# VECTOR $L_2$-GAIN AND A SMALL GAIN THEOREM FOR SWITCHED SYSTEMS

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Abstract: A concept of vector $L_2$-gain is presented for switched systems. Each subsystem is allowed to have individual input-output $L_2$-gain during any time interval when the subsystem is activated. Stability is derived from this vector $L_2$-gain for certain classes of control including output feedback. A small gain theorem for switched systems with vector $L_2$-gain is established which provides a tool for analyzing the behavior of feedback interconnected switched systems. The small gain condition is given in terms of the $L_2$-gains of the coupled activated subsystems.


Keywords: Switched systems, Vector $L_2$-gain, Stability, Feedback interconnection

## 1. INTRODUCTION

The $L_2$-gain analysis of control systems has been widely studied in the control area. Early works of $L_2$-gain analysis can be found in the framework of dissipativity (Hill and Moylan, 1980). The $L_2$-gain analysis in state space form has attracted even more attention. This form of $L_2$-gain not only provides a candidate for Lyapunov functions, but also greatly relates to the solutions of $H_\infty$ control problems.

As an important application of $L_2$-gain analysis, small-gain theorems play a fundamental role in stabilizing nonlinear feedback interconnected systems (van der Schaft, 2000; Haddad and Chellaboina, 2001; Helton and James, 1999; Teel, 1996). Besides, small gain theorems have found more applications such as robust nonlinear control (Jiang, 1999), input-to-state stability (Karafyllis and Tsinias, 2004), etc. Small gain theorems have also been generalized to variable $L_2$-gain settings to cover more general situations (Lin and Gong, 2004).

Switched systems, as an important class of hybrid systems, have drawn considerable attention in the control and computer communities in the last decade (DeCarlo *et al.*, 2000). Many results have appeared on stability and stabilization problems for such systems (Hespanha *et al.*, 2005; Michel and Hu, 1999; Peleties and DeCarlo, 1991; M. Zefran and Stein, 2001; Zhao and Dimirovski, 2004) and are summarized in the recent book (Liberzon, 2003). However, due to the hybrid nature of switched systems, the description

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of input-output relations becomes even more difficult. In this respect, there has been relatively little work up to now. Recently, (Hespanha, 2003) gave a method of computing the root-mean-square gains of switched linear systems. The $L_2$-gain was analyzed for switched symmetric linear systems under arbitrary switching (Zhai, 2003). Unfortunately, almost all existing related results employ a unified gain to measure all subsystems. Further, in terms of small gain theorems, no results have appeared for switched systems by now.

This paper gives a description of vector $L_2$-gain for switched systems. Each subsystem has its own input-output $L_2$-gain during any time interval when the subsystem is activated, which preserves the hybrid nature on input-output $L_2$-gain representation, and has less conservativity. This vector $L_2$-gain produces stability for a certain class of control including output feedback. A small gain theorem for switched systems with vector $L_2$-gain is established in terms of the condition on $L_2$-gains of the corresponding individual activated subsystems.

## 2. VECTOR $L_2$-GAIN

We consider a switched system of the form:

$$H: \quad \begin{aligned} \dot{x} &= f_\sigma(x, u_\sigma), \\ y &= h_\sigma(x) \end{aligned} \tag{1}$$

where $\sigma : R_+ \to M = \{1, 2, \cdots, m\}$ is a switching signal given or to be designed, $x \in R^n$ is the state, $u_i$ and $h_i(x)$ are respectively the control input vector and output vector of the $i$th subsystem and have appropriate dimensions. Furthermore, we suppose $f_i(0,0) = 0$, $h_i(0) = 0$, $i = 1, 2, \cdots, m$. Corresponding to the switching signal $\sigma$, we have the switching sequence

$$\begin{aligned} \Sigma = \{&x_0; (i_0, t_0), (i_1, t_1), \cdots, (i_j, t_j), \cdots, \\ &| i_j \in M, j \in N\}, \end{aligned} \tag{2}$$

in which $t_0$ is the initial time, $x_0$ is the initial state and $N$ is the set of nonnegative integers. When $t \in [t_k, t_{k+1})$, the $i_k$th subsystem is activated. For any $j \in M$, let $\Sigma_t(j) = \{t_{j_1}, t_{j_2}, \cdots, t_{j_k}, \cdots\}$ be the subsequence of $\{t_0, t_1, \cdots, t_k \cdots\}$ such that $t_{j_1} < t_{j_2} < \cdots < t_{j_k} < \cdots$ and the $j$th subsystem is activated on $[t_k, t_{k+1})$ if and only if $t_k \in \Sigma_t(j)$. In addition, we will need the following assumption.

**Assumption 2.1.** For any finite $T > t_0$, there exist a positive integer $K = K_T$, such that during the time interval $[t_0,\ T]$ the system (1) switches no more than $K$ times, independent of the initial states in a vicinity of the origin .

This assumption is adopted to rule out arbitrarily fast switching.

By $L_1[0,\ \infty)$ we denote the usual $L_1$ function space over $[0,\ \infty)$, that is, $\mu = \mu(t) \in L_1[0,\ \infty)$ if $\int_0^\infty |\ \mu(t)\ |\ dt < \infty$. Let $L_1^+[0,\ \infty)$ denote the subset of $L_1[0,\ \infty)$ consisting of all nonnegative functions.

For the system (1) the usual description of $L_2$-gain in state space form is still applicable. That is, the system (1) has $L_2$-gain $\gamma$ if there exist a positive definite function $S(x)$ such that for any $t > t_0$, it holds that

$$\begin{aligned} &S(x(t)) - S(x(t_0)) \\ &\leq \frac{1}{2} \int_{t_0}^{t} (\gamma^2 \|u_{\sigma(t)}(t)\|^2 - \|h_{\sigma(t)}(t)\|^2) dt. \end{aligned} \tag{3}$$

However, this property that is standard for general nonlinear systems is much too strong and restrictive for switched systems. In fact, for a switched system, each subsystem usually has its individual $L_2$-gain $\gamma_i$ when this subsystem is activated. Unifying $\{\gamma_i\}$ by taking the maximum, though possible, is obviously very conservative. This can be seen, for example, in the setting of a small gain theorem to be established where the product of $L_2$-gain $\gamma_i$ of the corresponding activated subsystems is less than one while that obtained by taking the maximum is greater than one. In addition, each individual subsystem has its own energy function $S_i(x)$ and a common energy function $S(x)$ for all subsystems may not exist or is difficult to find. Therefore, it is reasonable and necessary to introduce a version of $L_2$-gain for the system (1) revealing all $\gamma_i$ and $S_i(x)$. This can be done by introducing the vector $L_2$-gain as follows.

**Definition 2.1.** Let $\gamma_1, \gamma_2, \cdots, \gamma_m$ be positive constants. System (1) is said to have vector $L_2$-gain $\{\gamma_1, \gamma_2, \cdots, \gamma_m\}$ under the switching law $\Sigma$ if there exist positive definite continuous functions $S_1(x), S_2(x), \cdots, S_m(x)$ with $S_i(0) = 0$, and functions $\omega_j^i(u_i(t), t) \in L_1^+[0,\ \infty), 1 \leq i, j \leq m, i \neq j$, such that for $\forall u_i$ and $k = 0, 1, 2, \cdots$, it holds that

$$\begin{aligned} &S_{i_k}(x(t)) - S_{i_k}(x(s)) \\ &\leq \frac{1}{2} \int_{s}^{t} (\gamma_{i_k}^2 \|u_{i_k}(t)\|^2 - \|h_{i_k}(t)\|^2) dt, \\ &\qquad t_k \leq s \leq t < t_{k+1} \end{aligned} \tag{4}$$

$$\begin{aligned} &S_j(x(t_{k+1})) - S_j(x(t_k)) \\ &\leq \frac{1}{2} \int_{t_k}^{t_{k+1}} (\gamma_{i_k}^2 \|u_{i_k}(t)\|^2 - \|h_{i_k}(t)\|^2 \\ &\qquad + \omega_j^{i_k}(u_{i_k}(t), t)) dt, \quad j \neq i_k \end{aligned} \tag{5}$$

**Remark 2.1.** If the system (1) has $L_2$-gain $\gamma$ in the usual sense, then it has vector $L_2$-gain

$\{\gamma, \gamma, \cdots, \gamma\}$ with $\omega_j^i = 0, i \neq j$. Therefore, the concept of vector $L_2$-gain is a natural generalization of the usual one. Conversely, as pointed out earlier, the vector $L_2$-gain covers a more general situation, such as no existence of a common energy function $S(x)$. In the definition 2.1, (4) describes the usual $L_2$-gain property for each subsystem when being activated. While (5) gives relationship between the activated $i_k$th subsystem and any $j$th inactivated subsystem. When the $i_k$th subsystem is activated on $[t_k,\ t_{k+1})$, the energy $S_j(x)$ of the $j$th inactivated subsystem changes from $S_j(x(t_k))$ to $S_j(x(t_{k+1}))$. This can be viewed as the result of "energy" coming from the $i_k$th subsystem to the $j$th subsystem and is characterized by the "generalized supply rate" $\gamma_{i_k}^2\|u_{i_k}(t)\|^2 - \|h_{i_k}(t)\|^2 + \omega_j^{i_k}(u_{i_k}(t), t)$. Comparing (5) with (4) infers that the $j$th inactivated subsystem may absorb more energy from the $i_k$th activated subsystem than the $i_k$th subsystem from itself. This allows the description of vector $L_2$-gain to fit more systems. Intuitively however, absorbing too much energy from the $i_k$th subsystem will cause the instability of the $j$th subsystem though the energy of the $j$th subsystem may decrease to some extent when it is activated. This is avoided by letting $\omega_j^{i_k}(u_{i_k}(t), t) \in L_1^+[0,\ \infty)$, which says that when the $j$th subsystem is inactivated, the total increased "energy" coming to the $j$th subsystem from outside is bounded.

When the system (1) and $S_i(x)$ are all smooth, some algebraic conditions can be derived to characterize the vector $L_2$-gain. For example, consider the following smooth affine system

$$\begin{aligned} \dot{x} &= f_\sigma(x) + g_\sigma(x)u_\sigma, \\ y &= h_\sigma(x). \end{aligned} \tag{6}$$

It is not difficult to show that the system (6) has vector $L_2$-gain $\{\gamma_1, \gamma_2, \cdots, \gamma_m\}$ if the following conditions hold when $t_k \leq t < t_{k+1}$.

$$\frac{\partial S_{i_k}}{\partial x} f_{i_k} + \frac{1}{2}\frac{1}{\gamma_{i_k}^2}\frac{\partial S_{i_k}}{\partial x} g_{i_k} g_{i_k}^T \frac{\partial S_{i_k}^T}{\partial x} + \frac{1}{2} h_{i_k}^T h_{i_k} \leq 0 \tag{7}$$

$$\begin{aligned} &\frac{\partial S_j}{\partial x}(f_{i_k} + g_{i_k}\pi_{i_k}) - \frac{1}{2}\gamma_{i_k}^2\|\pi_{i_k}\|^2 \\ &+ \frac{1}{2} h_{i_k}^T h_{i_k} - \frac{1}{2}\omega_j^{i_k}(\pi_{i_k}, t) \leq 0, \end{aligned} \tag{8}$$

where $\pi_{i_k}$ is the solution to the following equation

$$\frac{\partial S_j}{\partial x} g_{i_k} - \gamma_{i_k}^2 \pi_{i_k}^T - \frac{1}{2}\frac{\partial \omega_j^{i_k}}{\partial u_{i_k}}|_{(u_{i_k} = \pi_{i_k})} = 0 \tag{9}$$

with the boundary condition $\pi_{i_k}(x)\ |_{x=0} = 0$.

## 3. STABILITY ANALYSIS

In this section, we will show the stability of a switched system with vector $L_2$-gain.

We first introduce the concept of asymptotic zero state detectability for nonlinear systems, which will be used to prove the asymptotic stability.

**Definition 3.1.** A system

$$\begin{aligned} \dot{x} &= f(x), \\ y &= h(x) \end{aligned} \tag{10}$$

is called asymptotically zero state detectable if for any $\epsilon > 0$, there exists $\delta > 0$, such that when $\| y(t+s) \| < \delta$ holds for some $t \geq 0$, $\Delta > 0$ and $0 \leq s \leq \Delta$, we have $\| x(t) \| < \epsilon$.

**Remark 3.1.** This asymptotic zero state detectability is a weaker version of small-time norm observability (Hespanha *et al.*, 2005).

**Theorem 3.1:** If the system (1) has vector $L_2$-gain $\{\gamma_1, \gamma_2, \cdots, \gamma_m\}$ under the switching law $\sum$, then, the origin is stable in the sense of Lyapunov for any control $u_i(t)$ satisfying

$$\| u_i(t) \|^2 \leq \frac{(1-\zeta_i^2)}{\gamma_i^2} \| h_i(t) \|^2 \tag{11}$$

for some $\zeta_i, 0 \leq \zeta_i \leq 1$. If in addition, $0 < \zeta_i \leq 1$, all $S_i$ are globally defined radially unbounded smooth functions, there exists at least one $j$ such that $\lim_{k\to\infty}(t_{j_k+1} - t_{j_k}) \neq 0$, and all subsystems of (1) are asymptotically zero state detectable, then, the origin is globally asymptotically stable.

**Proof.** For any constant $c > 0$, let $B(c) = \{x|\ \| x \| \leq c\}$, $r_i(c) = \min_x\{S_i(x)|\ \| x \| = c\}$ and $r(c) = \min_i\{r_i(c)\}$.

For any $\epsilon > 0$, since $\omega_j^i(u_i(t), t) \in L_1^+[0,\ \infty)$, there exists $T > 0$ such that for any $T_1, T_2$ , $T \leq T_1 \leq T_2 \leq \infty$, it holds that

$$\int_{T_1}^{T_2} \omega_j^i(u_i(t), t)dt < \frac{1}{m} r(\epsilon), \quad i, j \in M, i \neq j. \tag{12}$$

Assumption 2.1 says that the system (1) switches at most $K$ times on the time interval $[t_0,\ T]$ for some integer $K$. Thus, $t_K \geq T$, no matter where to start. Note that $S_i$ is positive definite and $S_i(0) = 0$, we can find $\delta_1 > 0$, $\delta_1 < \epsilon$, such that $S_i(x) < \frac{1}{2}r(\epsilon)$ when $x \in B(\delta_1)$. For this $\delta_1 > 0$, we can find $\delta_2 > 0$, $\delta_2 < \delta_1$ such that $S_i(x) < r(\delta_1)$ when $x \in B(\delta_2)$. Continuing this procedure, we finally have a sequence

$$\epsilon = \delta_0 > \delta_1 > \delta_2 > \cdots > \delta_K > 0$$

with the property:

$$\begin{aligned} &S_i(x) < r(\delta_p), \text{if } x \in B(\delta_{p+1}), 1 \leq p \leq K-1, \\ &S_i(x) < \frac{1}{2} r(\epsilon), \text{if } x \in B(\delta_1), \forall i \end{aligned} \tag{13}$$

Note that $S_{i_k}(x(t))$ decreases when the $i_k$th subsystem is activated, if we start within $B(\delta_K)$, we will stay in $B(\delta_1)$ as long as we switch no more than $K$ times and no matter how we switch. This implies $x(t) \in B(\delta_1), t \in [t_0,\ t_K]$ if $x(0) \in B(\delta_K)$. In particular, $S_i(x(t_K)) < \frac{1}{2}r(\epsilon), i \in M$.

Now, for any $j \in M$, let $t_{j_q} \in \Sigma_t(j)$ and $t_{j_q} > t_K$. Obviously, $j_q \geq K+1$. It is easy to deduce from (5) that

$$\begin{aligned} & S_j(x(t_{j_q})) - S_j(x(t_K)) \\ & = \sum_{\lambda=0}^{j_q-K-1} (S_j(x(t_{K+\lambda+1})) - S_j(x(t_{K+\lambda}))) \\ & \leq \frac{1}{2} \sum_{\lambda=0}^{j_q-K-1} \int_{t_{K+\lambda}}^{t_{K+\lambda+1}} (\gamma_{i_{K+\lambda}}^2 \|u_{i_{K+\lambda}}(t)\|^2 \\ & -\|h_{i_{K+\lambda}}(t)\|^2 + \omega_j^{i_{K+\lambda}}(u_{i_{K+\lambda}}(t),t))dt, \end{aligned} \tag{14}$$

where $\omega_j^j = 0$. Taking (11) into account we have

$$\begin{aligned} & S_j(x(t_{j_q})) - S_j(x(t_K)) \\ & \leq \frac{1}{2} \sum_{\lambda=0,\ i_K+\lambda\neq j}^{j_q-K-1} \int_{t_{K+\lambda}}^{t_{K+\lambda+1}} \omega_j^{i_{K+\lambda}}(u_{i_{K+\lambda}}(t),t)dt \\ & \leq \frac{1}{2} \sum_{i=1,\ i\neq j}^{m} \int_{t_K}^{\infty} \omega_j^i(u_i(t),t)dt < \frac{1}{2}r(\epsilon) \end{aligned} \tag{15}$$

Therefore,

$$S_j(x(t_{j_q})) \leq S_j(x(t_K)) + \frac{1}{2}r(\epsilon) < r(\epsilon)$$

Thus, $x(t) \in B(\epsilon)$ for all $t$ and stability follows.

Next, we show the asymptotic stability under the condition $0 < \zeta_i \leq 1$ and asymptotic zero state detectability. The following proof is motivated by (Hespanha *et al.*, 2005).

Substituting (11) into (4) we have

$$\begin{aligned} & \frac{1}{2}\zeta_{i_k}^2 \int_s^t \|h_{i_k}(t)\|^2 dt \\ & \leq S_{i_k}(x(s)) - S_{i_k}(x(t)), \quad t_k \leq s \leq t < t_{k+1} \end{aligned} \tag{16}$$

For the $j$ satisfying $\lim_{k\to\infty}(t_{j_k+1} - t_{j_k}) \neq 0$, we can select $\delta > 0$ such that the set $\Lambda = \{k | t_{j_k+1} - t_{j_k} \geq \delta\}$ is infinite. Define a function

$$\tilde{h}_j(t) = \begin{cases} h_j(x(t)), & t \in \bigcup_{k\in\Lambda} [t_{j_k},\ \ t_{j_k+1}) \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

For any $t > 0$, when $t_{j_k} \leq t < t_{j_k+1}$ for some $k \in \Lambda$, (16) gives

$$\begin{aligned} & \frac{1}{2}\zeta_j^2 \int_{t_0}^{t} \tilde{h}_j^T(t)\tilde{h}_j(t)dt \\ & \leq \sum_{p=1}^{k} \left(S_j(x(t_{j_p})) - S_j(x(t_{j_p+1}))\right) \\ & = S_j(x(t_{j_1})) - S_j(t_{j_k+1}) \\ & + \sum_{p=1}^{k-1} \left(S_j(x(t_{j_{p+1}})) - S_j(x(t_{j_p+1}))\right). \end{aligned} \tag{18}$$

It can be easily derived that

$$\begin{aligned} & S_j(x(t_{j_{p+1}})) - S_j(x(t_{j_p+1})) \\ & = \sum_{\lambda=1}^{j_{p+1}-j_p-1} \left(S_j(x(t_{j_p+1+\lambda})) - S_j(x(t_{j_p+\lambda}))\right) \\ & \leq \frac{1}{2} \sum_{\lambda=1}^{j_{p+1}-j_p-1} \int_{t_{j_p+\lambda}}^{t_{j_p+1+\lambda}} (-\zeta_{i_{j_p+\lambda}}^2 \|h_{i_{j_p+\lambda}}(t)\|^2 \\ & + \omega_j^{i_{j_p+\lambda}}(u_{i_{j_p+\lambda}}(t),t))dt \\ & \leq \frac{1}{2} \sum_{\lambda=1}^{j_{p+1}-j_p-1} \int_{t_{j_p+\lambda}}^{t_{j_p+1+\lambda}} (\omega_j^{i_{j_p+\lambda}}(u_{i_{j_p+\lambda}}(t),t))dt \end{aligned} \tag{19}$$

Therefore,

$$\begin{aligned} & \sum_{p=1}^{k-1} (S_j(x(t_{j_{p+1}})) - S_j(x(t_{j_p+1})) \\ & \leq \frac{1}{2} \sum_{\lambda=1}^{j_{p+1}-j_p-1} \sum_{p=1}^{k-1} \\ & \left( \int_{t_{j_p+\lambda}}^{t_{j_p+1+\lambda}} (\omega_j^{i_{j_p+\lambda}}(u_{i_{j_p+\lambda}}(t),t))dt \right) \\ & \leq \frac{1}{2} \sum_{i=1,\ i\neq j}^{m} \int_{t_0}^{\infty} (\omega_j^i(u_i(t),t))dt < \infty. \end{aligned} \tag{20}$$

When $t \notin [t_{j_k}, t_{j_k+1})$ for any $k \in \Lambda$, there exists $k \in \Lambda$ such that $t \geq t_{j_k+1}$ and $t < t_{j_q}$ for any $q \in \Lambda$ and $q > k$. In this case, we have $\tilde{h}_j(s) \equiv 0, s \in [t_{j_k+1}, t]$, and (18) still holds. It follows from (18) and (20) that $\int_{t_0}^{\infty} \tilde{h}_j^T(t)\tilde{h}_j(t)dt$ is finite. Now , we show $\tilde{h}_j(t) \to 0$ as $t \to \infty$. Suppose this is false, then there exist $\epsilon > 0$ and a sequence of time $t$, say, $q_1, q_2, \cdots,\ q_k \to \infty$, satisfying $\tilde{h}_j^T(q_i)\tilde{h}_j(q_i) \geq \epsilon, \forall i$. Note that (16) and (5) guarantee the boundedness of $x(t)$, and $\dot{x}(t) = f_\sigma(x(t), u_\sigma(t))$ is also bounded. Hence, $\tilde{h}_j(t)$ is uniformly continuous over $\bigcup_{k\in\Lambda}[t_{j_k},\ t_{j_k+1})$. In view of $t_{j_k+1} - t_{j_k} \geq \delta$, we have $\int_{t_0}^{\infty} \tilde{h}_j^T(t)\tilde{h}_j(t)dt = \infty$, which contradicts the fact that $\int_{t_0}^{\infty} \tilde{h}_j^T(t)\tilde{h}_j(t)dt$ is finite. Therefore, $h_j(t) \to 0$. So, $x(t_{j_k}) \to 0$ as $k \to \infty$ and $k \in \Lambda$ follows from the asymptotic zero state detectability of the $j$th subsystem, and this in turn implies $x(t) \to 0$ as $t \to \infty$ due to stability of the closed-loop system and continuity of $x(t)$.

**Remark 3.2.** The control can be chosen as linear output feedback of the form $u_i = G_i y_i$ with all the eigenvalues of the matrix $G_i^T G_i$ less than one.

## 4. SMALL-GAIN THEOREM

This section will establish a small-gain theorem for switched systems.

Suppose we have two switched systems:

$$H_1: \quad \begin{aligned} \dot{x} &= f_{\sigma_1}(x, u_{\sigma_1}), \\ y &= h_{\sigma_1}(x), \end{aligned} \tag{21}$$

and

$$H_2: \quad \begin{aligned} \dot{z} &= g_{\sigma_2}(z, v_{\sigma_2}), \\ w &= l_{\sigma_2}(z), \end{aligned} \tag{22}$$

where $\sigma_i : R_+ \to M_i = \{1, 2, \cdots, m_i\}, i = 1, 2$. The meaning of other variables are the same as those in the system (1).

Without loss of generality, we assume that the two switched systems have the same switching time sequence $\{t_0, t_1, \cdots, t_k, \cdots\}$ because otherwise we can put all the time instants of the two switching time sequences together and then reorder them and thus form a unified switching time sequence. When $t \in [t_k, t_{k+1})$, the $i_k^1$th and $i_k^2$th subsystems of $H_1$ and $H_2$ are activated respectively.

**Theorem 4.1.** Suppose that $H_1$ and $H_2$ have vector $L_2$-gains $\{\gamma_{11}, \gamma_{12}, \cdots, \gamma_{1m_1}\}$ with $S_{1i}, \omega_{1j}^i$ and $\{\gamma_{21}, \gamma_{22}, \cdots, \gamma_{2m_2}\}$ with $S_{2i}, \omega_{2j}^i$ respectively. If there exists $\alpha$ such that

$$\gamma_{1i_k^1} < \alpha < \frac{1}{\gamma_{2i_k^2}}, \tag{23}$$

then, the feedback interconnected system of $H_1$ and $H_2$ with

$$u_{\sigma_1} = -l_{\sigma_2}(z), \quad v_{\sigma_2} = h_{\sigma_1}(x) \tag{24}$$

is stable. If in addition, all $S_{1i}, S_{2i}$ are globally defined radially unbounded, there exists at least one interconnected subsystem having infinite activated time intervals with positive dwell time, and all subsystems of $H_1$ and $H_2$ are asymptotically zero state detectable, then, the feedback interconnected system is globally asymptotically stable.

**Proof.** We follow the method in (van der Schaft, 2000). For the interconnected system, let

$$S_{j_1j_2}(x, z) = S_{1j_1}(x) + \alpha^2 S_{2j_2}(z). \tag{25}$$

In particular, when $t \in [t_k, t_{k+1})$, the $i_k^1$th and $i_k^2$th subsystems of $H_1$ and $H_2$ are activated respectively, we have

$$S_{i_k^1 i_k^2}(x, z) = S_{1i_k^1}(x) + \alpha^2 S_{2i_k^2}(z). \tag{26}$$

When $t_k \leq s \leq t < t_{k+1}$ the vector $L_2$-gains of $H_1$ and $H_2$ give

$$\begin{aligned} &S_{1i_k^1}(x(t)) - S_{1i_k^1}(x(s)) \\ &\leq \frac{1}{2}\int_s^t (\gamma_{1i_k^1}^2 \|u_{i_k^1}(t)\|^2 - \|h_{i_k^1}(t)\|^2)dt, \end{aligned} \tag{27}$$

and

$$\begin{aligned} &S_{2i_k^2}(z(t)) - S_{2i_k^2}(z(s)) \\ &\leq \frac{1}{2}\int_s^t (\gamma_{2i_k^2}^2 \|v_{i_k^2}(t)\|^2 - \|l_{i_k^2}(t)\|^2)dt, \end{aligned} \tag{28}$$

Thus, the feedback relation (24) gives

$$\begin{aligned} &S_{i_k^1 i_k^2}(x(t), z(t)) - S_{i_k^1 i_k^2}(x(s), z(s)) \\ &\leq \frac{1}{2}\int_s^t \Big( (\alpha^2(\gamma_{2i_k^2})^2 - 1) \parallel h_{i_k^1}(\rho) \parallel^2 \\ &\quad + \Big((\gamma_{1i_k^1})^2 - \alpha^2\Big) \parallel l_{i_k^2}(\rho) \parallel^2 \Big) d\rho, \end{aligned} \tag{29}$$

Note that $\alpha^2(\gamma_{2i_k^2})^2 - 1 < 0$ and $(\gamma_{1i_k^1})^2 - \alpha^2 < 0$, $S_{i_k^1 i_k^2}(x(t), z(t))$ decreases on the corresponding active time interval. Now, we estimate the change of $S_{j_1j_2}$ on any time interval $[t_k,\ t_{k+1})$ when the $j_1$th subsystem of $H_1$ or $j_2$th subsystem of $H_2$ is inactivated. In this case, $(j_1, j_2) \neq (i_k^1, i_k^2)$ must hold.

A straightforward computation shows that

$$S_{j_1j_2}(x(t_{k+1}), z(t_{k+1})) - S_{j_1j_2}(x(t_k), z(t_k))$$

$$\leq \frac{1}{2}\int_{t_k}^{t_{k+1}} \Big((\alpha^2(\gamma_{2i_k^2})^2 - 1) \parallel h_{i_k^1}(t) \parallel^2$$

$$+ \Big((\gamma_{1i_k^1})^2 - \alpha^2\Big) \parallel l_{i_k^2}(t) \parallel^2\Big)\, dt \tag{30}$$

$$+ \frac{1}{2}\int_{t_k}^{t_{k+1}} (\omega_{1j_1}^{i_k^1}(u_{i_k^1}(t), t) + \alpha^2 \omega_{2j_2}^{i_k^2}(v_{i_k^2}(t), t))dt$$

$$\leq \frac{1}{2}\int_{t_k}^{t_{k+1}} (\omega_{1j_1}^{i_k^1}(u_{i_k^1}(t), t) + \alpha^2 \omega_{2j_2}^{i_k^2}(v_{i_k^2}(t), t))dt$$

Since $\omega_{1j_1}^{i_k^1} \in L_1^+[0, \infty)$ and $\omega_{2j_2}^{i_k^2} \in L_1^+[0, \infty)$, similarly to the proof of the theorem 3.1, we can complete the proof.

**Corollary 4.1.** The small gain condition (23) is automatically satisfied if all gains satisfy $\gamma_{1i} < 1$, $\gamma_{2j} < 1$. In this case $\alpha$ can be selected as 1.

## 5. CONCLUDING REMARKS

We have proposed a notion of vector $L_2$-gain for switched systems. The $L_2$-gain of each individual subsystem when being activated is preserved in the representation, which is helpful for understanding the individual input-output behavior. This vector $L_2$-gain could be regarded as a "generalized dissipativity". Unlike usual nonlinear systems for which a single dissipativity inequality is enough to describe the dissipative behavior, we have to describe the feature of both active and inactive subsystems, especially the impact of a activated subsystem on other inactivated subsystems. This is characterized, in this paper, by the functions $\omega_j^i$. Vector $L_2$-gain is also shown to produce stability for certain class of control, in particular, output feedback. A small gain theorem for switched systems with vector $L_2$-gain is established by using the small gain condition for the couple of activated subsystems. As for usual nonlinear control systems, this small gain theorem provides a tool for control of interconnected switched systems.

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