, write (1) as

$$\varphi=\bar{\varphi}+A_xw_1+A_uw_2+w_3+w_4+A_\nu w_5$$

where we defined $w_1=\delta_xz_1$, $w_2=\delta_uz_2$, $w_3=\Delta_xz_3$, $w_4=\Delta_uz_4$, $w_4=\delta_\nu z_5$; with the z_i's given by $z_1=1$, $z_2=1$, $z_3=H_{xx}w_1+H_{xu}w_2$, $z_4=H_{ux}w_1+H_{uu}w_2$, $z_5=1$. Then, (2), (3) are simply a restatement in vector format of the above positions. Finally, substituting (3) into (2), we have that $\varphi=\bar{\varphi}+L\Delta z$, and $z=R+H\Delta z$. From this latter expression we get $z=(I-H\Delta)^{-1}R$, which, substituted in the former, yields (5). □

Definition 1. (Scaling subspace). Let $\boldsymbol{\Delta}$ denote the subspace of matrices having block-diagonal structure. (e.g. the structure in (4)). The scaling subspace associated to $\boldsymbol{\Delta}$ is defined as

$$\mathcal{S}(\boldsymbol{\Delta})=\{(S,T):\forall\Delta\in\boldsymbol{\Delta},S\Delta=T\Delta\}.$$

It is easy to verify that the scaling subspace for the structure specified by (4) is constituted of matrix pairs (S,T) of the form

$$\begin{aligned}S&=\mathrm{diag}\,(\lambda_1I_{n_x},\lambda_2I_{n_u},S_x,S_u,\lambda_3I_{n_\nu})\\T&=\mathrm{diag}\,(\lambda_1,\lambda_2,S_x\otimes I_{n_x},S_u\otimes I_{n_u},\lambda_3)\end{aligned}$$

where $S_x = S_x^T \in \mathbb{R}^{n_x,n_x}$, $S_u = S_u^T \in \mathbb{R}^{n_u,n_u}$, and $\otimes$ denotes the Kronecker tensor product. [2]

2.2 Uncertain Linear Equations

Let $\mathcal{A} \in \mathbb{R}^{m,n}$, $\mathcal{Y} \in \mathbb{R}^{m}$, $L \in \mathbb{R}^{m,n_p}$, $R_\mathcal{A} \in \mathbb{R}^{n_q,n}$, $R_\mathcal{Y} \in \mathbb{R}^{n_q}$, $H \in \mathbb{R}^{n_q,n_p}$, and define

$$[\mathcal{A}(\Delta)\ \mathcal{Y}(\Delta)] = [\mathcal{A}\ \mathcal{Y}] + L\Delta(I - H\Delta)^{-1}[R_\mathcal{A}\ R_\mathcal{Y}], \tag{6}$$

where $\Delta \in \mathbf{\Delta}_1$, with

$$\mathbf{\Delta}_1 \doteq \{\Delta \in \mathbf{\Delta} \subseteq \mathbb{R}^{n_p,n_q} : \|\Delta\| \le 1\}.$$

Let further this linear fractional representation (LFR) be well-posed over $\mathbf{\Delta}_1$, meaning that $\det(I - H\Delta) \neq 0$, $\forall \Delta \in \mathbf{\Delta}_1$.

Given the data description in (6), we have the so-called uncertain linear equations (ULE) in the variable x

$$\mathcal{A}(\Delta)x = \mathcal{Y}(\Delta).$$

The set $\mathcal{X}$ defined below represent the set of all possible solutions (if any) to the above ULE

$$\mathcal{X} \doteq \{x :\ \mathcal{A}(\Delta)x = \mathcal{Y}(\Delta), \text{ for some } \Delta \in \mathbf{\Delta}_1\}.$$

To the uncertainty structure described by $\mathbf{\Delta}$, we associate a scaling pair $(S, T) \in \mathcal{S}(\mathbf{\Delta})$, as in Definition 1.

The following theorem provides conditions under which the set $\mathcal{X}$ is contained in a bounded ellipsoid $\mathcal{E}$, and exploits these conditions to determine a minimal (in the sense of the trace size measure) ellipsoid containing the solution set $\mathcal{X}$.

Theorem 1. (Calafiore and El Ghaoui, 2004). Let

$$\Psi \doteq [\mathcal{A}\ \ L\ \ y], \quad \Upsilon \doteq \begin{bmatrix} R_\mathcal{A} & H & R_\mathcal{Y} \\ 0_{n_p,n} & I_{n_p} & 0_{n_p,1} \end{bmatrix},$$

$$\Omega(S,T) \doteq \Upsilon^T \begin{bmatrix} T & 0 \\ 0 & -S \end{bmatrix} \Upsilon.$$

Let further the orthogonal complement $\Psi_\perp$ be chosen as $\Psi_\perp \doteq \left[\begin{array}{c|c} \Psi_{\perp 1} & \psi_{\perp 2} \\ \hline 0\cdots 0 & -1 \end{array}\right]$, where $\Psi_{\perp 1}$ is an orthogonal complement of $[\mathcal{A}\ \ L]$, and $\psi_{\perp 2}$ is any vector such that $[\mathcal{A}\ \ L]\psi_{\perp 2} = \mathcal{Y}$. (If no such $\psi_{\perp 2}$ exists, then the solution set is empty). If there exist $(S, T) \in \mathcal{S}(\mathbf{\Delta})$, $S \succeq 0$, and $P = P^T$, $\hat{x}$ such that the LMI

$$\left[\begin{array}{c|c} P & [I\ 0\ \hat{x}]\Psi_\perp \\ \hline \Psi_\perp^T [I\ 0\ \hat{x}]^T & \Psi_\perp^T\left(\mathrm{diag}\,(0,0,1) - \Omega(S,T)\right)\Psi_\perp \end{array}\right] \succeq 0 \tag{7}$$

is feasible, then the ellipsoid $\mathcal{E}(P, \hat{x})$ contains the solution set $\mathcal{X}$.

[2] For instance, if $n_x = 2$, we have $S_x = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}$, and $S_x \otimes I_{n_x}$ is equal to $\begin{bmatrix} s_{11}I_2 & s_{12}I_2 \\ s_{12}I_2 & s_{22}I_2 \end{bmatrix}$.

A minimal size ellipsoid can hence be determined by minimizing the trace of P subject to the LMI condition (7).

3. PROBLEM SETUP

Consider a nonlinear discrete-time dynamic system described by the recursive state equations

$$x(k+1) = f(x(k), u(k))$$

where $x(k) \in \mathbb{R}^{n_x}$ denotes the system state at time k, $u(k) \in \mathbb{R}^{n_u}$ is the input vector at time k, and f is twice differentiable.

We assume that at time k it is known that the state $x(k)$ belongs to a given ellipsoid $\mathcal{E}_x(k)$ of center $\bar{x}(k)$ and shape matrix $E(k)$, i.e.

$$x(k) = \bar{x}(k) + E_x(k)\delta_x(k),$$

for some vector $\delta_x(k)$ such that $\|\delta_x(k)\| \le 1$. Analogously, we assume that the possible inputs at time k lie in an ellipsoid $\mathcal{E}_u(k)$

$$u(k) = \bar{u}(k) + E_u(k)\delta_u(k),$$

for some vector $\delta_u(k)$ such that $\|\delta_u(k)\| \le 1$. Next, we expand function f in series around the nominal pair $\bar{u}(k), \bar{x}(k)$. The series expansion is computed up to the second order terms, while higher order contributions and process noise are taken into account via an additional disturbance term:

$$x(k+1) = \bar{x}(k+1) + A(k)\delta_x(k) + B_u(k)\delta_u(k) + h(\delta_x(k), \delta_u(k)) + B_\nu \nu(k)$$

where $\nu(k) \in \mathbb{R}^{n_\nu}$ is the additional process disturbance term, which is also assumed to be unknown but bounded as $\|\nu(k)\| \le 1$, $\bar{x}(k+1) \doteq f(\bar{x}(k), \bar{u}(k))$, and $A(k), B_u(k)$ are related to the Jacobians of f $A(k) \doteq \mathcal{J}_x(k)E_x(k)$, $B_u(k) \doteq \mathcal{J}_u(k)E_u(k)$, being

$$[\mathcal{J}_x(k)]_{i,j} \doteq \left.\frac{\partial f_i}{\partial x_j}\right|_{x=\bar{x}(k), u=\bar{u}(k)}, \quad i,j = 1,\ldots,n_x$$

$$[\mathcal{J}_u(k)]_{i,j} \doteq \left.\frac{\partial f_i}{\partial u_j}\right|_{x=\bar{x}(k), u=\bar{u}(k)}, \quad \begin{array}{c} i = 1,\ldots,n_x; \\ j = 1,\ldots,n_u. \end{array}$$

The second order terms are collected in vector $h(\delta_x(k), \delta_u(k))$, whose i-th component is

$$h_i(\delta_x(k), \delta_u(k)) = \begin{bmatrix} \delta_x(k) \\ \delta_u(k) \end{bmatrix}^T \begin{bmatrix} H_{xx}^{(i)}(k) & H_{xu}^{(i)}(k) \\ H_{xu}^{(i)T}(k) & H_{uu}^{(i)}(k) \end{bmatrix} \begin{bmatrix} \delta_x(k) \\ \delta_u(k) \end{bmatrix}$$

where the inner matrix above is related to the Hessians of f

$$\begin{bmatrix} H_{xx}^{(i)}(k) & H_{xu}^{(i)}(k) \\ H_{xu}^{(i)T}(k) & H_{uu}^{(i)}(k) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} E_x^T(k) & 0 \\ 0 & E_u^T(k) \end{bmatrix} \begin{bmatrix} \mathcal{H}_{xx}^{(i)}(k) & \mathcal{H}_{xu}^{(i)}(k) \\ \mathcal{H}_{xu}^{(i)T}(k) & \mathcal{H}_{uu}^{(i)}(k) \end{bmatrix} \sharp$$

being

$$\left[\mathcal{H}_{xx}^{(i)}(k)\right]_{\ell,j} = \left.\frac{\partial^2 f_i}{\partial x_\ell \partial x_j}\right|_{x=\bar{x}(k),u=\bar{u}(k)}, \quad \ell, j = 1, \dots, n_x$$

$$\left[\mathcal{H}_{xu}^{(i)}(k)\right]_{\ell,j} = \left.\frac{\partial^2 f_i}{\partial x_\ell \partial u_j}\right|_{x=\bar{x}(k),u=\bar{u}(k)}, \quad \begin{array}{l}\ell = 1, \dots, n_x; \\ j = 1, \dots, n_u\end{array}$$

$$\left[\mathcal{H}_{uu}^{(i)}(k)\right]_{\ell,j} = \left.\frac{\partial^2 f_i}{\partial u_\ell \partial u_j}\right|_{x=\bar{x}(k),u=\bar{u}(k)}, \quad \ell, j = 1, \dots, n_u.$$

Since we shall be concerned with one-step-ahead prediction and filtering, in the sequel we drop the explicit dependence on k, and denote $x(k)$ with x, $x(k+1)$ with x_+, etc. The state recursion is hence described compactly as

$$x_+ = \bar{x}_+ + A\delta_x + B_u\delta_u + h(\delta_x, \delta_u) + B_\nu \nu \quad (8)$$

where $\|\delta_x\| \leq 1$, $\|\delta_u\| \leq 1$, $\|\nu\| \leq 1$.

4. PREDICTION STEP

The prediction step consists in determining a "predicted" ellipsoid $\mathcal{E}_+$ of center $\hat{x}_+$ and shape matrix E_+ (or squared shape matrix $P_+ = E_+E_+^T$) that contains all the possible states x_+ consistent with (8).

To this end, we proceed in two steps. First, we express (8) in LFT form, using Lemma 1

$$x_+ = \bar{x}_+ + L\Delta(I - H\Delta)^{-1}R$$

where $\Delta \in \mathbf{\Delta}_1$, being $\mathbf{\Delta}_1 \doteq \{\Delta \in \mathbf{\Delta} : \|\Delta\| \leq 1\}$, and $\mathbf{\Delta}$ the structure subspace

$$\mathbf{\Delta} = \{\Delta : \Delta = \mathrm{diag}\,(\delta_x, \delta_u, \Delta_x, \Delta_u, \delta_\nu) \in \mathbb{R}^{n_w, n_z}\}$$

with $\Delta_x = \mathrm{diag}\,({\delta_x}^T, \dots, {\delta_x}^T) \in \mathbb{R}^{n_x, n_x^2}$, $\Delta_u = \mathrm{diag}\,({\delta_u}^T, \dots, {\delta_u}^T) \in \mathbb{R}^{n_x, n_x n_u}$, and $n_w \doteq 3n_x + n_u + n_\nu$, $n_z \doteq 3 + n_x^2 + n_x n_u$. The LFT data is

$$L = \left[\, A \;\; B_u \;\; I_{n_x} \;\; I_{n_x} \;\; B_\nu \,\right] \in \mathbb{R}^{n_x, n_w}$$

$$R = \begin{bmatrix} 1 \\ 1 \\ 0_{n_x^2,1} \\ 0_{n_x n_u,1} \\ 1 \end{bmatrix} \in \mathbb{R}^{n_z,1}$$

$$H = \begin{bmatrix} 0_{1,n_x} & 0_{1,n_u} & 0_{1,n_x} & 0_{1,n_x} & 0_{1,n_\nu} \\ 0_{1,n_x} & 0_{1,n_u} & 0_{1,n_x} & 0_{1,n_x} & 0_{1,n_\nu} \\ H_{xx} & H_{xu} & 0_{n_x^2,n_x} & 0_{n_x^2,n_x} & 0_{n_x^2,n_\nu} \\ H_{ux} & H_{uu} & 0_{n_x n_u,n_x} & 0_{n_x n_u,n_x} & 0_{n_x n_u,n_\nu} \\ 0_{1,n_x} & 0_{1,n_u} & 0_{1,n_x} & 0_{1,n_x} & 0_{1,n_\nu} \end{bmatrix}$$

with $H \in \mathbb{R}^{n_z,n_w}$. Then, we observe that the set of possible one-step reachable states x_+ coincides with the set of solutions of the uncertain linear equations (in the variable x_+)

$$I_{n_x} x_+ = \bar{x}_+ + L\Delta(I - H\Delta)^{-1}R.$$

The next lemma provides our key result for computing a minimal ellipsoid containing the set of one-step reachable states x_+.

Lemma 2. (Prediction step). If there exist $(S, T) \in \mathcal{S}(\mathbf{\Delta})$, $S \succeq 0$, and $P_+ = P_+^T$, $\hat{x}_+$ such that the LMI

$$\left[\begin{array}{c|cc} P_+ & -L & \bar{x}_+ - \hat{x}_+ \\ \hline * & S - H^T T H & H^T T R \\ * & * & R^T T R - 1 \end{array}\right] \succeq 0.$$

is feasible, then the ellipsoid $\mathcal{E}_+(P_+, \hat{x}_+)$ contains all the possible states x_+.

A minimal size ellipsoid can hence be determined by minimizing the trace of P_+ subject to the above LMI condition.

Proof. The set of reachable states coincides with the solution set of the ULE $\mathcal{A}(\Delta)x_+ = \mathcal{Y}(\Delta)$, with

$$[\mathcal{A}(\Delta) \;\; \mathcal{Y}(\Delta)] = [I_{n_x} \;\; \bar{x}_+] + L\Delta(I - H\Delta)^{-1}[0_{n_z,n_x} \;\; R].$$

A bounding ellipsoid for the ULE solution set is obtained by applying Theorem 1 to this ULE. In particular, in the situation at hand we have

$$\Psi = [I_{n_x} \;\; L \;\; \bar{x}_+], \quad \Upsilon = \begin{bmatrix} 0 & H & R \\ 0 & I & 0 \end{bmatrix}.$$

Hence, Ψ is full-rank and an orthogonal complement is explicitly given by

$$\Psi_\perp = \left[\begin{array}{c|c} -L & \bar{x}_+ \\ I & 0 \\ \hline 0 & -1 \end{array}\right].$$

Furthermore, $\Omega(S, T)$ results to be

$$\Omega(S,T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & H^T T H - S & H^T T R \\ 0 & R^T T H & R^T T R \end{bmatrix}$$

and we have

$$\Psi_\perp^T\,(\mathrm{diag}\,(0,0,1) - \Omega(S,T))\,\Psi_\perp = \begin{bmatrix} S - H^T T H & H^T T R \\ R^T T H & R^T T R - 1 \end{bmatrix}$$

from which the statement follows. □

5. MEASUREMENT UPDATE STEP

At time k, our forecast about the system state at $k+1$ is summarized by the 'predicted' ellipsoid $\mathcal{E}_+$ of center $\hat{x}_+$ and shape matrix E_+, which may be efficiently computed by means of Lemma 2. Then, at time instant $k+1$, a measurement y_+ related to x_+ becomes available. The purpose of the update step of the filter is to integrate the predicted knowledge about the state with the new information coming from the measurement equation.

Consider a non-linear measurement equation

$$y_+ = g(x_+, \xi_+)$$

where $x_+ \in \mathbb{R}^{n_x}$ denotes the true (and unknown) system state at time $k+1$, $\xi_+ \in \mathbb{R}^{n_\xi}$ is an input parameter at time $k+1$, $y_+ \in \mathbb{R}^{n_y}$ is the measurement, and g is twice differentiable.

At time $k+1$ it is known that the state x_+ belongs to the predicted ellipsoid $\mathcal{E}_+$, i.e.

$$x_+ = \hat{x}_+ + E_+\delta_x, \tag{9}$$

for some vector δ_x such that $\|\delta_x\| \leq 1$. Analogously, we assume that the input parameter ξ_+ lies in an ellipsoid $\mathcal{E}_\xi$

$$\xi_+ = \bar{\xi} + E_\xi \delta_\xi,$$

for some vector δ_ξ such that $\|\delta_\xi\| \leq 1$. Now, we expand function g in series around the nominal pair $\bar{\xi}, \hat{x}_+$. The series expansion is computed up to the second order terms, while higher order contributions and measurement noise are taken into account via an additional disturbance term:

$$y_+ = \bar{y}_+ + C(x_+ - \hat{x}_+) + D_\xi\delta_\xi + h(\delta_x, \delta_\xi) + D_v v \tag{10}$$

where $v \in \mathbb{R}^{n_v}$ is the additional measurement error term, which is also assumed to be unknown but bounded as $\|v\| \leq 1$, $\bar{y}_+ \doteq g(\hat{x}_+, \bar{\xi})$, and C, D_ξ are related to the Jacobians of g $C \doteq \mathcal{J}_x$, $D_\xi \doteq \mathcal{J}_\xi E_\xi$, being

$$[\mathcal{J}_x]_{i,j} \doteq \left.\frac{\partial g_i}{\partial x_j}\right|_{x=\hat{x}_+, \xi=\bar{\xi}}, \quad \begin{array}{l} i = 1, \ldots, n_y; \\ j = 1, \ldots, n_x \end{array}$$

$$[\mathcal{J}_\xi]_{i,j} \doteq \left.\frac{\partial g_i}{\partial \xi_j}\right|_{x=\hat{x}_+, \xi=\bar{\xi}}, \quad \begin{array}{l} i = 1, \ldots, n_y; \\ j = 1, \ldots, n_\xi. \end{array}$$

The second order terms are collected in vector $h(\delta_x, \delta_\xi)$, whose i-th component is

$$h_i(\delta_x, \delta_\xi) = \begin{bmatrix} \delta_x \\ \delta_\xi \end{bmatrix}^T \begin{bmatrix} H_{xx}^{(i)} & H_{x\xi}^{(i)} \\ H_{x\xi}^{(i)T} & H_{\xi\xi}^{(i)} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_\xi \end{bmatrix}$$

where the inner matrix above is related to the Hessians of g

$$\begin{bmatrix} H_{xx}^{(i)} & H_{x\xi}^{(i)} \\ H_{x\xi}^{(i)T} & H_{\xi\xi}^{(i)} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} E_+^T & 0 \\ 0 & E_\xi^T \end{bmatrix} \begin{bmatrix} \mathcal{H}_{xx}^{(i)} & \mathcal{H}_{x\xi}^{(i)} \\ \mathcal{H}_{x\xi}^{(i)T} & \mathcal{H}_{\xi\xi}^{(i)} \end{bmatrix} \begin{bmatrix} E_+ & 0 \\ 0 & E_\xi \end{bmatrix}$$

being

$$\left[\mathcal{H}_{xx}^{(i)}\right]_{\ell,j} = \left.\frac{\partial^2 g_i}{\partial x_\ell \partial x_j}\right|_{x=\hat{x}_+, \xi=\bar{\xi}}, \quad \ell, j = 1, \ldots, n_x$$

$$\left[\mathcal{H}_{x\xi}^{(i)}\right]_{\ell,j} = \left.\frac{\partial^2 g_i}{\partial x_\ell \partial \xi_j}\right|_{x=\hat{x}_+, \xi=\bar{\xi}}, \quad \begin{array}{l} \ell = 1, \ldots, n_x; \\ j = 1, \ldots, n_\xi \end{array}$$

$$\left[\mathcal{H}_{\xi\xi}^{(i)}\right]_{\ell,j} = \left.\frac{\partial^2 g_i}{\partial \xi_\ell \partial \xi_j}\right|_{x=\hat{x}_+, \xi=\bar{\xi}}, \quad \ell, j = 1, \ldots, n_\xi.$$

The measurement equation (10) can now be expressed as an uncertain linear equation in the variable x_+:

$$Cx_+ = (y_+ - \bar{y}_+ + C\hat{x}_+) - D_\xi\delta_\xi - h(\delta_x, \delta_\xi) - D_v v. \tag{11}$$

Applying Lemma 1 to the right-hand-side of (11), we have that

$$Cx_+ = \bar{\eta}_+ + L_y\Delta(I - H\Delta)^{-1}R \tag{12}$$

where

$$\Delta = \operatorname{diag}(\delta_x, \delta_\xi, \Delta_x, \Delta_\xi, \delta_v) \in \mathbb{R}^{n_w, n_z}$$

with $\Delta_x = \operatorname{diag}(\delta_x{}^T, \ldots, \delta_x{}^T) \in \mathbb{R}^{n_y, n_y n_x}$, $\Delta_\xi = \operatorname{diag}(\delta_\xi{}^T, \ldots, \delta_\xi{}^T) \in \mathbb{R}^{n_y, n_y n_\xi}$, $n_w \doteq n_x + n_\xi + 2n_y + n_v$, $n_z \doteq 3 + n_y n_x + n_y n_\xi$, and

$$\begin{aligned}
\bar{\eta}_+ &= y_+ - \bar{y}_+ + C\hat{x}_+ \\
L_y &= \begin{bmatrix} 0_{n_y,n_x} & -D_\xi & I_{n_y} & I_{n_y} & -D_v \end{bmatrix} \in \mathbb{R}^{n_y, n_w} \\
R &= \begin{bmatrix} 1 \\ 1 \\ 0_{n_x^2,1} \\ 0_{n_x n_\xi,1} \\ 1 \end{bmatrix} \in \mathbb{R}^{n_z,1} \\
H &= \begin{bmatrix} 0_{1,n_x} & 0_{1,n_\xi} & 0_{1,n_x} & 0_{1,n_x} & 0_{1,n_v} \\ 0_{1,n_x} & 0_{1,n_\xi} & 0_{1,n_x} & 0_{1,n_x} & 0_{1,n_v} \\ H_{xx} & H_{x\xi} & 0_{n_x^2,n_x} & 0_{n_x^2,n_x} & 0_{n_x^2,n_v} \\ H_{\xi x} & H_{\xi\xi} & 0_{n_x n_\xi,n_x} & 0_{n_x n_\xi,n_x} & 0_{n_x n_\xi,n_v} \\ 0_{1,n_x} & 0_{1,n_\xi} & 0_{1,n_x} & 0_{1,n_x} & 0_{1,n_v} \end{bmatrix}
\end{aligned}$$

with $H \in \mathbb{R}^{n_z, n_w}$. The set of states x_+ which are simultaneously consistent with the measurement equation (12) *and* the a-priori information (9) hence coincides with the set of solutions of the joint ULEs

$$\begin{aligned} Cx_+ &= \bar{\eta}_+ + L_y\Delta(I - H\Delta)^{-1}R \\ I_{n_x}x_+ &= \hat{x}_+ + E_+\delta_x. \end{aligned}$$

Notice further that, due to the particular structure of the problem at hand (e.g. $(I - H\Delta)$ is block lower-triangular, with identity blocks on the diagonal), we may write

$$E_+\delta_x = L_x\Delta(I - H\Delta)^{-1}R$$

with $L_x \doteq [E_x \;\; 0_{n_x,n_\xi} \;\; 0_{n_x,n_y} \;\; 0_{n_x,n_y} \;\; 0_{n_x,n_v}]$, and therefore the data of the joint ULEs $\mathcal{A}(\Delta)x_+ = \mathcal{Y}(\Delta)$ may be written in LFT format as

$$\begin{aligned} &[\mathcal{A}(\Delta) \;\; \mathcal{Y}(\Delta)] \\ &= \begin{bmatrix} C & \bar{\eta}_+ \\ I_{n_x} & \hat{x}_+ \end{bmatrix} + \begin{bmatrix} L_y \\ L_x \end{bmatrix} \Delta(I - H\Delta)^{-1}[0_{n_z,n_x} \;\; R]. \end{aligned} \tag{13}$$

Based on this representation, the next lemma provides a computationally efficient way of determining the 'filtered' ellipsoid that contains the states x_+ which are simultaneously consistent with the prediction and the measurement.

Lemma 3. (Measurement update). Given the predicted ellipsoid $\mathcal{E}_+(\hat{x}_+, E_+)$, the measurement data in the form (13), and assuming C is full-rank, a minimal size filtered ellipsoid $\mathcal{E}_{+|+}$ of center $\hat{x}_{+|+}$ and squared shape matrix $P_{+|+}$ is computed by solving the following SDP in the variables $\hat{x}_{+|+}, P_{+|+}$, $(S,T) \in \mathcal{S}(\mathbf{\Delta})$

$$\min \mathrm{Tr} P_{+|+} \text{ subject to: } S \succeq 0$$

$$\left[\begin{array}{c|cc} P_{+|+} & \Psi_{\perp 11} & \hat{x}_+ - \hat{x}_{+|+} \\ \hline * & Q_{11}(S,T) & Q_{12}(S,T) \\ * & Q_{21}(S,T) & Q_{22}(S,T) \end{array}\right] \succeq 0$$

where

$$\begin{bmatrix} Q_{11}(S,T) & Q_{12}(S,T) \\ Q_{21}(S,T) & Q_{22}(S,T) \end{bmatrix} \doteq$$

$$\begin{bmatrix} \Psi_{\perp 12} & \psi_{\perp 22} \\ 0 & -1 \end{bmatrix}^T \begin{bmatrix} S - H^T T H & -H^T T R \\ -R^T T H & 1 - R^T T R \end{bmatrix} \sharp$$

and $\Psi_{\perp 11} = \begin{bmatrix} I_{n_x} & 0_{n_x,n_y} & 0_{n_x,n_\xi} & 0_{n_x,n_v} \end{bmatrix}$,

$$\Psi_{\perp 12} = \begin{bmatrix} E_x^{-1} & 0_{n_x,n_y} & 0_{n_x,n_\xi} & 0_{n_x,n_v} \\ 0_{n_\xi,n_x} & 0_{n_\xi,n_y} & I_{n_\xi} & 0_{n_\xi,n_v} \\ -C & I_{n_y} & D_\xi & D_v \\ 0_{n_y,n_x} & -I_{n_y} & 0_{n_y,n_\xi} & 0_{n_y,n_v} \\ 0_{n_v,n_x} & 0_{n_v,n_y} & 0_{n_v,n_\xi} & I_{n_v} \end{bmatrix},$$

$$\psi_{\perp 22}^T = \left[0_{n_x}^T \; 0_{n_\xi}^T \; (y_+ - \bar{y}_+)^T \; 0_{n_y}^T \; 0_{n_v}^T \right].$$

Proof. The result is again obtained by applying Theorem 1 to the ULE (13). In this specific case, we need to determine the orthogonal complement $\Psi_{\perp 1}$ of $\begin{bmatrix} C & L_y \\ I_{n_x} & L_x \end{bmatrix}$ and vector $\psi_{\perp 2}$ such that

$$\begin{bmatrix} C & L_y \\ I_{n_x} & L_x \end{bmatrix} \psi_{\perp 2} = \begin{bmatrix} \bar{\eta}_+ \\ \hat{x}_+ \end{bmatrix}.$$

If C is full rank (i.e. $\mathrm{rank}\, C = n_y$), then it can be verified by direct inspection that $\Psi_{\perp 1} \doteq \begin{bmatrix} \Psi_{\perp 11} \\ \Psi_{\perp 12} \end{bmatrix}$ and $\psi_{\perp 2} \doteq \begin{bmatrix} \hat{x}_+ \\ \psi_{\perp 22} \end{bmatrix}$, with $\Psi_{\perp 11}, \Psi_{\perp 12}, \psi_{\perp 22}$ given in the statement of the lemma, indeed satisfy the two relations above. We also have

$$\Upsilon = \begin{bmatrix} 0_{n_z,n_x} & H & R \\ 0_{n_w,n_x} & I_{n_w} & 0_{n_w,1} \end{bmatrix}$$

and therefore

$$\mathrm{diag}\,(0,0,1) - \Omega(S,T) = \begin{bmatrix} S - H^T T H & -H^T T R \\ -R^T T H & 1 - R^T T R \end{bmatrix},$$

$$\Psi_\perp^T (\mathrm{diag}\,(0,0,1) - \Omega(S,T)) \Psi_\perp =$$
$$\begin{bmatrix} \Psi_{\perp 12} & \psi_{\perp 22} \\ 0 & -1 \end{bmatrix}^T \begin{bmatrix} S - H^T T H & -H^T T R \\ -R^T T H & 1 - R^T T R \end{bmatrix} \sharp,$$

$$\left[I \;\; 0 \mid \hat{x}_{+|+} \right] \Psi_\perp = \left[\Psi_{\perp 11} \mid \hat{x}_+ - \hat{x}_{+|+} \right] =$$
$$\left[I_{n_x} \;\; 0_{n_x,n_y} \;\; 0_{n_x,n_\xi} \;\; 0_{n_x,n_v} \middle| \hat{x}_+ - \hat{x}_{+|+} \right].$$

The statement of the lemma follows immediately from the above derivations. □

6. CONCLUSIONS

In this paper we presented the basic structure of a recursive algorithm that determines at each step an ellipsoidal membership set for the state of a nonlinear system, based on model predictions and measurement maps that are locally approximated up to the second order terms. At each step, the method requires solving two convex semidefinite programs whose numerical complexity essentially grows as $n_x^{3.5}$, according to the complexity analysis in (Calafiore and El Ghaoui, 2004).

Ongoing work is dedicated to the software implementation of the described algorithm and to the comparative analysis of its performance via numerical simulations and experiments.

REFERENCES

Bertsekas, D.P. and I.B. Rhodes (1971). Recursive state estimation with a set-membership description of the uncertainty. *IEEE Trans. Aut. Control* **16**, 117–128.

Calafiore, G. and L. El Ghaoui (2004). Ellipsoidal bounds for uncertain linear equations and dynamical systems. *Automatica* **40**(5), 773–787.

Chernousko, F.L. (1993). *State Estimation of Dynamic Systems.* CRC Press. Boca Raton, FL.

Di Marco, M., A. Garulli, A. Giannitrapani and A. Vicino (2004). A set theoretic approach to dynamic robot localization and mapping. *Autonomous Robots* **16**(1), 23–47.

Durieu, C., E. Walter and B. Polyak (2001). Multi-input multi-output ellipsoidal state bounding. *Journal of Optimization Theory and Applications* **111**(2), 273–303.

Einicke, G.A. and B. White (1999). Robust extended kalman filtering. *IEEE Transactions on Signal Processing* **47**(9), 2596–2599.

Fabrizi, E., G. Oriolo, S. Panzieri and G. Ulivi (1998). A kf-based localization algorithm for nonholonomic mobile robots. In: *Proceedings of the 6th IEEE Mediterranean Conference.* Alghero. pp. 130–135.

Jetto, L., S. Longhi and G. Venturini (1999). Development and experimental validation of an adaptive extended kalman filter for the localization of mobile robots. *IEEE Transactions on Robotics and Automation* **15**(2), 219–229.

Kurzhanski, A.B. and I. Valyi (1996). *Ellipsoidal Calculus for Estimation and Control.* Birkhauser. Boston.

Maskarov, D.G. and J.P. Norton (1996). State bounding with ellipsoidal set description of the uncertainty. *Int. J. Control* **65**(5), 847–866.

Schweppe, F.C. (1973). *Uncertain Dynamic Systems.* Prentice-Hall. New York.