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LFT REPRESENTATIONS FOR NONLINEAR MECHANICAL SYSTEMS

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Abstract: In this paper, we present a systematic methodology for constructing LFT representations for general mechanical systems derived via *Lagrange's equations.* The LFT representation allows for any nonlinear matrix second-order mechanical system to be transformed into an interconnection of an LTI system with a diagonal "uncertainty" block. This uncertainty block is, in fact, state-dependent. Sufficient conditions that ensure well-posedness of the LFT interconnection are given. Using such LFT representations, the stability properties of the system can then be analyzed using Linear Matrix Inequalities (LMIs).

Keywords: Mechanical Systems, Linear Fractional Transformations.

1. INTRODUCTION

A nonlinear system of the form

$$\dot{x} = f(x, u) \tag{1}$$

where f is a vector of rational polynomial functions of the states x and a linear function of u, has a *Linear Fractional Representation* (LFR) (Ghaoui and Scorletti, 1995; Cockburn and Morton, 1997). In this LFR representation, the system can be described as an interconnection of an LTI system with a diagonal "uncertainty" matrix containing the states. Such LFR/LFT representations are very common in robust control (Zhou *et al.*, 1996; Lambrechts *et al.*, 1993; Belcastro and Chang, 1998; Cheng and De Moor, 1994). In (Ghaoui and Scorletti, 1995) a state-feedback synthesis method is presented via a set of LMIs for systems written in LFR/LFT form. Stability conditions are then obtained by analyzing the properties of a differential inclusion related to an LFR. Subsequently, the problem is formulated as a convex optimization problem and solved using standard LMI techniques (Boyd *et al.*, 1994). Moreover, the feedback controller has a domain of attraction, which can be estimated a posteriori.

One of the main obstructions in the previous methodology is the absence of a *systematic* way of constructing such LFT/LFR representations for general non-linear systems having minimal dimension. Recent advances in this direction include, for example, the work of Cockburn and Morton (1997). Instead of insisting on general nonlinear systems, in this paper we restrict our attention to nonlinear mechanical systems that can be written in second-order matrix form as follows

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + K(q)q = u \tag{2}$$

where $M(q)$ represents the mass matrix, $C(q, \dot{q})$ represents damping and Coriolis terms and $K(q)$ is the stiffness matrix. The previous representation – derived via the application of Lagrange's equations – has much more structure that the general nonlinear system (1). For mechanical systems, the control design may benefit from a closer look at the structure of the equations describing the system under consideration. In this paper we use the special structure of Eq. (2) to construct LFT representations. These LFT/LFR representations

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can then be used for control design. Notice that in contrast to several other nonlinear methodologies (Isidori, 1995; Krstić *et al.*, 1995) this technique does not require any nonlinear transformation of the states that may result in a new set of system coordinates without an obvious physical meaning. As performance specifications are typically given in the original coordinates, casting these specification in the new coordinates is often a non-trivial exercise. This is an important issue that is often overlooked in nonlinear control design literature.

The notation of this paper is standard. For a real matrix P, $P > 0$ means that P is symmetric and positive-definite. A^T is the transpose of the matrix A, I_r denotes the identity matrix in $\mathbb{R}^{r\times r}$ with $I_0 = [\ ]$, the empty matrix, $0_{r_1\times r_2}$ denotes the $r_1 \times r_2$ zero matrix, $L_n \in \mathbb{R}^{n\times n}$ denotes a lower-triangular matrix with the following structure,

$$L_n = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

where $\mathrm{diag}(A, B)$ denotes the corresponding block-diagonal matrix. $\Upsilon_n^{i,j}$ is a $n\times n$ zero matrix except for the element (i, j), which is 1. If $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a *matrix of polynomials*, every entry of $A(q)$ is a multi-variable polynomial function of $q = (q_1, q_2, \dots, q_n) \in \Re^n$. The *degree of the variable* q_i in $A(q)$ is the highest exponent power to which this variable is being raised, among all the monomials of all the polynomial functions in the matrix $A(q)$. For m (monic) monomials $q_1^{i_{j,1}} q_2^{i_{j,2}} \cdots q_n^{i_{j,n}}$, $1 \le j \le m$ of the variable vector $q = (q_1, q_2, \dots, q_n) \in \Re^n$, we associate the diagonal matrix, $\Psi = \mathrm{diag}(\, I_{i_{1,1}} \otimes q_1, I_{i_{1,2}} \otimes q_2, \dots, I_{i_{2,1}} \otimes q_1, I_{i_{2,2}} \otimes q_2, \dots, I_{m,n} \otimes q_n\,)$.

1.1 *Lagrange's Equations*

In this paper, we consider holonomic mechanical systems having configuration variables equal in number to the degrees of freedom, say n. In addition, it is assumed that the kinematic and potential energies of the mechanical system are given in terms of quadratic forms and all nonconservative forces arise due to Rayleigh's dissipation function (Pars, 1965). To this end, let the generalized coordinates of the system be $q = [\, q_1, q_2, \dots, q_n \,]^T$, the generalized velocities of the system be $\dot q = [\, \dot q_1, \dot q_2, \dots, \dot q_n \,]^T$, and the corresponding generalized control forces be $u = [\, u_1, u_2, \dots, u_n \,]^T$. The kinematic and potential energies of the system are expressed as,

$$T = \frac{1}{2}\sum_{i,j=1}^{n} \dot q_i m_{ij}(q) \dot q_j = \frac{1}{2}\dot q^T M(q) \dot q \qquad (3a)$$

$$V = \frac{1}{2}\sum_{i,j=1}^{n} q_i s_{ij}(q) q_j = \frac{1}{2} q^T S(q) q \qquad (3b)$$

where $M : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $M(q) > 0$ for all $q \in \mathbb{R}^n$, $S : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, and $S(q)^T = S(q)$. In addition, the Rayleigh's dissipation function is a homogeneous quadratic form in terms of the velocities and its coefficients are functions of only the generalized coordinates q (Pars, 1965). That is,

$$R = \frac{1}{2}\sum_{i,j=1}^{n} \dot q_i r_{ij}(q) \dot q_j = \frac{1}{2}\dot q^T N(q) \dot q \qquad (4)$$

with $N : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, and $N(q) > 0$ for all $q \in \mathbb{R}^n$. Lagrange's equations describing systems with the Rayleigh's dissipation function R as the only source for nonconservative forces, except for the control forces are given by

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot q_i}\Big) - \frac{\partial L}{\partial q_i} + \frac{\partial R}{\partial \dot q_i} = u_i, \quad i = 1, \dots, n \quad (5)$$

where L is the Lagrangian function defined by $L = T - V$. The partial derivatives $\frac{\partial L}{\partial \dot q_i}$, $\frac{\partial L}{\partial q_i}$ and $\frac{\partial R}{\partial \dot q_i}$ can be calculated explicitly from Eq. (3) and (4) as follows,

$$\frac{\partial L}{\partial \dot q_i} = \sum_{j=1}^{n} m_{ij}(q)\dot q_j \qquad i = 1, \dots, n \qquad (6a)$$

$$\frac{\partial L}{\partial q_i} = \frac{1}{2}\sum_{j,l=1}^{n} \dot q_j \frac{\partial m_{jl}(q)}{\partial q_i}\dot q_l - \sum_{j=1}^{n} s_{ij}(q) q_j \qquad (6b)$$
$$-\frac{1}{2}\sum_{j,l=1}^{n} q_j \frac{\partial s_{jl}(q)}{\partial q_i} q_l, \quad i = 1, 2, \dots, n$$

$$\frac{\partial R}{\partial \dot q_i} = \sum_{j=1}^{n} r_{ij}(q)\dot q_j, \quad i = 1, \dots, n \qquad (6c)$$

Moreover, from Eq. (6a), one obtains,

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot q_i}\Big) = \sum_{j=1}^{n} m_{ij}(q)\ddot q_j + \sum_{j=1}^{n} \dot m_{ij}(q)\dot q_j$$

where $\dot m_{ij}(q) = \sum_{l=1}^{n} \frac{\partial m_{ij}(q)}{\partial q_l}\dot q_l$. We can then write n simultaneous differential equations describing the dynamics of the mechanical system as

$$\sum_{j=1}^{n} m_{ij}(q)\ddot q_j + \sum_{j=1}^{n} \dot m_{ij}(q)\dot q_j - \frac{1}{2}\sum_{j,l=1}^{n} \dot q_j \frac{\partial m_{jl}(q)}{\partial q_i}\dot q_l$$
$$+ \sum_{j=1}^{n} s_{ij}(q) q_j + \frac{1}{2}\sum_{j,l=1}^{n} q_j \frac{\partial s_{jl}(q)}{\partial q_i} q_l$$
$$+ \sum_{j=1}^{n} r_{ij}(q)\dot q_j = u_i, \qquad i = 1, \dots, n$$

Introduce now the matrices $\Pi_1(q,\dot q) \in \mathbb{R}^{n\times n}$ and $\Pi_2(q) \in \mathbb{R}^{n\times n}$ as follows

$$\Pi_1(q,\dot q) = \begin{bmatrix} \sum_{l=1}^{n} \frac{\partial m_{1l}(q)}{\partial q_1}\dot q_l & \cdots & \sum_{l=1}^{n} \frac{\partial m_{nl}(q)}{\partial q_1}\dot q_l \\ \vdots & \ddots & \vdots \\ \sum_{l=1}^{n} \frac{\partial m_{1l}(q)}{\partial q_n}\dot q_l & \cdots & \sum_{l=1}^{n} \frac{\partial m_{nl}(q)}{\partial q_n}\dot q_l \end{bmatrix},$$

$$\Pi_2(q) = \begin{bmatrix} \sum_{l=1}^{n} \frac{\partial s_{1l}(q)}{\partial q_1} q_l & \cdots & \sum_{l=1}^{n} \frac{\partial s_{nl}(q)}{\partial q_1} q_l \\ \vdots & \ddots & \vdots \\ \sum_{l=1}^{n} \frac{\partial s_{1l}(q)}{\partial q_n} q_l & \cdots & \sum_{l=1}^{n} \frac{\partial s_{nl}(q)}{\partial q_n} q_l \end{bmatrix}.$$

It turns out that $\sum_{j,l=1}^{n} \dot q_j \frac{\partial m_{jl}(q)}{\partial q_i}\dot q_l$ is equal to the ith element of the vector $\Pi_1(q,\dot q)\,\dot q$ and $\sum_{j,l=1}^{n} q_j \frac{\partial s_{jl}(q)}{\partial q_i} q_l$ is equal to the ith element of $\Pi_2(q)\dot q$. In order to put these n differential equations into a vector form, we first notice that

$$\frac{\partial(\dot q^T M(q)\dot q)}{\partial q} = \Pi_1(q,\dot q)\dot q \tag{7a}$$

$$\frac{\partial(q^T S(q) q)}{\partial q} = 2S(q)q + \Pi_2(q)q \tag{7b}$$

We then obtain the vector second-order form of the dynamic equation for the mechanical system as,

$$\begin{aligned} M(q)\ddot q + \big(\dot M(q) - \frac{1}{2}\Pi_1(q,\dot q) + N(q)\big)\dot q \\ + \big(S(q) + \frac{1}{2}\Pi_2(q)\big)q = u \end{aligned} \tag{8}$$

Comparing with Eq. (2), we obtain

$$M(q) = M(q) \tag{9a}$$

$$C(q,\dot q) = \dot M(q) - \frac{1}{2}\Pi_1(q,\dot q) + N(q) \tag{9b}$$

$$K(q) = S(q) + \frac{1}{2}\Pi_2(q) \tag{9c}$$

1.2 *Assumptions*

In the previous section we derived dynamic equations for a general (holonomic) mechanical system. In order to proceed, we make the following assumptions:

A1. $M(q)$ is a constant matrix, i.e, $M(q) = M$.

A2. $S(q)$ and $N(q)$ are matrices of polynomials. This implies that every entry in $S(q)$ and $N(q)$ is a multi-variable polynomial function of the generalized coordinates q. Assume that the highest degrees of the variables $q_1, q_2, \dots, q_n$ in either $S(q)$ or $N(q)$ are $k_1, k_2, \dots, k_n$, respectively. Let $|i| = i_1 + i_2 + \dots + i_n$. Then $S(q)$ and $N(q)$ can be written in the following form,

$$S(q) = S_0 + \sum_{|i|=1,\ i_j \le k_j}^{k_1+\dots+k_n} q_1^{i_1} q_2^{i_2} \dots q_n^{i_n} S^{i_1,\dots,i_n}$$

$$N(q) = N_0 + \sum_{|i|=1,\ i_j \le k_j}^{k_1+\dots+k_n} q_1^{i_1} q_2^{i_2} \dots q_n^{i_n} N^{i_1,\dots,i_n}$$

where $S_0, S^{i_1,i_2,\dots,i_n}, N_0, N^{i_1,i_2,\dots,i_n} \in \mathbb{R}^{n\times n}$ are constant matrices of the polynomial coefficients.

Since the mass matrix $M(q)$ is constant, we have $\dot M(q) = 0$ and $\Pi_1(q,\dot q) = 0$. Since $S(q)$ is a matrix of polynomials, the (i,j) entry of $\Pi_2(q)$

$$\sum_{l=1}^{n} \frac{\partial s_{jl}(q)}{\partial q_i} q_l$$

is also a multi-variable polynomial function of q. Therefore, $\Pi_2(q)$ is also a matrix of polynomials, and the degrees of q_1, q_2, $\dots$, q_n in $\Pi_2(q)$ are $k_1 + 1, k_2 + 1, \dots, k_n + 1$, respectively. Then $\Pi_2(q)$ can be expanded as,

$$\Pi_2(q) = \sum_{|i|=1,\ i_j \le k_j+1}^{k_1+\dots+k_n+n} q_1^{i_1} q_2^{i_2} \dots q_n^{i_n} \Phi^{i_1,\dots,i_n}$$

where $\Phi^{i_1,i_2,\dots,i_n} \in \mathbb{R}^{n\times n}$ are constant matrices of the polynomial coefficients with elements given by

$$\begin{aligned} \Phi_{r,p}^{i_1,i_2,\dots,i_n} = (i_r+1)(S_{1,p}^{i_1-1,i_2,\dots,i_r+1,\dots,i_n} + \\ S_{2,p}^{i_1,i_2-1,\dots,i_r+1,\dots,i_n} + \dots + S_{i_r,p}^{i_1,i_2,\dots,i_r,\dots,i_n} + \\ \dots + S_{n,p}^{i_1,i_2,\dots,i_r+1,\dots,i_n-1}) - S_{i_r,p}^{i_1,i_2,\dots,i_r,\dots,i_n} \end{aligned} \tag{10}$$

where $1 \le r,p \le n$. Notice that Φ is not unique. Defining now

$$K(q) := S(q) + \frac{1}{2}\Pi_2(q), \quad C(q) := N(q) \tag{11}$$

a mechanical system satisfying the Assumptions 1 and 2 can be written as,

$$M\ddot q + C(q)\dot q + K(q)q = u \tag{12}$$

Example 1.1 As an application of the previous analysis, let us study the following mechanical system. Assume that the kinetic energy, the potential energy, and the Rayleigh's dissipation function of the system are given by,

$$T = \frac{1}{2}(\dot q_1^2 + \dot q_2^2) \tag{13a}$$

$$V = \frac{1}{2}(q_1^4 + 2q_1q_2 + q_2^2) \tag{13b}$$

$$R = \frac{1}{2}(\dot q_1^2 + 2\dot q_1\dot q_2 + q_1^2 q_2 \dot q_2^2) \tag{13c}$$

where q_1 and q_2 are the generalized coordinates and the system is a two-degree of freedom system. The corresponding M, $S(q)$ and $N(q)$ matrices can be chosen as

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S(q) = \begin{bmatrix} q_1^2 & 1 \\ 1 & 1 \end{bmatrix}, \quad N(q) = \begin{bmatrix} 1 & 1 \\ 1 & q_1^2 q_2 \end{bmatrix}$$

Applying Lagrange's equation to this system, the dynamics of the system can be easily obtained in second-order form,

$$\ddot{q}_1 + 2q_1^3 + q_2 + \dot{q}_1 + \dot{q}_2 = u_1 \tag{14a}$$

$$\ddot{q}_2 + q_1 + q_2 + \dot{q}_1 + q_1^2 q_2 \dot{q}_2 = u_2 \tag{14b}$$

Let $q_3 = \dot{q}_1$ and $q_4 = \dot{q}_2$ and define $x := [q_1, q_2, q_3, q_4]^T$, be the state of the system. Then we can put the dynamics of the system into state space form, as follows

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2q_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q_1^2 q_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

with $u = [u_1, u_2]^T$. Introducing the constant matrices A, B and a state-dependent matrix $\tilde{A}(q)$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{15a}$$

$$\tilde{A}(q) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2q_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q_1^2 q_2 \end{bmatrix} \tag{15b}$$

The dynamics of system can be conveniently written as,

$$\dot{x} = Ax + \tilde{A}(q)x + Bu \tag{16}$$

2. LFTS FOR MECHANICAL SYSTEMS

In the matrix second-order system of Eq. (12), recall that $K : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $C : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ are matrices of polynomials of the states. Let us decompose $K(q)$ and $C(q)$ in the following manner,

$$\begin{gathered} K(q) = \tilde{K}(q) + K_0, \quad C(q) = \tilde{C}(q) + C_0 \\ K_0 = S_0, \quad C_0 = N_0 \end{gathered} \tag{17}$$

where $K_0 \in \mathbb{R}^{n\times n}$, $C_0 \in \mathbb{R}^{n\times n}$, $\tilde{K} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $\tilde{C} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $\tilde{C}(0) = 0$, $\tilde{K}(0) = 0$. The state-dependent matrices $\tilde{K}(q)$ and $\tilde{C}(q)$ may be written out explicitly as,

$$\begin{aligned} \tilde{K}(q) &= \sum_{|i|=1,\ i_j \le k_j}^{k_1+\cdots+k_n} q_1^{i_1} \cdots q_n^{i_n} S^{i_1,\ldots,i_n} \\ &+ \frac{1}{2} \sum_{|i|=1,\ i_j \le k_j+1}^{k_1+\cdots+k_n+n} q_1^{i_1} \ldots q_n^{i_n} \Phi^{i_1,\ldots,i_n} \end{aligned} \tag{18a}$$

$$\tilde{C}(q) = \sum_{|i|=1,\ i_j \le k_j}^{k_1+\cdots+k_n} q_1^{i_1} \cdots q_n^{i_n} N^{i_1,\ldots,i_n} \tag{18b}$$

In order to put the system into first order from, we now let $x = [q,\ \dot{q}]^T$ and $\mathcal{W} = M^{-1}$,

$$A = \begin{bmatrix} 0 & I_n \\ -\mathcal{W}K_0 & -\mathcal{W}C_0 \end{bmatrix} \tag{19a}$$

$$\tilde{A}(q) = \begin{bmatrix} 0 & 0 \\ -\mathcal{W}\tilde{K}(q) & -\mathcal{W}\tilde{C}(q) \end{bmatrix} \tag{19b}$$

$$B = \begin{bmatrix} 0 \\ I_n \end{bmatrix} \tag{19c}$$

where $x \in \mathbb{R}^{2n}$, $A \in \mathbb{R}^{2n\times 2n}$, $\tilde{A} : \mathbb{R}^n \to \mathbb{R}^{2n\times 2n}$, $B \in \mathbb{R}^{2n}$. Then Eqs. (12) can be rewritten in state-space form as

$$\dot{x} = Ax + \tilde{A}(q)x + Bu \tag{20}$$

Since $\tilde{K}(q)$ and $\tilde{C}(q)$ are matrices of polynomials, $\tilde{A}(q)$ is also a matrix of polynomials. It can be easily established that the degrees of the variables $q_1, q_2, \ldots, q_n$ in $\tilde{A}(q)$ are $k_1+1, k_2+1, \ldots, k_n+1$ respectively. Specifically, $\tilde{A}(q)$ takes the form,

$$\tilde{A}(q) = \sum_{|i|=1,\ i_j \le k_j+1}^{k_1+\cdots+k_n+n} q_1^{i_1} \cdots q_n^{i_n} A^{i_1,\ldots,i_n} \tag{21}$$

where $A^{i_1,i_2,\ldots,i_n} \in \mathbb{R}^{2n\times 2n}$ is given by

$$\begin{aligned} A_{r,p}^{i_1,\ldots,i_n} &= 0, \qquad 1 \le r \le n,\ 1 \le p \le 2n \\ A_{r,p}^{i_1,\ldots,i_n} &= -\sum_{l=1}^{n} \mathcal{W}_{r,l} S_{l,p}^{i_1,\ldots,i_n} - \frac{1}{2}\sum_{l=1}^{n} \mathcal{W}_{r,l} \Phi_{l,p}^{i_1,\ldots,i_n} \\ &\quad \text{with} \quad n < r \le 2n, \quad 1 \le p \le n \\ A_{r,p}^{i_1,\ldots,i_n} &= -\sum_{l=1}^{n} \mathcal{W}_{r,l} N_{l,p}^{i_1,\ldots,i_n} \\ &\quad \text{with} \quad n < r \le 2n, \quad n < p \le 2n \end{aligned}$$

From the previous derivation, it follows immediately that the pair $(A,\ B)$ is controllable and that $\tilde{A}(0) = 0$. Accordingly, Eq. (20) can be put in the following LFT form (Zhou *et al.*, 1996),

$$\begin{aligned} \dot{y} &= Ay + B_p p + Bu \\ q &= C_q y + D_{qp} p \\ p &= \Delta(q) q \end{aligned} \tag{23}$$

where $\Delta(q) \in \Psi$ is a diagonal matrix and its diagonal elements are the states of the system, and where $B_p \in \mathbb{R}^{2n\times m}$ and $C_q \in \mathbb{R}^{m\times 2n}$ satisfy

$$B_p \Delta(q)(I_m - D_{qp}\Delta(q))^{-1} C_q = \tilde{A}(q) \tag{24}$$

for some constant matrix $D_{qp} \in \mathbb{R}^{m\times m}$.

2.1 *Well-Posedness*

The LFT model (23) is well posed inside a set $\Omega \in \mathbb{R}^{2n}$ if for any $q \in \Omega$, the matrix $(I_m - D_{qp}\Delta(q))$ is invertible (Ghaoui and Scorletti, 1995). The following lemma shows that for any monomial in $\tilde{A}(q)$, we can construct an LFT with a special structure for the matrix D_{qp}.

Lemma 2.1. For an arbitrary monomial of $\tilde{A}(q)$, say, $\gamma(q) = q_1^{\ell_1} q_2^{\ell_2} \ldots q_n^{\ell_n}$ with $\ell = \ell_1 + \ell_2 + \ldots + \ell_n$, there exists $B_{(i)} \in \mathbb{R}^{\ell}$, $C_{(i)} \in \mathbb{R}^{\ell \times n}$, a low triangular matrix $D_{(i)} = L_\ell$ and a diagonal matrix $\Delta_{(i)}(q) = \operatorname{diag}(q_1 I_{\ell_1}, q_2 I_{\ell_2}, \ldots, q_n I_{\ell_n})$, such that $\gamma(q) = B_{(i)} \Delta_{(i)}(q)(I_l - D_{(i)}\Delta_{(i)}(q))^{-1} C_{(i)}$.

Proof. Without loss of the generality, we may assume that $\ell_n \neq 0$. Define the vector $\mathcal{G} := (\ell_1, \ell_2, \ldots, \ell_n) \in \mathbb{N}_0^n$ and associate the following matrices to $\mathcal{G}$

$$\begin{aligned}
\mathcal{E}_1(\mathcal{G}; q) &= \operatorname{diag}(q_1 I_{\ell_1}, q_2 I_{\ell_2}, \ldots, q_n I_{\ell_n}), \quad \ell_n \neq 0 \\
\mathcal{E}_2(\mathcal{G}; q) &= \left[I_{\ell-1} \vdots 0_{(\ell-1)\times 1} \right] \mathcal{E}_1(\mathcal{G}; q) \begin{bmatrix} I_{\ell-1} \\ \cdots\cdots \\ 0_{1\times(\ell-1)} \end{bmatrix}
\end{aligned}$$

In addition, define the state-dependent matrix

$$\mathcal{S}(q) = \begin{bmatrix} 0_{1\times\ell} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \mathcal{E}_2(\mathcal{G}, q) \vdots 0_{(\ell-1)\times 1} \end{bmatrix} \tag{26}$$

A simple calculation shows that $I_\ell - D_{(i)}\Delta_{(i)}(q) = I_\ell - \mathcal{S}(q)$ which can be written explicitly as,

$$\begin{bmatrix}
1 & & & & & & & & & & & & \\
-q_1 & 1 & & & & & & & & & & & \\
 & \ddots & \ddots & & & & & & & & & & \\
 & & -q_1 & 1 & & & & & & & & & \\
 & & & -q_2 & 1 & & & & & & & & \\
 & & & & \ddots & \ddots & & & & & & & \\
 & & & & & -q_2 & 1 & & & & & & \\
 & & & & & & -q_3 & 1 & & & & & \\
 & & & & & & & \ddots & \ddots & & & & \\
 & & & & & & & & & -q_n & 1
\end{bmatrix}$$

Since the determinant of $I_\ell - D_{(i)}\Delta_{(i)}(q)$ is always one, its inverse exists and a simple calculation shows that the inverse is also in lower-triangular form, the diagonal elements are all 1, and the sub-diagonal elements are the same as the negative of the diagonal elements in $\mathcal{E}_2(\mathcal{G}; q)$. The inverse $(I_\ell - D_{(i)}\Delta_{(i)}(q))^{-1}$ can thus be written explicitly as,

$$\begin{bmatrix}
1 & & & & & & & & & \\
q_1 & & & & & & & & & \\
q_1^2 & 1 & & & & & & & & \\
\vdots & \ddots & \ddots & & & & & & & \\
q_1^{l_1} & \cdots & q_1 & 1 & & & & & & \\
q_1^{l_1} q_2 & \cdots & & q_2 & 1 & & & & & \\
\vdots & \vdots & & & \ddots & \ddots & & & & \\
q_1^{l_1} q_2^{l_2} & \cdots & & & & q_2 & 1 & & & \\
q_1^{l_1} q_2^{l_2} q_3 & \cdots & & & & & q_3 & 1 & & \\
\vdots & \vdots & & & & & & \ddots & \ddots & \\
q_1^{l_1} q_2^{l_2} \ldots q_n^{l_n - 1} & \cdots & & & & \cdots & & & -q_n & 1
\end{bmatrix}$$

Defining $B_{(i)} := [0 \ 0 \ \cdots \ 0 \ 1]$, and $C_{(i)} := [1 \ 0 \ \cdots \ 0 \ 0]^T$ we conclude that

$$\gamma(q) = B_{(i)} \Delta_{(i)}(q)(I_l - D_{(i)}\Delta_{(i)}(q))^{-1} C_{(i)}.$$

This means that every monomial of $\tilde{A}(q)$ can be put in an LFT form with a lower triangular matrix $D_{(i)} = L_\ell$. ■

Remark 2.1 The previous LFT representation is not unique. Therefore, the decomposition for $\gamma(q)$ provided in Lemma 2.1 involves a matrix $\Delta_{(i)}(q)$ is not necessary of minimal dimension. Model reduction techniques can be applied to reduce the order of $\Delta_{(i)}(q)$ block, as long as the lower triangular structure of D_{qp} is not affected (Beck *et al.*, 1996; Beck and D'Andrea, 1998).

Example 2.1 Consider the simple monomial $\gamma(q) = q_1^3 q_2^3 q_3^2$, with $\ell_1 = 3$, $\ell_2 = 3$, $\ell_3 = 2$. We have $D_{(i)} = L_8$ and $\Delta_{(i)}(q) = \operatorname{diag}(q_1 I_3, q_2 I_3, q_3 I_2)$, $B_{(i)} = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 1]$, $C_{(i)} = [1\ 0\ 0\ 0\ 0\ 0\ 0\ 0]^T$. Then $I_8 - D_{(i)}\Delta_{(i)}(q)$ assumes the form,

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -q_1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -q_1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -q_2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -q_2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -q_2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -q_3 & 1
\end{bmatrix}$$

Since the determinant of $I_8 - D_{(i)}\Delta_{(i)}(q)$ is 1, its inverse $(I_8 - D_{(i)}\Delta_{(i)}(q))^{-1}$ exists and retains a lower-triangular form,

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
{q_1}^2 & q_1 & 1 & 0 & 0 & 0 & 0 & 0 \\
{q_1}^3 & {q_1}^2 & q_1 & 1 & 0 & 0 & 0 & 0 \\
{q_1}^3 q_2 & {q_1}^2 q_2 & q_1 q_2 & q_2 & 1 & 0 & 0 & 0 \\
{q_1}^3 q_2^2 & {q_1}^2 q_2^2 & q_1 {q_2}^2 & {q_2}^2 & q_2 & 1 & 0 & 0 \\
{q_1}^3 q_2^3 & q_1^2 {q_2}^3 & q_1 {q_2}^3 & {q_2}^3 & {q_2}^2 & q_2 & 1 & 0 \\
q_1^3 q_2^3 q_3 & q_1^2 {q_2}^3 q_3 & q_1 {q_2}^3 q_3 & {q_2}^3 q_3 & {q_2}^2 q_3 & q_2 q_3 & q_3 & 1
\end{bmatrix}$$

After multiplying with $\Delta_{(i)}(q)$, we obtain, $\Delta_{(i)}(q)(I_8 - D_{(i)}\Delta_{(i)}(q))^{-1}$, which is given by

$$\begin{bmatrix}
q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
{q_1}^2 & q_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
{q_1}^3 & {q_1}^2 & q_1 & 0 & 0 & 0 & 0 & 0 \\
{q_1}^3 q_2 & {q_1}^2 q_2 & q_1 q_2 & q_2 & 0 & 0 & 0 & 0 \\
{q_1}^3 q_2^2 & {q_1}^2 {q_2}^2 & q_1 {q_2}^2 & {q_2}^2 & q_2 & 0 & 0 & 0 \\
{q_1}^3 {q_2}^3 & {q_1}^2 {q_2}^3 & q_1 {q_2}^3 & {q_2}^3 & {q_2}^2 & q_2 & 0 & 0 \\
{q_1}^3 {q_2}^3 q_3 & {q_1}^2 {q_2}^3 q_3 & q_1 {q_2}^3 q_3 & {q_2}^3 q_3 & {q_2}^2 q_3 & q_2 q_3 & q_3 & 0 \\
{q_1}^3 {q_2}^3 q_3^2 & q_1^2 {q_2}^3 q_3^2 & q_1 {q_2}^3 q_3^2 & {q_2}^3 q_3^2 & {q_2}^2 q_3^2 & q_2 {q_3}^2 & q_3^2 & q_3
\end{bmatrix}$$

From the previous matrix it is obvious that $\gamma(q)$ is equal to the $(8, 1)$ element of $\Delta_{(i)}(q)(I_8 - D_{(i)}\Delta_{(i)}(q))^{-1}$ and therefore we have that $\gamma(q) = B_{(i)}\Delta_{(i)}(q)(I_8 - D_{(i)}\Delta_{(i)}(q))^{-1} C_{(i)}$.

Lemma 2.1 shows that every monomial of $\tilde{A}(q)$ has an LFT representation with a lower triangular matrix $D_{(i)}$. To construct $\tilde{A}(q)$ using the LFT representation in Lemma 2.1, consider a monomial

$\gamma(q) = q_1^{\ell_1} q_2^{\ell_2} \ldots q_n^{\ell_n}$ and an element of $\tilde{A}(q)_{r,p}^{\ell_1,\ldots,\ell_n}$, let

$$\begin{aligned} B^{\ell_1,\ldots,\ell_n} &= \tilde{A}(q)_{r,p}^{\ell_1,\ldots,\ell_n} \Upsilon_n^{r,n} \\ C^{\ell_1,\ldots,\ell_n} &= \Upsilon_n^{1,p} \\ D^{\ell_1,\ldots,\ell_n} &= L_\ell \\ \Delta^{\ell_1,\ldots,\ell_n} &= \mathrm{diag}(q_1 I_{\ell_1}, \ldots, q_n I_{\ell_n}) \end{aligned}$$

then

$$\begin{aligned} B^{\ell_1,\ldots,\ell_n} \Delta^{\ell_1,\ldots,\ell_n}(q)(I_l - D^{\ell_1,\ldots,\ell_n} \Delta^{\ell_1,\ldots,\ell_n}(q))^{-1} \\ C^{\ell_1,\ldots,\ell_n} = \gamma(q) \tilde{A}(q)_{r,p}^{\ell_1,\ldots,\ell_n} \Upsilon_n^{r,p} \end{aligned}$$

By properly stacking all elements together, B_p, C_q, $\Delta(q)$ and a lower triangular matrix $D_{qp} = L_m$, can be constructed to satisfy Eq. (24). Let us continue with Example (1.1) to demonstrates how this will work out.

Example 2.2 In Example 1.1, we have derived the nonlinear state space equations (16) for the mechanical system given in (14). In Eqs. (15), $\tilde{A}(q)$ is in a matrix of polynomials, which we can express using an LFT model. There are two monomials in $\tilde{A}$ namely, q_1^2 and $q_1^2 q_2$. Define $\Delta(q) = \mathrm{diag}(q_1, q_1, q_1, q_1, q_2)$ and

$$B_p = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad C_q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D_{qp} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

A straightforward calculation shows that

$$B_p \Delta (I_5 - D_{qp} \Delta)^{-1} C_q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2q_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q_1^2 q_2 \end{bmatrix}$$

which is equal to $\tilde{A}(q)$ in Eqs.(15).

With the lower triangular structure of D_{qp}, we have $(D_{qp}\Delta)^j = (\Delta D_{qp})^j = 0$ for $j \geq m$. This nilpotency property can be used to simplify the calculation of the inverse $(I_m - D_{qp}\Delta(q))^{-1}$. Under some mild assumptions on the size of $\Delta(q)$, and with D_{qp} a lower triangular matrix, we have that

$$(I_m - D_{qp}\Delta(q))^{-1} = I_m + \sum_{j=1}^{m-1} (D_{qp}\Delta(q))^j \quad (29)$$

Therefore, one can express the inverse with a finite series of matrices of polynomials. In addition, since D_{qp} is a lower triangular matrix with zero diagonal elements, $(I_m - D_{qp}\Delta(q))$ is always invertible. Hence, the LFT model defined by Eqs. (23) is well posed everywhere.

3. CONCLUSIONS

In this paper we present a methodology for systematically constructing LFT representations for second-order nonlinear mechanical systems. The "uncertainty" block of this LFT is a state-dependent structured (diagonal) block. Current research deals with the elimination of Assumptions 1 and 2 and with use of the proposed LFTs for stability analysis and control design for second-order nonlinear mechanical systems.

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