ML IDENTIFICATION OF CLOSED-LOOP SYSTEMS IN A SPECIFIED FREQUENCY BAND

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Abstract: In classical time domain Box-Jenkins identification discrete-time plant and noise models are estimated using sampled input/output signals. The frequency content of the input/output samples covers uniformly the whole unit circle in a natural way, even in case of prefiltering. In Ljung (1999) the time domain Box-Jenkins framework has been extended to frequency domain data captured in open loop on arbitrary frequency grids. In this paper we handle the closed loop case. Contrary to the classical time domain case it is shown that the controller should be either known or estimated.

Keywords**:** system identification, frequency-domain method, closed loop, parameter estimation, noise characterization.

1. INTRODUCTION

Since the frequency content of a sampled signal covers the whole unit circle, the classical time domain Box-Jenkins approach (Box and Jenkins, 1970) identifies the discrete-time plant and noise models from DC (zero Hz) to Nyquist (half the sampling frequency). Often one is only interested in the plant characteristics on a fraction of the unit circle, or one would like to remove the effect of slow trends and/or high frequency disturbances. The classical approach consists in applying a prefilter to the input/output data (Ljung, 1999). The prefiltering does not affect the input/output relation, and is equivalent to dividing the noise model by the prefilter characteristics. However, to preserve the efficiency (open/closed loop) and the consistency (closed loop only) of the plant estimates, the parametric noise model should be flexible enough to follow the prefiltered error spectrum accurately (Ljung, 1999). As such, it will try to cancel the effect of the prefilter. Hence, through the prefilter/noise model selection a compromise must be made between the suppression of the undesired frequency band(s) and the loss in efficiency and/or consistency of the plant estimates. These conflicting demands can be avoided by performing the filtering in the frequency domain: the plant and noise models are identified in the frequency band(s) of interest only.

In (Ljung, 1999) a frequency domain Box-Jenkins framework has been developed for data collected in open loop. The proposed frequency domain maximum likelihood (ML) solution can handle arbitrary frequency grids (parts of the unit circle). In this paper the frequency domain maximum likelihood solution is extended to the closed loop case. A surprising result is that the controller should be either known or estimated. The former has already been mentioned in McKelvey (2000).

2. LINEAR TIME INVARIANT MODELS

Assuming that the input of the plant $u(t)$ and the driving white noise source $e(t)$ are piecewise constant (= zero-order-hold), the output of the plant $y(t)$ can be written as

$$y(t) = G(q)u(t) + H(q)e(t) \quad (1)$$

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where q is the backward shift operator ($qx(t) = x(t-1)$), and where $G(z^{-1})$, $H(z^{-1})$ are related to the true underlying continuous-time systems $G_c(s)$, $H_c(s)$ by

$$X_1(z^{-1}) = (1-z^{-1})Z\{X_{1c}(s)/s\} \qquad (2)$$

with $Z\{\ \}$ the Z-transform and $X_1 = G, H$ (Middleton and Goodwin, 1990). Taking the discrete Fourier transform (DFT) of the input and output samples $u(t)$, $y(t)$, $t = 0, 1, \ldots, N-1$, relationship (1) can be rewritten as

$$\begin{aligned} Y(k) = {} & G(z_k^{-1})U(k) + T_G(z_k^{-1}) + \\ & H(z_k^{-1})E(k) + T_H(z_k^{-1}) \end{aligned} \qquad (3)$$

where $z_k = \exp(j2\pi k f_s/N)$ with $f_s = 1/T_s$ the sampling frequency, and

$$X(k) = N^{-1/2}\sum_{t=0}^{N-1} x(t)z_k^{-t} \qquad (4)$$

the DFT of $x(t)$ with $X = Y, U, E$ and $x = y, u, e$ (Pintelon and Schoukens, 2001). The plant G, plant transient T_G, noise H, and noise transient T_H transfer functions are rational functions of z^{-1}

$$\begin{aligned} G(z^{-1}) &= \frac{B(z^{-1})}{A(z^{-1})},\ T_G(z^{-1}) = \frac{I_G(z^{-1})}{A(z^{-1})} \\ H(z^{-1}) &= \frac{C(z^{-1})}{D(z^{-1})},\ T_H(z^{-1}) = \frac{I_H(z^{-1})}{D(z^{-1})} \end{aligned} \qquad (5)$$

where $X_1(z^{-1}) = \sum_{r=0}^{n_{x_1}} x_{1_r} z^{-r}$, with $X_1 = A$, B, C, D, I_G, I_H and $x_1 = a$, b, c, d, i_g, i_h, $n_{i_g} = \max(n_a, n_b) - 1$ and $n_{i_h} = \max(n_c, n_d) - 1$. The numerator coefficients i_{g_r} and i_{h_r} of T_G and T_H depend on the initial and final conditions of the experiment and decrease as an $O(N^{-1/2})$ as $N \to \infty$ (Pintelon and Schoukens, 2001). Hence, for N sufficiently large, the transient terms T_G and T_H in (3) can be neglected w.r.t. $G(z_k^{-1})U(k)$ and $H(\Omega_k)E(k)$.

The DFT $E(k)$ of the driving white noise source $e(t)$ has the following properties. Since $e(t)$ is zero mean white (uncorrelated over t) noise, it follows that $E(k)$ (4) is zero mean white (uncorrelated over k) noise with $\mathrm{var}(E(k)) = \mathrm{var}(e(t)) = \sigma^2$ and $\mathcal{E}\{E^2(k)\} = 0$ (= circular complex distributed) (Pintelon and Schoukens, 2001). If $e(t)$ is normally distributed, then $E(k)$ is circular complex normally distributed. If $e(t)$ is independent and identically distributed with existing moments of any order, then $E(k)$ is asymptotically ($N \to \infty$) independent, circular complex normally distributed (see Pintelon and Schoukens, 2001, Lemma 14.24). All these properties motivate the following assumption in the frequency domain.

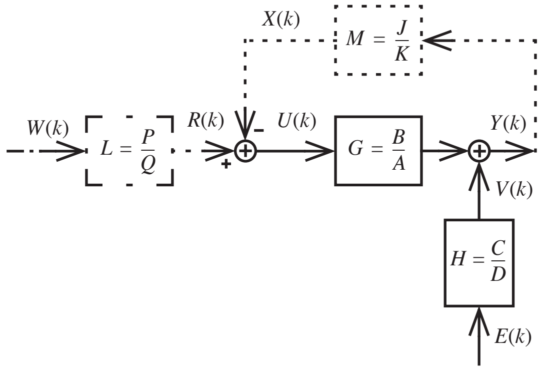

Fig. 1. Identification in open loop (solid lines only), closed loop with known controller (solid and dashed lines only), and closed loop with unknown controller (solid, dashed, and dash-dot lines). G, H, M, and L are the plant, the noise, the controller, and the signal transfer functions.

Assumption 1 (Plant/noise model)
The observed plant input $U(k)$ and plant output $Y(k)$ frequency domain data are related as

$$Y(k) = G(z_k^{-1})U(k) + H(z_k^{-1})E(k) \qquad (6)$$

where $G(z^{-1})$ and $H(z^{-1})$ are defined in (5). $E(k)$ is independent (over k), circular complex ($\mathcal{E}\{E^2(k)\} = 0$) normally distributed noise, with zero mean, and variance $\lambda = \mathcal{E}\{|E(k)|^2\}$. □

3. CLOSED LOOP FRAMEWORK

The closed loop set up of Fig. 1 is defined by the following assumptions.

Assumption 2 (Closed loop)
The input/output data $U(k)$, $Y(k)$ are related to the reference signal $R(k)$ and the driving white noise source $E(k)$ as

$$\begin{aligned} U(k) &= \frac{R(k)}{1 + G(z_k^{-1})M(z_k^{-1})} - \frac{M(z_k^{-1})H(z_k^{-1})E(k)}{1 + G(z_k^{-1})M(z_k^{-1})} \\ Y(k) &= \frac{G(z_k^{-1})R(k)}{1 + G(z_k^{-1})M(z_k^{-1})} + \frac{H(z_k^{-1})E(k)}{1 + G(z_k^{-1})M(z_k^{-1})} \end{aligned} \qquad (7)$$

where $G(z^{-1})$, $H(z^{-1})$ and $M(z^{-1})$ are rational transfer functions in z^{-1}. □

Assumption 3 (Independence reference signal and process noise)
The reference signal $R(k)$ is independent of the process noise $V(k)$. □

It will be shown that the maximum likelihood estimator needs the controller knowledge if only a part of the frequency grid is considered (see Sections 4 and 5). If the controller or the reference signal are known, then the closed loop problem can be reformulated to an equivalent open loop problem. (see Section 4). If the controller and the reference signal are unknown, then the controller must be estimated (see Section 5). The reference signal acts as then as a disturbance for the identification of the controller, and it should be modelled as filtered white noise (see Fig. 1). Similarly to the process noise and the plant model (see Section 2) the reference signal $R(k)$ and the output of the controller $X(k)$ are written as

$$\begin{aligned} R(k) &= L(z_k^{-1})W(k) + T_L(z_k^{-1}) \\ X(k) &= M(z_k^{-1})Y(k) + T_M(z_k^{-1}) \end{aligned} \tag{8}$$

where the signal $L(z^{-1}) = P(z^{-1})/Q(z^{-1})$, the signal transient $T_L(z^{-1}) = I_L(z^{-1})/Q(z^{-1})$, the controller $M(z^{-1}) = J(z^{-1})/K(z^{-1})$, and the controller transient $T_M(z^{-1}) = I_M(z^{-1})/K(z^{-1})$ transfer functions are rational forms in z^{-1}.

4. IDENTIFICATION IN CLOSED LOOP WITH KNOWN CONTROLLER

4.1 Maximum likelihood cost function

Consider the parametric models $G(z^{-1}, \theta)$ and $H(z^{-1}, \theta)$ (5), with $\theta = [a^T, b^T, c^T, d^T]^T$

$$a^T = [a_0, a_1, \ldots, a_{n_a}], \ldots, d^T = [d_0, \ldots, d_{n_d}] \tag{9}$$

and assume that the frequency domain data $U(k)$, $Y(k)$ is available at frequencies $k = 1, 2, \ldots, F$, covering a part of the whole frequency range $[0, f_s/2]$. Assume furthermore that the controller is known.

Assumption 4 (Known controller)
The controller transfer function $M_0(z^{-1})$ is known. □

Theorem 1 (Log-likelihood - known controller): Under Assumptions 1-4 the negative Gaussian log-likelihood function is, within a constant, given by

$$\sum_{k=1}^{F} \log(\lambda |S(z_k^{-1}, \theta)|^2) + \frac{1}{\lambda}\sum_{k=1}^{F} |\varepsilon_G(z_k^{-1}, \theta)|^2 \tag{10}$$

with $\lambda = \mathrm{var}(E(k)) = \mathcal{E}\{|E(k)|^2\}$, and

$$\begin{aligned} S(z^{-1}, \theta) &= H(z^{-1}, \theta)/(1 + G(z^{-1}, \theta)M_0(z^{-1})) \\ \varepsilon_G(z_k^{-1}, \theta) &= (Y(k) - G(z_k^{-1}, \theta)U(k))/H(z_k^{-1}, \theta) \end{aligned} \tag{11}$$

At DC ($f_k = 0$) and Nyquist ($f_k = f_s/2$) the sums in (10) are multiplied by 1/2.

Proof: see Appendix I. □

Notes: (i) For $M_0 = 0$ (10) reduces to the open loop result in Ljung (1999) on p. 230. (ii) Eq. (10) is mentioned in McKelvey (2000). (iii) For frequency sets covering uniformly the unit circle the first sum in eq. (10) with $\lambda = 1$ is asymptotically ($F \to \infty$) zero, and the ML-solution reduces to the classical time-domain result. However, since the plant and noise models are identified at a particular frequency band (part of the whole frequency range $[0, f_s/2]$), the first sum in (10) is not zero. It explains why the controller must be known.

4.2 Maximum likelihood estimator

The parametric models $G(z^{-1}, \theta)$ and $H(z^{-1}, \theta)E(k)$ in (6) are overparametrized and, therefore, θ should be constrained. According to the particular model structure, one (OE), two (ARMA, ARMAX), or three (BJ) parameter constraints are needed. Since the cost function (10) contains exactly the same parameter ambiguities as $G(z^{-1}, \theta)$ and $H(z_k^{-1}, \theta)E(k)$ in (6), the estimated models $G(z^{-1}, \hat{\theta})$ and $\hat{\lambda}^{1/2}H(z^{-1}, \hat{\theta})$, with $\hat{\theta}$, $\hat{\lambda}$ the minimizers of (10), are independent of the particular parameter constraint(s) chosen (Pintelon and Schoukens, 2001).

Since cost function (10), is a function of $|H(z^{-1}, \theta)|^2$, no distinction can be made between noise models which only differ in poles and/or zeros that are mirrored w.r.t. the unit circle. This global identifiability problem is avoided by restricting the allowable poles/zeros positions of the noise model to the stable region of the z-domain.

Assumption 5 (Constraint noise model)
$H^{-1}(z^{-1}, \theta)$ is a stable transfer function. The poles of $H(z^{-1}, \theta)$ that are not in common with $G(z^{-1}, \theta)$ are stable.

Theorem 2 (ML estimator - known controller): Under Assumptions 1-5 the Gaussian maximum likelihood (ML) estimator $\hat{\theta}(Z)$ of the plant and noise model parameters minimizes

$$\begin{aligned} V_F(\theta, Z) &= F^{-1}\sum_{k=1}^{F} |\varepsilon_G(z_k^{-1}, \theta)g_F(\theta)|^2 \\ g_F(\theta) &= \exp\left(F^{-1}\sum_{k=1}^{F} \log S(z_k^{-1}, \theta)\right) \end{aligned} \tag{12}$$

w.r.t. θ. $S(z^{-1}, \theta)$, $\varepsilon_G(z_k^{-1}, \theta)$ are defined in (11).

Proof: Calculating the derivative of (10) w.r.t. λ gives

$$\lambda(\theta) = F^{-1}\sum_{k=1}^{F}\left|\varepsilon_G(z_k^{-1}, \theta)\right|^2 \quad (13)$$

Eliminating λ in (10) using (13) and taking the exponential function gives (12) within a θ-independent constant. □

As a result the minimizer of (12) can be calculated in a numerical stable way via the iterative Newton-Gauss and Levenberg-Marquardt methods. Theorem 2 describes the ML estimator starting from frequency domain data $U(k)$, $Y(k)$ (Assumptions 1-5) described by model (6). If the raw data are time domain signals then (6) is asymptotically (number of time domain samples $N \to \infty$) valid. To improve the finite sample behaviour of the estimate $\hat{\theta}(Z)$, model (6) is replaced by (3). This results in the same cost function (12) where the prediction error $\varepsilon_G(z_k^{-1}, \theta)$ is replaced by

$$\begin{aligned}\varepsilon_G(z_k^{-1}, \theta) = H^{-1}(z_k^{-1}, \theta)[Y(k) - G(z_k^{-1}, \theta)U(k) \\ - T_G(z_k^{-1}, \theta) - T_H(z_k^{-1}, \theta)]\end{aligned} \quad (14)$$

5. IDENTIFICATION IN CLOSED LOOP WITH UNKNOWN CONTROLLER

5.1. Maximum likelihood cost function

From Section 4 it follows that if the controller is unknown, it must be estimated to avoid a bias error in the plant model. When identifying simultaneously the plant and the controller, $Y(k)$ is a noisy observation of the true plant output, and $U(k)$ is a noisy observation of the true controller output. Hence, similarly to the identification of the plant, the following assumptions are needed to identify the controller.

Assumption 6 (Controller/signal model)
The observed controller input $Y(k)$ and controller output $-U(k)$ frequency domain data are related as

$$-U(k) = M(z_k^{-1})Y(k) - L(z_k^{-1})W(k) \quad (15)$$

where $M(z^{-1})$ and $L(z^{-1})$ are rational functions in z^{-1}. $W(k)$ is independent (over k), circular complex ($\mathcal{E}\{W^2(k)\} = 0$) normally distributed noise, with zero mean, and variance μ. □

Consider now the parametric transfer function models $G(z^{-1}, \theta)$, $H(z^{-1}, \theta)$, $L(z^{-1}, \theta)$, and $M(z^{-1}, \theta)$ with

$$\theta = [a^T, b^T, c^T, d^T, j^T, k^T, p^T, q^T] \quad (16)$$

where j, k and p, q are the numerator and denominator coefficients of $L(z^{-1})$ and $M(z^{-1})$ respectively, and assume that the frequency domain data $U(k)$, $Y(k)$ is available at frequencies $k = 1, 2, \ldots, F$.

Theorem 3 (Log-likelihood - unknown controller): Under Assumptions 1-3, 6, the negative Gaussian log-likelihood function is, within a constant, given by

$$\begin{aligned}\sum_{k=1}^{F}\log(\lambda\mu|T(\Omega_k, \theta)|^2) + \frac{1}{\lambda}\sum_{k=1}^{F}|\varepsilon_G(\Omega_k, \theta)|^2 \\ + \frac{1}{\mu}\sum_{k=1}^{F}|\varepsilon_M(\Omega_k, \theta)|^2\end{aligned} \quad (17)$$

with $\lambda = \mathrm{var}(E(k))$, $\mu = \mathrm{var}(W(k))$,

$$\begin{aligned}T(z^{-1}, \theta) &= \frac{H(z^{-1}, \theta)L(z^{-1}, \theta)}{1 + G(z^{-1}, \theta)M(z^{-1}, \theta)} \\ \varepsilon_M(z_k^{-1}, \theta) &= (M(z_k^{-1}, \theta)Y(k) + U(k))/L(z_k^{-1}, \theta)\end{aligned} \quad (18)$$

and where $\varepsilon_G(z_k^{-1}, \theta)$ is defined in (11). At DC ($f_k = 0$) and Nyquist ($f_k = f_s/2$) the sums in (17) are multiplied by 1/2.

Proof: see Appendix II. □

Notes (i) The identification of the plant/noise models is coupled with the identification the controller/signal models through the transfer function $T(z^{-1}, \theta)$ (18). (ii) For frequency sets covering uniformly the unit circle the first sum in eq. (17) with $\lambda\mu = 1$ is asymptotically ($F \to \infty$) zero, and the ML-solution reduces to the classical joint input-output approach (Ljung, 1999; Söderström and Stoica, 1989).

5.2. Maximum likelihood estimator

Following exactly the same lines of Section 4.2 we obtain the following result.

Assumption 7 (Constraint signal model)
The signal model $L(z^{-1}, \theta)$ is a stable and inversely stable transfer function.

Theorem 4 (ML estimator - unknown controller): Under Assumptions 1-7 the Gaussian maximum likelihood (ML) estimator $\hat{\theta}(Z)$ of the plant, noise, controller and signal model parameters minimizes

$$\begin{aligned}V_F(\theta, Z) &= |h_F(\theta)|^2 V_G(\theta, Z)V_M(\theta, Z) \\ V_G(\theta, Z) &= F^{-1}\sum_{k=1}^{F}|\varepsilon_G(z_k^{-1}, \theta)|^2 \\ V_M(\theta, Z) &= F^{-1}\sum_{k=1}^{F}|\varepsilon_M(z_k^{-1}, \theta)|^2 \\ h_F(\theta) &= \exp(F^{-1}\sum_{k=1}^{F}\log T(z_k^{-1}, \theta))\end{aligned} \quad (19)$$

w.r.t. θ. $\varepsilon_G(z_k^{-1}, \theta)$, $\varepsilon_M(z_k^{-1}, \theta)$ and $T(z^{-1}, \theta)$ are defined in (11) and (18). At DC ($f_k = 0$) and Nyquist ($f_k = f_s/2$) the sums in (19) are multiplied by 1/2.

Proof: follow exactly the same lines of the proof of Theorem 2. □

In case of time domain data ε_G and ε_M in (19) are extended with the plant, noise, controller and signal transient terms (follow the lines of Section 4.2).

6. SIMULATIONS

The simulation set up consists of a second-order plant model $G_0(z^{-1})$ ($n_a = 2$, $n_b = 1$), and a second-order monic noise model $H_0(z^{-1})$ ($n_c = n_d = 2$), in a unity feedback setting ($M_0 = 1$). The reference signal $r(t)$ and the driving white noise source $e(t)$ are white Gaussian DT noise processes. The input/output DFT spectra $U(k)$ and $Y(k)$ are calculated from $N = 4000$ time domain samples, and DFT lines $k = 199, 200, \ldots, 1199$ ($F = 1001$) are used to identify the plant and noise model parameters. For each of the hundred runs of the Monte-Carlo simulation the three following estimates are calculated:

1. ML estimate (12) with $M_0 = 0$ (we pretend that the data was gathered in open loop),
2. ML estimate (12) with $M_0 = 1$ (true controller model is used),
3. ML estimate (19) (the unknown controller is identified).

The following model structure is used $n_a = 2$, $n_b = 1$, $b_0 = 0$, $n_{i_g} = 1$, $n_c = n_d = 2$, $n_{i_h} = 1$, $n_j = n_k = 0$, and $n_p = n_q = 0$. From Fig. 2 it can be seen that the plant and noise model estimates with $M_0 = 0$ are biased (the difference between the true model and the mean estimated model is significantly larger than the 95% uncertainty bound of the mean estimate), while no bias can be detected in those with the known $M_0 = 1$ and the estimated controller. The mean estimated value of the controller transfer function equals 0.99995 with a standard deviation of 8×10^{-4}.

7. CONCLUSION

A surprising consequence of Theorem 1 is that the knowledge of the controller contributes to the knowledge of the plant and noise models ($M \neq M_0$ in (12) leads to biased estimates), which is not the case for the time domain prediction error method (see Ljung, 1999). The apparent contradiction can be explained by the fact that cutting out a part of the unit circle corresponds to non-causal filtering in the time

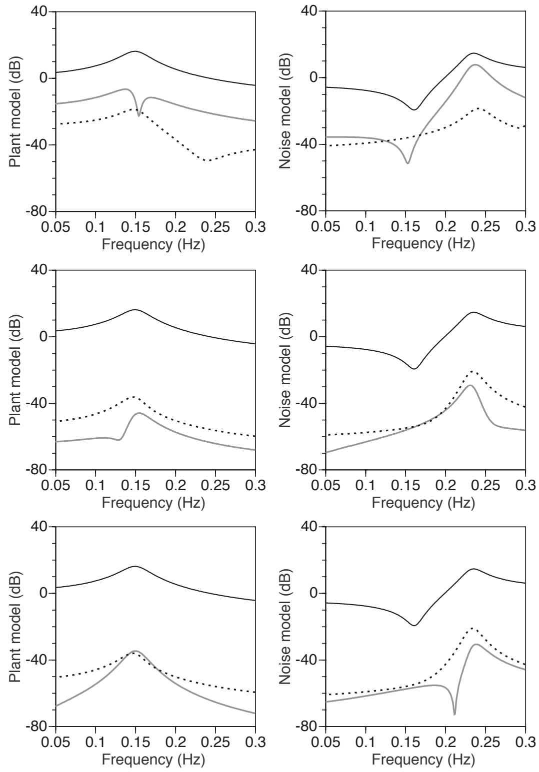

Fig. 2. Closed loop simulation ($f_s = 1$ Hz): comparison between the true and the mean estimated models. Left figures: plant models. Right figures: noise models. Top row: open loop estimates, (12) with controller transfer function set to zero. Middle row: closed loop estimates with known controller, (12) with true controller transfer function. Bottom row: closed loop estimates with unknown controller (19). Black line: true model. Gray line: complex difference between true and mean estimated model. Dashed line: 95% confidence bound mean estimate.

domain (e.g. convolution with a sinc-function). The latter invalidates the classical construction of the likelihood function based on time domain data captured in feedback (Caines, 1988). Indeed, the construction of the likelihood function relies on the basic assumption that the plant input $u(t)$ does not depend on future samples of the process noise $v(t)$. This assumption is violated for non-causal processes $v(t)$.

From Theorem 4 it follows that identifying the plant characteristics in closed loop without knowledge of the controller is much more complex than in case the controller is known. Indeed, four transfer functions must be estimated when the controller is unknown: beside the plant and noise characteristics also the controller and signal transfer functions. Hence, it is strongly recommended to store also the reference signal in a feedback experiment.

Replacing everywhere z_k^{-1} by $s_k = j\omega_k$ and using the concept of band-limited (BL) white continuous-time noise within a BL-measurement set up (see Pintelon *et al.*, 2004), all the results of this paper remain valid for the continuous-time case.

APPENDIX I

Under Assumptions 1-4 $Y(k)$ (6) is independent (over k), circular complex normally distributed. To construct the likelihood function ($U(k)$ is known exactly) it is sufficient to calculate the mean and variance of $Y(k)$ given the model parameters θ and the variance λ of the driving white noise source. In closed loop the process noise $V(k)$ is correlated with the input of the plant $U(k)$ (Assumption 2), and is independent of the reference signal $R(k)$ (Assumption 3). Therefore, the expected values in the mean and variance calculation of $Y(k)$ should be conditioned on $R(k)$. The latter is known since the controller $M_0(\Omega)$ is known (Assumption 4) and since $Y(k)$ and $U(k)$ are known exactly (no measurement errors). Using (7) we find

$$\begin{aligned} \mathcal{E}\{Y(k)|R(k), \theta, \lambda\} &= \frac{G(z_k^{-1}, \theta)R(k)}{1 + G(z_k^{-1}, \theta)M_0(z_k^{-1})} \\ &\equiv Y(k, \theta) \\ \mathrm{var}(Y(k)|R(k), \theta, \lambda) &= \mathrm{var}\left(\frac{H(z_k^{-1}, \theta)E(k)}{1 + G(z_k^{-1}, \theta)M_0(z_k^{-1})}\right) \\ &= \lambda\left|S(z_k^{-1}, \theta)\right|^2 \end{aligned} \tag{20}$$

where $S(z^{-1}, \theta)$ is defined in (11). Hence,

$$f_{Y(k)}(Y(k)|R(k), \theta, \lambda) = \begin{cases} \dfrac{e^{-\frac{|Y(k) - Y(k, \theta)|^2}{\lambda|S(z_k^{-1}, \theta)|^2}}}{\pi\lambda\left|S(z_k^{-1}, \theta)\right|^2}, & k \neq 0, \dfrac{N}{2} \\ \dfrac{e^{-\frac{|Y(k) - Y(k, \theta)|^2}{2\lambda|S(z_k^{-1}, \theta)|^2}}}{\sqrt{2\pi\lambda\left|S(z_k^{-1}, \theta)\right|^2}}, & k = 0, \dfrac{N}{2} \end{cases} \tag{21}$$

because $Y(k)$ is real at DC ($k = 0$) and Nyquist ($k = N/2$) and circular complex elsewhere. Using

$$f_Y(Y|R, \theta, \lambda) = \prod_{k=1}^{F} f_{Y(k)}(Y(k)|R(k), \theta, \lambda) \tag{22}$$

(independence of $Y(k)$ over k) with f_Y the likelihood of the output data $Y(1)$, $Y(2)$, …, $Y(F)$, and elaborating the exponent in (21),

$$\frac{Y(k) - Y(k, \theta)}{S(z_k^{-1}, \theta)} = \frac{Y(k) - G(z_k^{-1}, \theta)U(k)}{H(z_k^{-1}, \theta)} \tag{23}$$

finally proves (10). The factor 1/2 at DC and Nyquist in the sums of (10) stems from (21).

APPENDIX II

Under Assumptions 1-3, 6 the vector $Z(k) = [Y(k), U(k)]^T$ is independent (over k), circular complex normally distributed for any frequency different of DC ($f_k = 0$) and Nyquist ($f_k = f_s/2$), while it is normally distributed at DC and Nyquist. Using (7) and (15) we find for any k

$$\begin{aligned} &\mathcal{E}\{Z(k)|\theta, \lambda, \mu\} = 0 \\ &C_{Z(k)} = \mathcal{E}\{Z^H(k)Z(k)|\theta, \lambda, \mu\} \\ &\quad = \frac{1}{|1 + GM|^2}\begin{bmatrix} \mu|GL|^2 + \lambda|H|^2 & \mu GL - \lambda\overline{M}|H|^2 \\ \mu\overline{GL} - \lambda M|H|^2 & \mu|L|^2 + \lambda|HM|^2 \end{bmatrix} \end{aligned} \tag{24}$$

where $X_1 = X_1(z_k^{-1}, \theta)$, $X_1 = G$, H, M, and L. After some calculations we get

$$\begin{aligned} \det C_{Z(k)} &= \lambda\mu\left|T(z_k^{-1}, \theta)\right|^2 \\ Z^H(k)C_{Z(k)}^{-1}Z(k) &= \frac{\left|\varepsilon_G(z_k^{-1}, \theta)\right|^2}{\lambda} + \frac{\left|\varepsilon_M(z_k^{-1}, \theta)\right|^2}{\mu} \end{aligned} \tag{25}$$

where $\varepsilon_G(z_k^{-1}, \theta)$, $T(z_k^{-1}, \theta)$, and $\varepsilon_M(z_k^{-1}, \theta)$ are defined in (11) and (18) respectively. The rest of the proof follows the same lines of Appendix I (use (21) and (22)).

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