INTERCONNECTION OF THE KRONECKER FORM AND SPECIAL COORDINATE BASIS OF GENERAL MULTIVARIABLE LINEAR SYSTEMS

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Abstract: This paper establishes a straightforward interconnection between the Kronecker canonical form and the special coordinate basis of linear systems. Such an interconnection enables the computation of the Kronecker canonical form, and as a by-product, the Smith form, of the system matrix of general multivariable time-invariant linear systems. The overall procedure involves the transformation of a given system in the state-space description into the special coordinate basis, which is capable of explicitly displaying all the system structural properties, such as finite and infinite zero structures, as well as system invertibility structures. The computation of the Kronecker canonical form and Smith form of the system matrix is rather simple and straightforward once the given system is put under the special coordinate basis. The procedure is applicable to proper systems and singular systems.

Keywords: Kronecker canonical form, Smith form, singular systems.

1. INTRODUCTION

The Kronecker canonical form has been extensively used in the literature to capture the invariant indices and structural properties of linear systems. It is now well understood that the system structural properties play a crucial role in the design of control systems. In this paper, we consider a multivariable linear time-invariant system characterized by

$$\Sigma \ : \ \begin{cases} E\ \dot{x} = A\ x + B\ u, \\ \quad y = C\ x + D\ u, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are respectively the state, input and output of the given system. $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$ are constant matrices. Σ is said to be singular if rank$(E) < n$. Otherwise, it is said to be a proper system. It is well understood in the literature that the structural properties of Σ, such as the finite and infinite zero structures, as well as the system invertibility structures, can be fully captured by its (Rosenbrock) system matrix defined as follows (see, *e.g.*, Rosenbrock (1970)):

$$P_\Sigma(s) = \begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix}. \tag{2}$$

We recall that two pencils $sM_1 - N_1$ and $sM_2 - N_2$ of dimensions $m \times n$ are said to be equivalent if there exist constant nonsingular matrices $\tilde{P}$ and $\tilde{Q}$ of appropriate dimensions such that

$$\tilde{Q}(sM_1 - N_1)\tilde{P} = sM_2 - N_2. \tag{3}$$

It was shown in Gantmacher (1959) that any pencil $sM - N$ can be reduced to a canonical quasi-diagonal form, which is given by

$$\tilde{Q}(sM - N)\tilde{P} = \text{blkdiag}\big\{sI - J, \\ L_{l_1}, \dots, L_{l_{p_b}}, R_{r_1}, \dots, R_{r_{m_c}}, I - sH, 0\big\} \tag{4}$$

In the context of this paper, we will focus on

$$sM - N = \begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix} = P_\Sigma(s), \qquad (5)$$

the (Rosenbrock) system matrix pencil associated with Σ. In (4), the last term, *i.e.*, 0, is corresponding to the case when there are redundant columns or rows associated with the input matrices and measurement matrices. J is in Jordan canonical form, and $sI - J$ has the following $\Sigma_{i=1}^{\delta} d_i$ pencils as its diagonal blocks,

$$sI_{m_{i,j}} - J_{m_{i,j}}(\beta_i) := \begin{bmatrix} s-\beta_i & -1 & & \\ & \ddots & \ddots & \\ & & s-\beta_i & -1 \\ & & & s-\beta_i \end{bmatrix}, \quad (6)$$

$j = 1, 2, \ldots, d_i$, $i = 1, 2, \ldots, \delta$. L_{l_i}, $i = 1, 2, \cdots, p_{\mathrm{b}}$, is an $(l_i + 1) \times l_i$ bidiagonal pencil, *i.e.*,

$$L_{l_i} := \begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix}. \qquad (7)$$

R_{r_i}, $i = 1, 2, \cdots, m_{\mathrm{c}}$, is an $r_i \times (r_i + 1)$ bidiagonal pencil, *i.e.*,

$$R_{r_i} := \begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix}. \qquad (8)$$

Finally, H is nilpotent and in Jordan canonical form, and $I - sH$ has the following d pencils as its diagonal blocks,

$$I_{n_j+1} - sJ_{n_j+1}(0) := \begin{bmatrix} 1 & -s & & \\ & \ddots & \ddots & \\ & & 1 & -s \\ & & & 1 \end{bmatrix}, \qquad (9)$$

$j = 1, 2, \ldots, d$. Then, $\{(s - \beta_i)^{m_{i,j}}, j = 1, 2, \ldots, d_i\}$ are finite elementary divisors at β_i, $i = 1, 2, \ldots, \delta$. The index sets $\{r_1, r_2, \ldots, r_{m_{\mathrm{c}}}\}$ and $\{l_1, l_2, \ldots, l_{p_{\mathrm{b}}}\}$ are right and left minimal indices, respectively. Lastly, $\{(1/s)^{n_j}, j = 1, 2, \ldots, d\}$ are the infinite elementary divisors. The definition of structural invariants of Σ is based on the invariant indices of its system pencil. In particular, the right and left invertibility indices are respectively the right and left minimal indices of the system pencil, the finite and infinite zero structures of the given system are related to the finite and infinite elementary divisors of the system pencil. We note that these invariant indices are related to the invariant lists of Morse (1973) as well. To be precise, the finite elementary divisors are related to the list $\mathbf{I}_1$ of Morse, the right and left minimal indices are respectively corresponding to the $\mathbf{I}_2$ and $\mathbf{I}_3$ lists of Morse, and finally, the infinite elementary divisors are related to the $\mathbf{I}_4$ list of Morse.

The Smith form of the system matrix is another way to capture the invariant zero structure of the given system Σ. We recall the definition of the Smith form from the classical text of Rosenbrock and Storey (1970). Given a polynomial matrix $A(s)$, there exist unimodular transformations $M(s)$ and $N(s)$ such that

$$S(s) = M(s)A(s)N(s) = \begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix}, \qquad (10)$$

where

$$D(s) = \mathrm{diag}\Big\{p_1(s), p_2(s), \cdots, p_r(s)\Big\}, \qquad (11)$$

and where each $p_i(s)$, $i = 1, 2, \cdots, r$, is a monic polynomial and $p_i(s)$ is a factor of $p_{i+1}(s)$, $i = 1, 2, \cdots, r-1$. Note that a unimodular matrix is a square polynomial matrix whose determinant is a nonzero constant. $S(s)$ of (10) is called the Smith canonical form or Smith form of $A(s)$. We will show in this paper that it is straightforward to obtain the Smith form of $P_\Sigma(s)$ once it is transformed into the Kronecker canonical form.

Traditionally, the computation of the Kronecker canonical form was carried out through certain iterative reduction schemes (see, for example, Beelen and Dooren (1988), Gantmacher (1959), Lin (1988), Puerta *et al.* (2002) and Dooren (1979),), of which some were based on the reduction of the system matrix to a generalized Schur form (see, for example, Demmel and Kagstrom (1993*a*), Demmel and Kagstrom (1993*b*)). The main objective of this paper is to establish a straightforward interconnection between the Kronecker canonical form and the special coordinate basis of linear systems of Sannuti and Saberi (1987). We will show that it is simple to derive a constructive procedure for computing the Kronecker canonical form, and as a by-product, the Smith form, of $P_\Sigma(s)$ by utilizing the special coordinate basis technique, which was originally proposed by Sannuti and Saberi (1987), and was recently completed by Chen (1998), in which all the system structural properties of the special coordinate basis were rigorously justified. The software realization of the special coordinate basis and other related decomposition techniques required is readily available in Lin *et al.* (2004). Thus, the additional cost for computing the canonical forms mentioned above is very minimal.

The rest of the paper is organized as follows: In Section 2, we present the main results of this paper, *i.e.*, the computational procedures for the Kronecker canonical form and Smith form of the system matrix, $P_\Sigma(s)$. The interconnection of the Kronecker canonical form and the special coordinate basis will be clearly displayed in the procedure. The results will be illustrated by a numerical example in Section 3. Finally, some concluding remarks will be drawn in Section 4.

2. COMPUTATION OF KRONECKER AND SMITH FORMS OF THE SYSTEM MATRIX

Before proceeding to present our main results, we first show that the computation of the Kronecker canonical form and Smith form of the system pencil of singular systems can be done by converting the singular system into an auxiliary proper system. This can be done as follows. Without loss of generality, we assume that E is in the form of,

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tag{12}$$

and thus A, B and C can be partitioned accordingly as

$$A=\begin{bmatrix} A_{\rm nn} & A_{\rm ns} \\ A_{\rm sn} & A_{\rm ss} \end{bmatrix}, \quad B=\begin{bmatrix} B_{\rm n} \\ B_{\rm s} \end{bmatrix}, \quad C=[\, C_{\rm n} \quad C_{\rm s}\,]. \tag{13}$$

Rewriting the system pencil of (5) as

$$P_\Sigma(s)=\left[\begin{array}{c|cc} sI-A_{\rm nn} & -A_{\rm ns} & -B_{\rm n} \\ \hline -A_{\rm sn} & -A_{\rm ss} & -B_{\rm s} \\ C_{\rm n} & C_{\rm s} & D \end{array}\right]=\begin{bmatrix} sI-A_{\rm x} & -B_{\rm x} \\ C_{\rm x} & D_{\rm x} \end{bmatrix}, \tag{14}$$

it is simple to see that the invariant indices of Σ are equivalent to those of a proper system characterized by $(A_{\rm x}, B_{\rm x}, C_{\rm x}, D_{\rm x})$. Thus, without loss of generality, we focus on the computation of the Kronecker form and Smith form of the system matrix of a proper system characterized by

$$\Sigma \;:\; \begin{cases} \dot{x} = A\,x + B\,u, \\ y = C\,x + D\,u, \end{cases} \tag{15}$$

i.e., the following matrix pencil,

$$P_\Sigma(s) = \begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix}, \tag{16}$$

throughout the reminder of this manuscript. We next recall that the Kronecker canonical form of the system matrix of Σ, *i.e.*, $P_\Sigma(s)$, is invariant under nonsingular state, input and output transformations, $\Gamma_{\rm s}$, $\Gamma_{\rm i}$ and $\Gamma_{\rm o}$, and is invariant under any state feedback and output injection. Such a fact follows directly from the following manipulation:

$$\begin{aligned} UP_\Sigma(s)V &= \begin{bmatrix} \Gamma_{\rm s}^{-1} & -\tilde{K}\Gamma_{\rm o}^{-1} \\ 0 & \Gamma_{\rm o}^{-1} \end{bmatrix} \begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} \begin{bmatrix} \Gamma_{\rm s} & 0 \\ \Gamma_{\rm i}\tilde{F} & \Gamma_{\rm i} \end{bmatrix} \\ &= \begin{bmatrix} sI-(\tilde{A}+\tilde{B}\tilde{F}+\tilde{K}\tilde{C}+\tilde{K}\tilde{D}\tilde{F}) & -(\tilde{B}+\tilde{K}\tilde{D}) \\ \tilde{C}+\tilde{D}\tilde{F} & \tilde{D} \end{bmatrix} \\ &= \begin{bmatrix} sI-A_{\rm KF} & -B_{\rm K} \\ C_{\rm F} & \tilde{D} \end{bmatrix}, \end{aligned} \tag{17}$$

where $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ is the transformed system and is given by

$$\tilde{A} = \Gamma_{\rm s}^{-1}A\Gamma_{\rm s}, \quad \tilde{B} = \Gamma_{\rm s}^{-1}B\Gamma_{\rm i}, \tag{18}$$

and

$$\tilde{C} = \Gamma_{\rm o}^{-1}C\Gamma_{\rm s}, \quad \tilde{D} = \Gamma_{\rm o}^{-1}D\Gamma_{\rm i}, \tag{19}$$

$\tilde{F}$ and $\tilde{K}$ are respectively the state feedback and output injection gain matrices under the coordinate of the transformed system, and finally, $\Sigma_{\rm KF}$ characterized by the quadruple $(A_{\rm KF}, B_{\rm K}, C_{\rm F}, \tilde{D})$ is the resulting transformed system under the state feedback and output injection laws.

We are now ready to show that the Kronecker canonical form of $P_\Sigma(s)$ can be obtained neatly through the special coordinate basis of Σ. The following is a step-by-step algorithm that generates the required nonsingular transformations U and V for the canonical form:

STEP KCF.1. *Computation of the special coordinate basis of* Σ.

Apply the result of Sannuti and Saberi (1987) (see also Chen (2000)) to find nonsingular state, input and output transformations, $\Gamma_{\rm s}\in\mathbb{C}^{n\times n}$, $\Gamma_{\rm i}\in\mathbb{R}^{m\times m}$ and $\Gamma_{\rm o}\in\mathbb{R}^{p\times p}$, such that the given system Σ of (15) is transformed into the special coordinate basis as given in Theorem 2.4.1 of Chen (2000) or in the following compact form:

$$\tilde{A} = \Gamma_{\rm s}^{-1}A\Gamma_{\rm s} = B_0C_0 + \begin{bmatrix} A_{\rm aa} & L_{\rm ab}C_{\rm b} & 0 & L_{\rm ad}C_{\rm d} \\ 0 & A_{\rm bb} & 0 & L_{\rm bd}C_{\rm d} \\ B_{\rm c}E_{\rm ca} & L_{\rm cb}C_{\rm b} & A_{\rm cc} & L_{\rm cd}C_{\rm d} \\ B_{\rm d}E_{\rm da} & B_{\rm d}E_{\rm db} & B_{\rm d}E_{\rm dc} & A_{\rm dd}^*+B_{\rm d}E_{\rm dd}+L_{\rm dd}C_{\rm d} \end{bmatrix},$$

$$\tilde{B} = \Gamma_{\rm s}^{-1}B\Gamma_{\rm i} = [\,B_0 \quad B_1\,] = \begin{bmatrix} B_{\rm 0a} & 0 & 0 \\ B_{\rm 0b} & 0 & 0 \\ B_{\rm 0c} & 0 & B_{\rm c} \\ B_{\rm 0d} & B_{\rm d} & 0 \end{bmatrix},$$

$$\tilde{C} = \Gamma_{\rm o}^{-1}C\Gamma_{\rm s} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} C_{\rm 0a} & C_{\rm 0b} & C_{\rm 0c} & C_{\rm 0d} \\ 0 & 0 & 0 & C_{\rm d} \\ 0 & C_b & 0 & 0 \end{bmatrix},$$

and

$$\tilde{D} = \Gamma_{\rm o}^{-1}D\Gamma_{\rm i} = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $A_{\rm dd}^*\in\mathbb{R}^{n_{\rm d}\times n_{\rm d}}$, $B_{\rm d}\in\mathbb{R}^{n_{\rm d}\times m_{\rm d}}$ and $C_{\rm d}\in\mathbb{R}^{m_{\rm d}\times n_{\rm d}}$ have the following forms,

$$A_{\rm dd}^* = \text{blkdiag}\left\{A_{q_1},\cdots,A_{q_{m_{\rm d}}}\right\}, \tag{20}$$

$$B_{\rm d} = \text{blkdiag}\left\{B_{q_1},\cdots,B_{q_{m_{\rm d}}}\right\}, \tag{21}$$

and

$$C_{\rm d} = \text{blkdiag}\left\{C_{q_1},\cdots,C_{q_{m_{\rm d}}}\right\}, \tag{22}$$

with A_{q_i}, B_{q_i} and C_{q_i}, $i=1,2,\cdots,m_{\rm d}$, being given as follows:

$$A_{q_i}=\begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i}=\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [1 \;\; 0]. \tag{23}$$

Also, we assume that $A_{\rm aa}\in\mathbb{C}^{n_{\rm a}\times n_{\rm a}}$ is already in the Jordan canonical form, *i.e.*,

$$A_{\rm aa} = \text{blkdiag}\left\{J_{\rm a,1}, J_{\rm a,2},\cdots,J_{{\rm a},k}\right\}, \tag{24}$$

where $J_{\mathrm{a},i}$, $i = 1, 2, \cdots, k$, are some $n_i \times n_i$ Jordan blocks:

$$J_{\mathrm{a},i} = \mathrm{diag}\Big\{\alpha_i, \alpha_i, \cdots, \alpha_i\Big\} + \begin{bmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{bmatrix}, \quad (25)$$

and $(A_{\mathrm{bb}}, C_{\mathrm{b}})$, with $A_{\mathrm{bb}} \in \mathbb{R}^{n_{\mathrm{b}} \times n_{\mathrm{b}}}$ and $C_{\mathrm{b}} \in \mathbb{R}^{p_{\mathrm{b}} \times n_{\mathrm{b}}}$, is in the form of the observability structural decomposition (see, for example, Brunovsky (1970), Theorem 2.3.1 of Chen (2000), and Luenberger (1967) for its dual version), *i.e.*,

$$\begin{aligned} A_{\mathrm{bb}} &= A^*_{\mathrm{bb}} + L_{\mathrm{bb}} C_{\mathrm{b}} \\ &= \mathrm{blkdiag}\Big\{A_{\mathrm{bb},1}, \cdots, A_{\mathrm{bb},p_{\mathrm{b}}}\Big\} + L_{\mathrm{bb}} C_{\mathrm{b}}, \end{aligned} \quad (26)$$

and

$$C_{\mathrm{b}} = \mathrm{blkdiag}\Big\{C_{\mathrm{b},1}, \cdots, C_{\mathrm{b},p_{\mathrm{b}}}\Big\}, \quad (27)$$

with

$$A_{\mathrm{bb},i} = \begin{bmatrix} 0 & I_{l_i-1} \\ 0 & 0 \end{bmatrix}, \quad C_{\mathrm{b},i} = [1 \;\; 0], \quad i = 1, \cdots, p_{\mathrm{b}}. (28)$$

Finally, $(A_{\mathrm{cc}}, B_{\mathrm{c}})$, with $A_{\mathrm{cc}} \in \mathbb{R}^{n_{\mathrm{c}} \times n_{\mathrm{c}}}$ and $B_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}} \times m_{\mathrm{c}}}$, is assumed to be in the form of the controllability structural decomposition, *i.e.*,

$$\begin{aligned} A_{\mathrm{cc}} &= A^*_{\mathrm{cc}} + B_{\mathrm{c}} E_{\mathrm{cc}} \\ &= \mathrm{blkdiag}\Big\{A_{\mathrm{cc},1}, \cdots, A_{\mathrm{cc},m_{\mathrm{c}}}\Big\} + B_{\mathrm{c}} E_{\mathrm{cc}}, \end{aligned} \quad (29)$$

and

$$B_{\mathrm{c}} = \mathrm{blkdiag}\Big\{B_{\mathrm{c},1}, \cdots, B_{\mathrm{c},m_{\mathrm{c}}}\Big\}, \quad (30)$$

with

$$A_{\mathrm{cc},i} = \begin{bmatrix} 0 & I_{l_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{\mathrm{c},i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad i = 1, \cdots, m_{\mathrm{c}}. (31)$$

STEP KCF.2. *Determination of state feedback and output injection laws.*

Let

$$\tilde{F} = -\begin{bmatrix} C_{0\mathrm{a}} & C_{0\mathrm{b}} & C_{0\mathrm{c}} & C_{0\mathrm{d}} \\ E_{\mathrm{da}} & E_{\mathrm{db}} & E_{\mathrm{dc}} & E_{\mathrm{dd}} \\ E_{\mathrm{ca}} & 0 & E_{\mathrm{cc}} & 0 \end{bmatrix}, \quad (32)$$

and

$$\tilde{K} = -\begin{bmatrix} B_{0\mathrm{a}} & L_{\mathrm{ad}} & L_{\mathrm{ab}} \\ B_{0\mathrm{b}} & L_{\mathrm{bd}} & L_{\mathrm{bb}} \\ B_{0\mathrm{c}} & L_{\mathrm{cd}} & L_{\mathrm{cb}} \\ B_{0\mathrm{d}} & L_{\mathrm{dd}} & 0 \end{bmatrix}. \quad (33)$$

It is straightforward to verify that the resulting Σ_{KF} is characterized by

$$A_{\mathrm{KF}} = \begin{bmatrix} A_{\mathrm{aa}} & 0 & 0 & 0 \\ 0 & A^*_{\mathrm{bb}} & 0 & 0 \\ 0 & 0 & A^*_{\mathrm{cc}} & 0 \\ 0 & 0 & 0 & A^*_{\mathrm{dd}} \end{bmatrix}, \quad B_{\mathrm{K}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_{\mathrm{c}} \\ 0 & B_{\mathrm{d}} & 0 \end{bmatrix}, (34)$$

and

$$C_{\mathrm{F}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{\mathrm{d}} \\ 0 & C_{\mathrm{b}} & 0 & 0 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (35)$$

STEP KCF.3. *Finishing touches.*

It is now simple to verify that the (Rosenbrock) system matrix associated with Σ_{KF} has the following form:

(1) The corresponding term associated with $J_{\mathrm{a},i}$ is given by

$$sI - J_{\mathrm{a},i} = \begin{bmatrix} s-\alpha_i & -1 & & \\ & \ddots & \ddots & \\ & & s-\alpha_i & -1 \\ & & & s-\alpha_i \end{bmatrix} \quad (36)$$

which is already in the format of (6).

(2) The corresponding term associated with the pair $(A_{\mathrm{bb},i}, C_{\mathrm{b},i})$ is given by

$$\begin{bmatrix} -1 & 0 \\ 0 & I_{l_i} \end{bmatrix} \begin{bmatrix} C_{\mathrm{b},i} \\ sI - A_{\mathrm{bb},i} \end{bmatrix} = \begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix} \quad (37)$$

which is in the format of (7).

(3) The corresponding term associated with the pair $(A_{\mathrm{cc},i}, B_{\mathrm{c},i})$ is given by

$$[\, sI - A_{\mathrm{cc},i} \quad -B_{\mathrm{c},i}] = \begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix} \quad (38)$$

which is in the format of (8).

(4) Lastly, the corresponding term associated with $(A_{q_i}, B_{q_i}, C_{q_i})$ is given by

$$\begin{bmatrix} sI - A_{q_i} & -B_{q_i} \\ C_{q_i} & 0 \end{bmatrix} = \begin{bmatrix} s & -1 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & s & -1 & 0 \\ & & & s & -1 \\ 1 & \cdots & 0 & 0 & 0 \end{bmatrix}. \quad (39)$$

Let

$$U_{q_i} = \begin{bmatrix} 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & -1 \end{bmatrix}, V_{q_i} = -\begin{bmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \iddots & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix} \quad (40)$$

Then, we have

$$U_{q_i} \begin{bmatrix} sI - A_{q_i} & -B_{q_i} \\ C_{q_i} & 0 \end{bmatrix} V_{q_i} = \begin{bmatrix} 1 & -s & & \\ & \ddots & \ddots & \\ & & 1 & -s \\ & & & 1 \end{bmatrix} \quad (41)$$

which is now in the format of (9).

The Kronecker canonical form of the system matrix of Σ_{KF}, or equivalently the system matrix of Σ, *i.e.*, (16), can then be obtained by taking into account the additional transformations given in (37) and (40) together with some appropriate permutation transformations. This completes the algorithm.

Next, we proceed to compute the Smith form of the system matrix, $P_\Sigma(s)$. In what follows, we will show that it is also straightforward to obtain the Smith form of $P_\Sigma(s)$ by using the special coordinate basis technique.

STEP SMITH.1. *Determination of the Kronecker form of $P_\Sigma(s)$.*
Utilize the special coordinate basis of Σ to determine the Kronecker canonical form of $P_\Sigma(s)$ as given in the previous algorithm. However, for the computation of the Smith form of $P_\Sigma(s)$, we need not to decompose $A_{\rm aa}$ into the Jordan canonical form, which might involve complex transformations. Instead, we leave $A_{\rm aa}$ as a real-valued matrix. Note that the transformations involved in the Kronecker canonical form are constant and nonsingular, and thus unimodular.

STEP SMITH.2. *Determination of unimodular transformations.*

(1) Using the procedure given in the proof of Theorem 7.4 in Chapter 3 of Rosenbrock and Storey (1970), it is straightforward to show that the term $sI - J_{{\rm a},i}$ in (36) can be deduced to the following Smith form:

$$(sI-J_{{\rm a},i}) \rightleftharpoons \text{diag}\Big\{\overbrace{1,\cdots,1}^{n_i-1},(s-\alpha_i)^{n_i}\Big\} \quad (42)$$

In general, following the procedure given in Rosenbrock and Storey (1970), we can compute two unimodular transformations $M_{\rm a}(s)$ and $N_{\rm a}(s)$ such that $sI - A_{\rm aa}$ is transformed into the Smith form, *i.e.*,

$$M_{\rm a}(s)(sI-A_{\rm aa})N_{\rm a}(s)=\Big\{p_{{\rm a},1}(s),\cdots,p_{{\rm a},n_{\rm a}}(s)\Big\} \quad (43)$$

Clearly, these polynomials are related to the invariant zero structures of the given system Σ.

(2) The term corresponding to $(A_{{\rm bb},i}, C_{{\rm b},i})$ given in (37) has a constant Smith form:

$$\begin{bmatrix} I_{l_i} \\ 0 \end{bmatrix} = -\begin{bmatrix} 1 & & & \\ s & \ddots & & \\ \vdots & \ddots & \ddots & \\ s^{l_i} & \cdots & s & 1 \end{bmatrix} \left(\begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix} \right) I_{l_i} \quad (44)$$

Note that the first term on the right-hand side of the above equation is a unimodular matrix.

(3) Similarly, the Smith form for the term corresponding to $(A_{{\rm cc},i}, B_{{\rm c},i})$ given in (38) is also a constant matrix:

$$[I_{r_i} \quad 0] = I_{r_i} \left(\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix} \right) N_{r_i}(s) \quad (45)$$

where

$$N_{r_i}(s) = -\begin{bmatrix} 1 & & & \\ s & \ddots & & \\ \vdots & \ddots & \ddots & \\ s^{r_i} & \cdots & s & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ I_{r_i} & 0 \end{bmatrix} \quad (46)$$

is a unimodular matrix.

(4) Lastly, the Smith form for the term corresponding to $(A_{q_i}, B_{q_i}, C_{q_i})$ given in (40) is an identity matrix:

$$I_{q_i+1} = \left(\begin{bmatrix} 1 & -s & & \\ & \ddots & \ddots & \\ & & \ddots & -s \\ & & & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & s & \cdots & s^{q_i} \\ & \ddots & \ddots & \vdots \\ & & \ddots & s \\ & & & 1 \end{bmatrix} \quad (47)$$

Once again, the last term of the equation above is a unimodular matrix.

Finally, in view of (43) to (47) together with some appropriate permutation transformations, it is now straightforward to obtain unimodular transformations $M(s)$ and $N(s)$ such that

$$M(s)P_\Sigma(s)N(s) = \begin{bmatrix} D_\Sigma(s) & 0 \\ 0 & 0 \end{bmatrix}, \quad (48)$$

where

$$D_\Sigma(s) = \text{diag}\Big\{\overbrace{1,\cdots,1}^{n_{\rm bcd}}, p_{{\rm a},1}(s),\cdots,p_{{\rm a},n_{\rm a}}(s)\Big\} \quad (49)$$

and where $n_{\rm bcd} = n_{\rm b} + n_{\rm c} + n_{\rm d} + m_0 + m_{\rm d}$.

3. ILLUSTRATIVE EXAMPLE

We illustrate the results of Section 2 with the following example.

Example 3.1. Consider system characterized by (15) with

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 2 & 1 & 1 \\ -1 & 3 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (50)$$

and

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad (51)$$

which is already in the form of the special coordinate basis with an invariant zero at 1, and $n_{\rm a} = n_{\rm b} = n_{\rm c} = n_{\rm d} = 1$. Following the algorithm given in Steps KCF.1 to KCF.3, we obtain

$$\tilde{F} = \begin{bmatrix} 1 & -3 & -1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \\ -1 & -2 \\ 0 & 0 \end{bmatrix},$$

$$A_{\rm KF} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $B_{\rm K} = B$, $C_{\rm F} = C$, $\tilde{D} = D$, and the required two nonsingular transformations,

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$
V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -3 & -1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 & 0 & 0 \end{bmatrix},
$$

which transform $P_\Sigma(s)$ into the Kronecker canonical form, *i.e.*,

$$
UP_\Sigma(s)V = \left[\begin{array}{c|c|cc|cc} s-1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & s & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & -s \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right].
$$

Next, following the algorithm given in Steps SMITH.1 and SMITH.2, we obtain two unimodular matrices,

$$
M(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & s-1 \end{bmatrix},
$$

and

$$
N(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & s-1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 & s-1 \end{bmatrix},
$$

with $\det[M(s)] = -1$ and $\det[N(s)] = 1$, which convert $P_\Sigma(s)$ into the Smith form, *i.e.*,

$$
M(s)P_\Sigma(s)N(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & s-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
$$

Clearly, the polynomial in the entry (4,4) of the above Smith form of $P_\Sigma(s)$, *i.e.*, $s-1$, results from the invariant zero of Σ.

4. CONCLUSION

In this paper, we have presented a computational procedure for computing the well known Kronecker canonical form and Smith form of the system matrix of a general multivariable linear system, either proper or singular. The proposed method is based the structural decomposition techniques of linear systems, namely, the special coordinate basis. The interconnection between the Kronecker canonical form and the special coordinate basis has been established in a straightforward manner. The results have been implemented in an m-function in MATLAB.

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