CONSTRAINED DECONVOLUTION WITH FILTERING

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Abstract: The present paper deals with deconvolution for linear systems within a discrete time framework. The originality of the work stands in the integration of both a filtering step and a level constraint (positivity constraint) within the estimation algorithm. After setting the structure of the filter, an iterative scheme based on Lagrange multipliers optimization technique is developed. The regularized optimization criterion is based on the fit to filtered data principle, the positive signal to be restored is considered as the square of an exogenous unconstrained signal. An example shows the enhancement brought by this approach.

Keywords: Inverse dynamic problem, deconvolution, signal processing, optimal estimation, iterative methods.

1. INTRODUCTION

The deconvolution problem is widely encountered in the engineering literature. The range of applications is covering seismology in Chi (1998), spectroscopy in Jain (1989), endocrinology in De Nicolao (1995), image processing in Ichioka, *et al.* (1981), mechanical structural dynamics in Fasana (1997) to mention a few. This entails the practical aspect of deconvolution.

Therefore, in order to obtain physically meaningful results, one has to consider *a priori* information in the design (e. g. positivity constraints in hormone time series estimation in De Nicolao (1995)). The introduction of level constraints in the design of deconvolution techniques gives rise to nonlinear algorithms. The optimal solution to this nonlinear problem is obtain by iterative means (steepest descent in Ichioka, *et al.* (1981), conjugate gradient in De Nicolao (1995), Sekko, *et al.* (1996), Thomas (1991)).

Another topic of interest is the noise sensitivity reduction within the restored signal. As a matter of fact, available data for signal restoration are generally noise corrupted. Most of estimation technique do not handle the measurement noise. Recent techniques have been developed in order to insert an optimal filtering step within the estimation algorithm in Neveux, *et al.* (2000), Sekko, *et al.* (1996) and Sekko (1999).

The combination of both level constraint and filtering has been treated in the continuous time case (Sekko, *et al.* (1996)) and in the discrete time framework (Neveux, *et al.* (2000)). In the latter, the constrained estimation was a sub-optimal constrained algorithm. Indeed, in order to keep the linearity of the estimate expression, an unconstrained estimate was computed and then projected onto the real positive value set. In the present paper, an optimal constrained technique with integrated filtering is presented in the discrete time framework.

2. STATEMENT OF THE PROBLEM

Let $u(t)$ be the causal unknown signal to be restored and $y_m(t)$ the noisy measured output of a process supposed to be linear and described by its impulse response $h(t)$ (fig. 1). This process is supposed to be stable and causal.

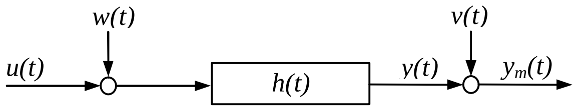

Fig. 1. The distortion process

Therefore, after discretization, the input-output relation can be written as :

$$y_m(k) = \sum_{i=0}^{N} h(i).\big(u(k\text{-}i)+w(k\text{-}i)\big)+v(k) \qquad (1)$$

where :

- $h(i)$, $u(i)$, $w(i)$, $y_m(i)$ and $v(i)$ are the discretised counter part of $h(t)$, $u(t)$, $w(t)$, $y_m(t)$ and $v(t)$ respectively ;
- $w(t)$ and $v(t)$ are supposed to be independent gaussian white noises with variance s_w^2 and s_v^2 respectively.

In addition, the signals u, w and v are supposed to be mutually uncorrelated.

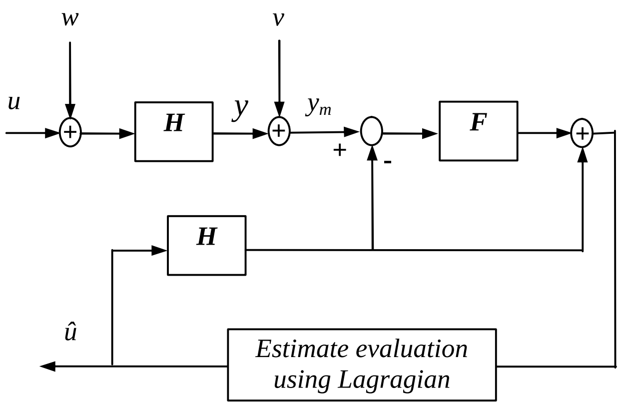

Fig. 2. Block diagram of the proposed method

So after $M + 1$ recorded samples, with $M \geq N$, one gets the matrix relation :

$$\underline{y} = H.\big(\underline{u} + \underline{w}\big) \qquad (2)$$

$$\underline{y_m} = \underline{y} + \underline{v} \qquad (3)$$

with :

$$\underline{y_m} = \big[y_m(0) \quad y_m(1) \quad \cdots \quad y_m(M)\big]^T$$

$$H = \begin{bmatrix} h(0) & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & h(0) & \ddots & & & & \vdots \\ \vdots & & \ddots & \ddots & & & \vdots \\ h(N) & & & \ddots & \ddots & & \vdots \\ 0 & \ddots & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & \ddots & 0 \\ 0 & \cdots & 0 & h(N) & \cdots & \cdots & h(0) \end{bmatrix} \qquad (4)$$

$$H \in \Re^{(M+1)x(M+1)}$$

The objective of the present paper is to set an optimal deconvolution filter with a positivity constraint. Optimality is considered as the integration of the constraint within the iterative algorithm of restoration of the input signal $u(t)$ (see fig. 2).

3. MAIN RESULTS

3.1. Optimal filter design

In this section, the signal u is supposed to be known. Then, from the knowledge of u, y_m and the variance of the noises, it is possible to define an optimal filter. Consequently, the following criterion is defined :

$$J_f = E\Big\{\big(\underline{\hat{y}} - \underline{y}\big)^T.\big(\underline{\hat{y}} - \underline{y}\big)\Big\} \qquad (6)$$

with the structural constraint for the optimal linear filter :

$$\underline{\hat{y}} = F.\underline{y_m} + G.\underline{u} \qquad (7)$$

The minimization of criterion (6) leads to the following expressions for the gain matrices F and G :

$$\begin{cases} F = \left(H.H^T + ß.I\right)^{-1}.H.H^T \\ G = (I - F).H \end{cases} \qquad (8)$$

with : $ß = \dfrac{s_v^2}{s_w^2}$

Proof

The criterion (6) can be rewritten just as follows :

$$J_f = E\Big\{\big(\underline{\hat{y}} - \underline{y}\big)^T\big(\underline{\hat{y}} - \underline{y}\big)\Big\} = Trace\Big\{E\Big\{\big(\underline{\hat{y}} - \underline{y}\big)\big(\underline{\hat{y}} - \underline{y}\big)^T\Big\}\Big\} \qquad (9)$$

The innovation which corresponds to the error of estimation is defined as :

$$\underline{e} = \underline{\hat{y}} - \underline{y} \qquad (10)$$

Using equations (2), (7) and (10), one obtains the expression :

$$\underline{e} = [(F - I)H + G]\underline{u} + (F - I)H\,\underline{w} + F\underline{v} \qquad (11)$$

So :

$$J_f = Trace\begin{Bmatrix} [(F-I)H+G]E\{\underline{u}\underline{u}^T\}[(F-I)H+G]^T + \ldots \\ \ldots + [(F-I)H]E\{\underline{w}\underline{w}^T\}[(F-I)H]^T + \ldots \\ \ldots + FE\{\underline{v}\underline{v}^T\}F^T \end{Bmatrix} \qquad (12)$$

The estimate should be unbiased, then it turns out that :

$$E\{\underline{e}\} = [(F-I)H+G]E\{\underline{u}\} + [(F-I)H]E\{\underline{w}\} + FE\{\underline{v}\} = 0 \qquad (13)$$

Then, from the initial assumptions, the expression of *G* is easily obtained :

$$G = (I - F)H \qquad (14)$$

Introducing (14) in equation (12), one obtains the new expression of the criterion :

$$J_f = Trace\{[(F-I)H]E\{\underline{w}\underline{w}^T\}[(F-I)H]^T + FE\{\underline{v}\underline{v}^T\}F^T\} \qquad (15)$$

The optimal value of *F* is such that :

$$\frac{\P J_f}{\P F} = 0 \qquad (16)$$

From equations (15) and (16), the following expression is obtained :

$$\frac{\P J_f}{\P F} = 2[(F-I)H]E^T\{\underline{w}\underline{w}^T\}H^T + 2FE^T\{\underline{v}\underline{v}^T\} = 0 \qquad (17)$$

So one can extract the expression of *F* :

$$F = \left[HE^T\{\underline{w}\underline{w}^T\}H^T + E^T\{\underline{v}\underline{v}^T\}\right]^{-1} HE^T\{\underline{w}\underline{w}^T\}H^T \qquad (18)$$

But, by definition :

$$E^T\{\underline{w}\underline{w}^T\} = s_w^2 I \quad and \quad E^T\{\underline{v}\underline{v}^T\} = s_v^2 I \qquad (19)$$

Then, the final expression of *F* is obtained :

$$F = \left(HH^T + bI\right)^{-1} HH^T \quad with: \quad b = \frac{s_v^2}{s_w^2} \qquad (20)$$

This completes the proof.

3.2. Constrained deconvolution using Lagrange multipliers

The constrained deconvolution problem can be set as follows:

$$\underline{\hat{u}} = \arg\min_{\underline{u}} \left\{ \left\| \underline{\hat{y}} - H.\underline{u} \right\|_2 \text{ with } \underline{u} \in \Omega \right\} \qquad (21)$$

For the expression of the filtered data, one gets the following modified criterion:

$$\underline{\hat{u}} = \arg\min_{\underline{u}} \left\{ \left\| F.\underline{y_m} - F.H.\underline{u} \right\|_2 \text{ with } \underline{u} \in \Omega \right\} \qquad (22)$$

where $\|.\|_2$ stands for the euclidian norm and is a subset of $\Re^{M+1}$ where the constraint is satisfied. Considering a positivity constraint one sets: $\Omega \equiv \Re_+^{M+1}$.

The optimization technique based on the Lagrangian multipliers are used to find the solution to the constrained problem set in criterion (21). In order to introduced the positivity constraint into the optimization, the signal u is supposed to be the square of an unconstrained synthetic signal.
Hence, the optimization technique can be described as:

$$\begin{cases} \Psi_0(\underline{u}) = \dfrac{1}{2}\left(F.\underline{y_m} - F.H.\underline{u}\right)^T.\left(F.\underline{y_m} - F.H.\underline{u}\right) \\ \underline{u} = \{\underline{q}_i^2\} \\ \nabla\Psi_0 = \{2.\underline{l}_i.\underline{q}_i\} \end{cases} \qquad (23)$$

with: $\underline{l} = (F.H)^T.(F.\underline{y_m}-F.H.\underline{u})$
where $\underline{l}$ is the Lagrange multipliers vector, ∇ is the gradient operator.

Remark: The positivity constraint can be introduced differently within the optimization scheme. For example, Sekko *et al.* (1996) have proposed the following function as a projection operator in continuous time framework:

$$u = 0.5[\ln(2\cosh q) + q] \qquad (24)$$

Thus, the formulation (23) of the optimal solution has to be modified in order to match with the positivity projection operator.

4. EXAMPLE

In order to test the proposed method, let us consider a synthetic system described by the following transfer function :

$$H(s) = \frac{1}{s^2 + 2s + 10} \qquad (25)$$

The objective is the reconstruct the synthetic positive input signal from the output (fig. 3). Such signal can be found in calorimetric reactor testing.

To obtain meaningful simulation results, for noise *w* and *v*, 20 different realization have been considered. Hence, the presented results showed in fig. 4 and fig. 5 represent the mean behavior of the proposed deconvolution filter.

The estimation error has been quantified in term of the mean square error in dB defined as:

$$MSE_{\text{dB}} = 10 \log_{10} \| u - \hat{u} \|_2 \qquad (25)$$

The results obtained with the proposed technique show that for any SNR, it is possible to find a parameter b such that the input estimation is enhanced (fig. 4). For each SNR value, an optimal value of the parameter b can be estimated, says b_{opt}. From fig. 5, it is clear that as the SNR is increasing, the enhancement is lessening. This, clearly sets the fact that as noise is not significant in regard with the output signal, it is not necessary to insert a filtering step into the restoration technique.

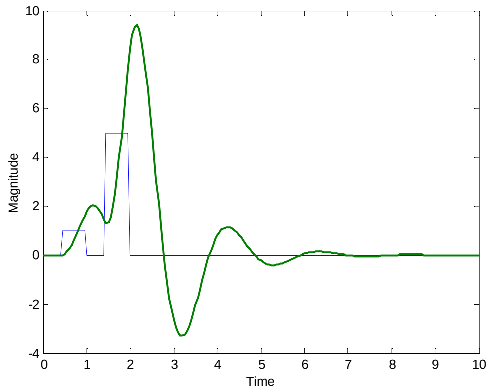

Fig. 3. Input signal (solid line) – Noise free output signal (bold solid line)

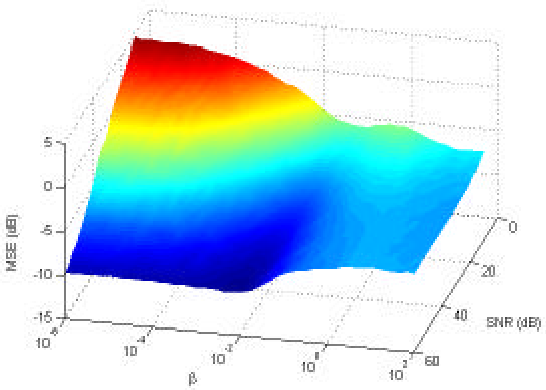

Fig. 4. Evolution of MSE_{dB} with b and SNR

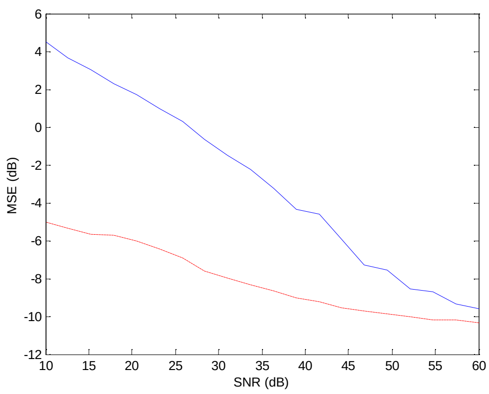

Fig. 5. Evolution of MSE_{dB}, no filtering step (solid line) – filtering step with b_{opt} (dashed line)

5. CONCLUSIONS

An iterative constrained method for the inverse problem of deconvolution has been presented. In inverse problem, the measurement noise corrupting the data is the source of instability. The proposed method tackled the problem of data filtering by integrating an optimal filter within the structure of the estimator. As a matter of fact, an optimality criterion based on the filtered data has been set up. The positivity constraint under consideration has been introduced in the optimization scheme considering the signal to be restored as the square on an unconstrained synthetic signal. The optimization scheme is based on Lagrangian multipliers. However, other method can be used in order to considered level constraint in the signal to be restored. The technique showed its ability to bring an important enhancement for low SNR.

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