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INVERSE PROBLEMS WITH CONSTRAINTS

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Abstract:
Many design questions in telecommunications, signal processing and control can be formulated as inverse problems. Moreover, there is a surprising degree of similarity in the solutions to the problems from diverse areas. Here we will outline some basic approaches to solving inverse problems and draw connections between different areas of application. Inverse problems become particularly interesting (and difficult) when the solution is required to satisfy hard and/or soft constraints. Recent research results on this topic will be outlined and the impact on practical problems discussed.

1. INTRODUCTION

Most (and arguably all) algorithmic problems in Signal Processing, Telecommunications and Control can be viewed as inverse problems of various types. A rough description of an inverse problem is as follows: given an *outcome* (say, a set of measurements) determine the input that leads to (or lead to) that outcome. This class of problem has a common mathematical foundation but takes on slightly different forms in different fields; e.g.

Signal Processing - Say we are given a set of river levels with errors in location (x, y) and depth (z), and we wish to determine an underlying model of the river depth (Moore and Grayson, 1991), (Moore *et al.*, 1991), (Lane *et al.*, 1994). This can be achieved by carrying out spatial-temporal smoothing of the data accounting for spatial correlations, time evolution and the errors in all measured variables. Note that here we are inverting the procedure of using a model to predict depths.

Telecommunications - Given the output of an unknown and noisy telecommunications channel, determine both the channel characteristics and the transmitted signal (Ding and Li, 1998). Note that here we are inverting the received data to determine a channel model and the transmitted signal.

Control - Given a desired output behaviour for a process, determine the best choice of manipulated variables to drive the process to achieve that desired behaviour (Goodwin *et al.*, 2001*b*). Note that here we are inverting the model of the process to determine the best driving control signal.

These kinds of problem are germane to most (arguably all) algorithm design problems in signal processing, telecommunications and control. The problems are made more difficult when ill-conditioned (i.e. when the forward mapping is near a singularity) or when the inverse must satisfy constraints (e.g. the input is known to take only a finite set of values or we require that internal variables not exceed specified levels).

The solution of inverse problems has been at the center of developments in signal processing (Bertero and Boccaci, 1998), (Sabatier, 1987) telecommunications (Qureshi, 1985) and control (Goodwin *et al.*, 2001*b*) for, at least, the last 50 years. However, the three areas have evolved somewhat disjointedly and each now has its own set of design paradigms. Also, the importance of inversion has often not been explicitly highlighted. Furthermore, different fields have tended to emphasize different aspects. Thus stability has been a major issue in control whereas performance issues have dominated telecommunications and implementation has been paramount in signal processing. Our goal here is to expose the common features of these problems and suggest how cross-fertilization of ideas may lead to improved insights.

2. INVERSE PROBLEMS IN ESTIMATION AND CONTROL

The close connection between inverse problems in estimation and control is highlighted in Figure 1.

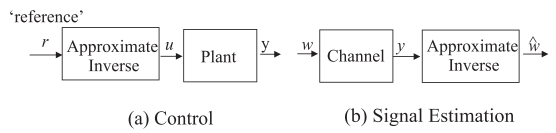

Fig. 1. Inverse problems

In the control context, we usually augment the "reference" signal, r, by other signals (typically derived from an observer) which reflect disturbances and modelling errors. This is a robustifying issue that does not alter the conceptual accuracy of the core principle shown in Figure 1 (a). For example, the class of *all* stabilizing controllers for a stable plant is shown in Figure 2. The "model" block in Figure 2 acts as an observer for plant states, disturbances, etc. In the absence of disturbances and model errors, Figure 1(a) and Figure 2 are identical.

One minor difference between the schemes in Figures 1(a) and 1(b) is that control uses a right inverse whereas estimation uses a left inverse. However, these problems are simply related by a transpose since, in general M.I.M.O. case, $y = Gu$, if and only if, $y^T = u^T G^T$.

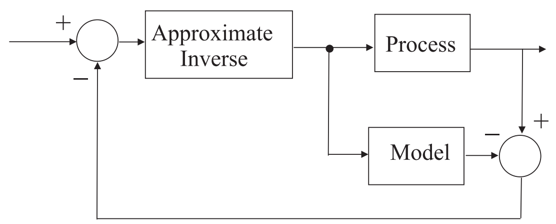

Fig. 2. All stabilizing control laws

To be a little more specific, say that the model of a S.I.S.O. process or channel takes the following form:

$$G(q) = q^{-d}(g_o + G^{'}(q)) \qquad (1)$$

Where q^{-1} is the unit delay operator, g_0 is a static gain and $G^{'}(q)$ is a strictly proper transfer function. Then, assuming that $G(q)$ is minimum phase, the approximate inverses in Figures 1 and 2 can be implemented in the prototype form shown in Figure 3.

When the static nonlinear block (N/L) in Figure 3 is replaced by a gain of unity, then it is readily

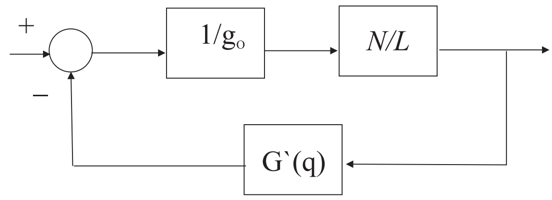

Fig. 3. Ad-hoc inverse

verified that the scheme generates $[q^d G(q)]^{-1}$. Under these conditions, the control law on the left reduces to the well known Smith Predictor (Smith, 1958) or Minimum Variance Control Law (Astrom, 1970) (under specific assumptions about the nature of the disturbances).

3. ADDING CONSTRAINTS TO INVERSE PROBLEMS

The prototype inverse scheme shown in Figure 3 is "constraint-free". There are several reasons why one may wish to add constraints to the inverse problem. Some of these are:

(i) Stability Direct inversion as in Figures 1 to 3 may not preserve stability. Of course, it is well known (e.g., in the linear case) that to ensure stability, various interpolation constraints must be added to the inverse. For example, say that the process is stable, then one needs to avoid any inverse which reflects the singularity associated with unstable zeros into unstable poles. This usually means that some form of approximation is introduced into the inverse calculation. For example, we could ask what transfer function is closest to the inverse in a mean square sense subject to a stability constraint. This leads to an inverse in which unstable zeros are reflected through the stability boundary (Kwakernaak and Sivan, 1972). Similarly, if one solves the estimation problem of Figure 1(b) in a mean square sense with vanishingly small measurement noise, then the dual inverse is obtained (Astrom, 1970).

(ii) Robustness At a conceptual level, the issue of robustness can be cast as requiring an approximate inverse which is not overly sensitive to the fidelity of the model. This requires some form of regularization of the inversion procedure (Goodwin and Welsh, 2001), (Welsh and Goodwin, 2002). For example, in the control context, one should exercise caution when inverting zeros that are close to the stability boundary or which give rise to high gains in particular frequency ranges.

(iii) Amplitude Constraints In practical problems, one frequently requires that the inverse re-

spect certain hard (i.e. amplitude) constraints. For example, say that the signal u in Figure 1(a) is required to lie in the range $-1 \leq u \leq 1$. One simple solution to solve this problem is to replace the nonlinear element in Figure 3 by a saturation function. Indeed, this leads to the most common form of anti-windup solution to S.I.S.O. constrained control problems (Bernstein and Michel, 1995), (Hanus *et al.*, 1987), (Hippe and Wurmthaler, 1997), (Kapoor *et al.*, 1996), (Lozier, 1956), (Kothare *et al.*, 1994). A related problem is when the signal u is required to be one of two levels $\{+1, -1\}$ or more. In this case, it seems reasonable to replace the nonlinear block in Figure 1(a) by a sign function (or quantizer). This leads to an anti-windup form of on-off control and its generalizations (Quevedo *et al.*, 2002).

It is also helpful to introduce the term, "constraint horizon one", to describe this type of strategy. The reason for the use of this terminology is that the constraints are dealt with one time step at a time. Note that this does not imply that the inverse has zero memory. Indeed, the feedback path in Figure 3, which can be thought of as an observer for the system states, gives rise to memory in the inverse (De Doná *et al.*, 2000).

Turning to the estimation problem of Figure 1(b), it is common in digital communications that the transmitted signal, ω is a scalar belonging to a finite alphabet e.g. $\omega \in \Omega = \{s_1, \ldots, s_n\}$ or specifically $\omega = \pm 1$ when $n = 2$. It then makes sense to place a similar restriction on $\hat{\omega}$ in Figure 1 (b). This again suggests the use of the sign function (or quantizer)in Figure 2. Indeed, this is a very well known solution commonly called a Decision Feedback Equalizer (Ding and Li, 1998). Interestingly, the above discussion shows that the Decision Feedback Equalizer problem and the finite set control problem are closely related. More will be said on this topic below.

Naturally, one can go beyond the simple kinds of constraints mentioned above. For example, in the control problem, we sometimes require that certain internal states of the system not exceed specified levels. "Anti-windup", or "constraint horizon one", solutions can also be developed for these kinds of problems (Rojas and Goodwin, 2002).

(iv) The "Cost" of Constraints Naturally, adding constraints to an inverse problem will reduce the achievable performance. This is a topic of considerable practical importance and has attracted continuous interest since the pioneering work of Bode (Bode, 1945) on the impact of stability constraints on sensitivity integrals in the frequency domain. The results on performance limitations due to constraints can be divided into two classes:

(i) limitations which hold for *all* possible designs which respect the constraints, e.g., Bode's sensitivity integrals (Bode, 1945), (Seron *et al.*, 1997). These reflect the inherent trade-offs involved in constrained designs.
(ii) limitations which give lower bounds on the *best* possible performance, e.g., for L_2 performance (Qui and Davison, 1993).

Some results also exist on the connections between these types of limitation – see (Middleton and Braslavsky, 2000). Most existing work holds for linear systems subject to stability constraints. However, there exists some recent work on extensions to nonlinear systems (Braslavsky *et al.*, 1999), (Pérez *et al.*, 2001), (Goodwin *et al.*, 2000) and (Iglesias, 2001).

4. SOLUTIONS VIA OPTIMIZATION

The prototype solution to the approximate inverse problems shown in Figure 3 is ad-hoc. We are thus led to ask if one can cast the inverse approximation problem as an optimization problem. Depending on the criterion, one is lead to various standard optimization problems such as L_1, L_2, L_∞ and combinations thereof (Zhou *et al.*, 1998).

With hard constraints, optimization becomes a "natural" way to proceed. Indeed, one can readily set up either the control or estimation problems as constrained optimization problems. This is standard in the control area where it leads to, so called, Model Predictive Control (Richalet *et al.*, 1976), (Garcia *et al.*, 1989), (Rawlings *et al.*, 1994), (Morari and Lee, 1997), (Mayne *et al.*, 2000), (Qin and Badgwell, 1997), (Mayne, 2000), (Michalska and Mayne, 1993). The dual problem in estimation is known as Receding Horizon Estimation (Michalska and Mayne, 1995), (Muske *et al.*, 1993), (Robertson *et al.*, 1996), (Rao *et al.*, 2001). Both problems have been the subject of intense research interest in recent years.

In the next two sections we will give slightly more detail on the formulation of control and estimation problems for constrained linear systems.

5. CONSTRAINED CONTROL

Model Predictive Control (or MPC) is a control algorithm based on solving an open-loop *constrained* optimal control problem (Mayne *et al.*, 2000). A *receding horizon* approach is then used, to implement the control law. This can be summarised in the following steps:

(i) At time k and for the current state $x(k)$, solve, (typically on-line), an open-loop optimal control problem over some future interval, taking into account the *current* and *future* hard constraints on inputs and states.

(ii) Apply the first step in the optimal control sequence.

(iii) Repeat the procedure at time $(k+1)$ using the current state $x(k+1)$.

The solution is converted into a closed-loop strategy by using the measured value of $x(k)$ as the current state. When $x(k)$ is not directly measured, then one can obtain a closed-loop policy by replacing $x(k)$ by an estimate provided by some form of observer (Pérez and Goodwin, 2001). In a general nonlinear setting, the method is as follows.

Given a model

$$x(\ell+1) = f(x(\ell), u(\ell)), \qquad x(k) = x\,, \tag{2}$$

the MPC at event (x,k) is computed by solving a constrained optimal control problem:

$$\mathcal{P}_N(x): \qquad V_N^{\text{OPT}}(x) = \min_{\mathbf{u}\in\mathcal{U}_N} V_N(x,\mathbf{u})\,,$$

where

$$\mathbf{u} = \{u(k), u(k+1), \dots, u(k+N-1)\}\,,$$
$$V_N(x,\mathbf{u}) = \sum_{\ell=k}^{k+N-1} L(x(\ell),u(\ell)) + F(x(k+N))\,,$$

and $\mathcal{U}_N$ is the set of $\mathbf{u}$ that satisfy the constraints over the entire interval $[k, k+N-1]$:

$$u(\ell) \in \mathbb{U} \qquad \ell = k, k+1, \dots, k+N-1\,,$$
$$x(\ell) \in \mathbb{X} \qquad \ell = k, k+1, \dots, k+N\,,$$

together with the terminal constraint

$$x(k+N) \in \mathbb{X}_f\,.$$

Usually, $\mathbb{U} \subset \mathbb{R}^m$ is convex and compact, $\mathbb{X} \subset \mathbb{R}^n$ is convex and closed, and $\mathbb{X}_f$ is a set that can be appropriately selected to achieve stability (Mayne *et al.*, 2000).

In the above formulation, the model and cost are time invariant. Hence, one obtains a time-invariant feedback control law. In particular, we can set $k=0$ in the open loop control problem without loss of generality. Then at event (x,k) we solve:

$$\mathcal{P}_N(x): \qquad V_N^{\text{OPT}}(x) = \min_{\mathbf{u}\in\mathcal{U}_N} V_N(x,\mathbf{u})\,, \tag{3}$$

where

$$\mathbf{u} = \{u(0), u(1), \dots, u(N-1)\}\,, \tag{4}$$
$$V_N(x,\mathbf{u}) = \sum_{\ell=0}^{N-1} L(x(\ell),u(\ell)) + F(x(N))\,, \tag{5}$$

subject to

$$u(\ell) \in \mathbb{U} \qquad \ell = 0, 1, \dots, N-1\,, \tag{6}$$
$$x(\ell) \in \mathbb{X} \qquad \ell = 0, 1, \dots, N\,, \tag{7}$$
$$x(N) \in \mathbb{X}_f\,. \tag{8}$$

Let the minimising open-loop control sequence be

$$\mathbf{u}^{\text{OPT}}(x) = \{u^{\text{OPT}}(0;x), u^{\text{OPT}}(1;x), \dots, u^{\text{OPT}}(N-1;x)\}\,; \tag{9}$$

then the actual control applied at time k is the first element of this sequence, i.e.,

$$u = u^{\text{OPT}}(0;x)\,. \tag{10}$$

Time is then stepped forward one instant, and the above procedure is repeated for another N-step-ahead optimisation horizon. The first input of the new N-step-ahead input sequence is then applied. The above procedure is repeated endlessly.

The above MPC *implicitly* defines a time-invariant control policy $\mathcal{K}_N : \mathbb{X} \to \mathbb{U}$ of the form

$$\mathcal{K}_N(x) = u^{\text{OPT}}(0;x)\,. \tag{11}$$

It is usual in MPC to compute *numerically*, at event (x,k), the optimal control move $\mathcal{K}_N(x)$ rather than pre-computing the control law $\mathcal{K}_N(\cdot)$. However, as we shall argue below, there are cases where $\mathcal{K}_N(\cdot)$ can be pre-computed and the control law implemented via evaluation of $\mathcal{K}_N(\cdot)$. When the system is linear, the model can be expressed as

$$x(\ell+1) = Ax(\ell) + Bu(\ell)\,, \tag{12}$$
$$y(\ell) = Cx(\ell)\,, \tag{13}$$

where $x(\ell) \in \mathbb{R}^n$, $u(\ell) \in \mathbb{R}^m$, and $y(\ell) \in \mathbb{R}^m$.

Reference tracking and disturbance rejection can be readily added to the basic setup described above. Thus, consider the problem of tracking a constant setpoint y_s and rejecting a time-varying output disturbance $\{d(\ell)\}$. We can formulate this as a regulation problem by defining the error

$$e(\ell) = y(\ell) + (d(\ell) - y_s)\,. \tag{14}$$

It is convenient to make no distinction between the output disturbance and the setpoint. We thus define an "equivalent" output disturbance d_e as the signal

$$d_e(\ell) = d(\ell) - y_s\,. \tag{15}$$

Without loss of generality, we take the current time as 0.

Assuming knowledge of the external signal d_e and the current state measurement $x(0) = x$, then the M-move control sequence $\mathbf{u} = \{u(0), u(1), \dots, u(M-1)\}$ is defined by minimising the finite-horizon cost:

$$
\begin{aligned}
V_{N,M}(x,\mathbf{u}) =& [x(N)-x_s]^{\mathrm{T}} Q_f [x(N)-x_s] \\
&+ \sum_{\ell=0}^{N-1} e^{\mathrm{T}}(\ell) Q e(\ell) \\
&+ \sum_{\ell=0}^{M-1} [u(\ell)-u_s]^{\mathrm{T}} R [u(\ell)-u_s] ,
\end{aligned} \tag{16}
$$

where $Q \geq 0$, $R > 0$, $Q_f \geq 0$. In (16), N is the prediction horizon and $M \leq N$ is the control horizon. The quantities, u_s and x_s, are steady state values defined below.

We assume that $d_e(\ell)$ in (15) contains time-varying components as well as a constant (steady-state) component, denoted by $\bar{d}_e$. Then, from (15),

$$
\bar{d}_e = \bar{d} - y_s , \tag{17}
$$

where $\bar{d}$ is the constant component of the output disturbance $d(\ell)$.

In equation (16), we then let u_s and x_s be the steady state values of u and x:

$$
u_s = -[C(I-A)^{-1}B]^{-1}\bar{d}_e , \tag{18}
$$

$$
x_s = (I-A)^{-1} B u_s . \tag{19}
$$

The minimisation of (16) is performed on the assumption that the control reaches its steady state value after M steps, that is $u(\ell) = u_s$, $\forall \ell \geq M$. We also assume that the setpoint y_s and the corresponding input and steady state values, u_s and x_s, are feasible, i.e., they satisfy the required constraints. The dynamic optimisation problem described above is transformed into a *nondynamic constrained quadratic program*. To see this, we start by writing, from (12) and using $Bu_s = (I - A)x_s$: [1]

$$
\mathbf{x} - \mathbf{x}_s = \Gamma\,\mathbf{u} + \Omega\,x - \overline{\mathbf{x}}_s , \tag{20}
$$

where

$$
\mathbf{x} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} ; \quad
\mathbf{x}_s = \begin{bmatrix} x_s \\ x_s \\ \vdots \\ x_s \end{bmatrix} ; \quad
\mathbf{u} = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(M-1) \end{bmatrix} ; \tag{21}
$$

$$
\Omega = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} ; \overline{\mathbf{x}}_s = \begin{bmatrix} x_s \\ x_s \\ \vdots \\ x_s \\ Ax_s \\ \vdots \\ A^{N-M}x_s \end{bmatrix} ;
$$

$$
\Gamma = \begin{bmatrix}
B & 0 & \dots & 0 & 0 \\
AB & B & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A^{M-1}B & A^{M-2}B & \dots & AB & B \\
A^{M}B & A^{M-1}B & \dots & A^2B & AB \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A^{N-1}B & A^{N-2}B & \dots & \dots & A^{N-M}B
\end{bmatrix} .
$$

Then, using (20), (21) and

$$
\mathbf{Q} = \mathrm{diag}[C^{\mathrm{T}}QC, \dots, C^{\mathrm{T}}QC, Q_f] , \tag{22}
$$

$$
\mathbf{R} = \mathrm{diag}[R, \dots, R] , \tag{23}
$$

$$
\mathbf{u}_s = [u_s^{\mathrm{T}}\ u_s^{\mathrm{T}}\ \dots\ u_s^{\mathrm{T}}]^{\mathrm{T}} , \tag{24}
$$

$$
\begin{aligned}
\mathbf{D} =& [d_e(0)^{\mathrm{T}} - \bar{d}_e^{\mathrm{T}}, d_e(1)^{\mathrm{T}} - \bar{d}_e^{\mathrm{T}}, \dots d_e(N-1)^{\mathrm{T}} \\
&- \bar{d}_e^{\mathrm{T}},\ 0]^{\mathrm{T}} ,
\end{aligned} \tag{25}
$$

$$
Z = \mathrm{diag}[C^{\mathrm{T}}Q, C^{\mathrm{T}}Q, \dots C^{\mathrm{T}}Q] , \tag{26}
$$

we can express (16) as

$$
\begin{aligned}
V_{N,M}(x,\mathbf{u}) =& e^{\mathrm{T}}(0)Qe(0) \\
&+ (\mathbf{x}-\mathbf{x}_s)^{\mathrm{T}}\mathbf{Q}(\mathbf{x}-\mathbf{x}_s) \\
&+ (\mathbf{u}-\mathbf{u}_s)^{\mathrm{T}}\mathbf{R}(\mathbf{u}-\mathbf{u}_s) \\
&+ 2(\mathbf{x}-\mathbf{x}_s)^{\mathrm{T}}Z\mathbf{D} \\
&+ \mathbf{D}^{\mathrm{T}}\,\mathrm{diag}[Q, Q, \dots, Q]\mathbf{D} \\
=& \overline{V} + \mathbf{u}^{\mathrm{T}}W\mathbf{u} + 2\mathbf{u}^{\mathrm{T}}(Fx+H) .
\end{aligned} \tag{27}
$$

In (27), $\overline{V}$ is independent of $\mathbf{u}$ and

$$
W = \Gamma^{\mathrm{T}}\mathbf{Q}\Gamma + \mathbf{R} , \qquad F = \Gamma^{\mathrm{T}}\mathbf{Q}\,\Omega \tag{28}
$$

$$
H = -\Gamma^{\mathrm{T}}\mathbf{Q}\,\overline{\mathbf{x}}_s - \mathbf{R}\mathbf{u}_s + \Gamma^{\mathrm{T}}Z\mathbf{D} . \tag{29}
$$

Next we show how constraints can be introduced into the problem formulation. Magnitude and rate constraints on the plant *input* and *output* can be expressed as follows, for $\ell = 0, 1, \dots, T-1$:

$$
\begin{aligned}
u_{\min} \leq u(\ell) \leq u_{\max} , \\
y_{\min} \leq y(\ell) \leq y_{\max} , \\
\delta u_{\min} \leq u(\ell) - u(\ell-1) \leq \delta u_{\max} .
\end{aligned} \tag{30}
$$

These constraints can be written as *linear* constraints on $\mathbf{u}$ of the form

$$
L\mathbf{u} \leq K . \tag{31}
$$

We note that optimization of (27) subject to (31) is a standard Quadratic Programming Problem (Bazaraa and Shetty, 1979). It is also possible to include finite alphabet constraints as detailed in (Quevedo *et al.*, 2002). In these various optimization problems it is possible to utilize different values for the prediction horizon, N, input horizon, M, and constraint horizon, T. Typically, one chooses $O < T \leq M \leq N$. The case, $T = 1$, corresponds to the "constraint horizon one" solution.

[1] With a slight abuse of notation, we use **bold** letters to denote both sequences and "piles" of vectors.

6. CONSTRAINED STATE ESTIMATION

A similar procedure to that described in section 5 can be used for constrained state, estimation problems (Michalska and Mayne, 1995), (Muske *et al.*, 1993), (Robertson *et al.*, 1996). To illustrate, consider a linear discrete time system of the form:

$$x_{k+1} = Ax_k + B\omega_k \tag{32}$$
$$y_k = Cx_k + \nu_k \tag{33}$$

where $\{\omega_k\}$, $\{\nu_k\}$ are white noise sequences of covariance Q and R respectively around mean values of $\{\bar{\omega}_k\}$ and zero. We also assume that x_o has a-priori mean $\hat{x}_{o|o}$ and covariance $P_{o|o}$. If we assume Gaussian distributions for the errors, then the negative log-likelihood function given data $\{y_1, \ldots, y_N\}$ can be written

$$\begin{aligned} J =& \frac{1}{2}(\hat{x}_o - \hat{x}_{o|o})^{\mathrm{T}} P_{o|o}^{-1}(\hat{x}_o - \hat{x}_{o|o}) \\ &+ \frac{1}{2}\sum_{k=1}^{N}(y_k - C\hat{x}_k)^{\mathrm{T}} R^{-1}(y_k - C\hat{x}_k) \\ &+ \frac{1}{2}\sum_{k=0}^{N-1}(\hat{\omega}_k - \bar{\omega}_k)^{\mathrm{T}} Q^{-1}(\hat{\omega}_k - \bar{\omega}_k) \end{aligned} \tag{34}$$

where $\hat{x}_k$, $\hat{\omega}_k$ are related by

$$\hat{x}_{k+1} = A\hat{x}_k + \hat{\omega}_k \quad ; \quad k = 0, \ldots, N-1 \tag{35}$$

In the unconstrained case, it is well known that optimizing (34) subject to (35) with respect to $\{\hat{x}_o,\ \hat{\omega}_o,\ ,\ldots,\hat{\omega}_{N-1}\}$ leads to the Kalman Filter. The problem posed above can be solved in a variety of ways including (forward) Dynamic Programming etc.. We can also readily recast the above fixed interval estimation problem into the familiar framework of quadratic optimization. Specifically, let

$$Z = \begin{bmatrix} \hat{x}_o \\ \hat{\omega}_o \\ \vdots \\ \hat{\omega}_{N-1} \end{bmatrix} \tag{36}$$

Then the cost J in (34) can be expressed as:

$$J = \frac{1}{2}Z^{\mathrm{T}} H Z + d^{\mathrm{T}} Z \tag{37}$$

where

$$H = \left[L^{\mathrm{T}} P_{o|o}^{-1} L + M^{\mathrm{T}} \mathbf{R}^{1} M + S^{\mathrm{T}} \Omega^{-1} S\right] \tag{38}$$

$$d = -\left[L^{\mathrm{T}} P_{o|o}^{-1} \hat{x}_{o|o} + M^{\mathrm{T}} \mathbf{R}^{-1} Y + S^{\mathrm{T}} \Omega^{-1} \overline{W}\right] \tag{39}$$

with

$$L = [I, 0, \ldots, 0]^{\mathrm{T}} \;;\; S = \begin{bmatrix} 0 & I & & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & I \end{bmatrix} \;; \tag{40}$$

$$M = \begin{bmatrix} CA & CB & & \\ CA^2 & CAB & CB & \\ \vdots & & & \ddots \\ CA^N & \cdots & \cdots & CB \end{bmatrix} \tag{41}$$

$$\mathbf{R}^{-1} = \mathrm{diag}\left[R^{-1}\right] \;;\; \Omega^{-1} = \mathrm{diag}\left[Q^{-1}\right] \;; \tag{42}$$

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} ; \overline{W} = \begin{bmatrix} \hat{\omega}_o \\ \vdots \\ \hat{\omega}_{N-1} \end{bmatrix} \tag{43}$$

Linear inequality constraints on $\{\hat{x}_k\ ;\ k = 0, \ldots, N\}$ and/or $\{\hat{\omega}_k\ ;\ k = 0, \ldots, N-1\}$ can be expressed as

$$\Lambda Z \leq g \tag{44}$$

where the above inequality should be interpreted componentwise. This problem is then in the form of a standard QP problem (Bazaraa and Shetty, 1979) as in section 5.

7. DUALITY ISSUES

Both of the problems in section 5 and 6 can readily be written in dual forms. For example, the Lagrangian dual of problem (37) subject to (44) can be expressed as:

Maximize

$$\frac{1}{2}\lambda^{\mathrm{T}} D\lambda + \lambda^{\mathrm{T}}\gamma \tag{45}$$

subject to

$$\lambda \geq 0 \tag{46}$$

where

$$D = -\Lambda H^{-1}\Lambda^{\mathrm{T}} \tag{47}$$
$$\gamma = -g - \Lambda H^{-1} d \tag{48}$$

Indeed, it is instructive to expose the duality in detail wherein one sees that the dual of a finite horizon constrained state estimation problem is a finite horizon constrained control problem and vice versa.

As explained in section 5, in control, one typically optimises over a *future* horizon $\{k+1, \ldots, k+N\}$ with respect to the controls $\{u_k, \ldots, u_{k+N-1}\}$. One then implements u_k and the state evolves to x_{k+1}. One then repeats the optimization based on the horizon $\{k+2, \ldots, k+N+1\}$ and so on. Thus, the state evolution from time k to $k+1$ provides a natural way of interconnecting the various finite horizon problems.

In estimation, one typically optimizes over a *past* horizon $\{k-N, \ldots, k\}$ given data $\{y_{k-N+1}, \ldots, y_k\}$ leading to the current estimate $\hat{x}_{k|k}$. Based on the cost function given in (37), this begs the question of where $\hat{x}_{k-N|k-N}$ comes from. A natural choice is to set this equal to the final estimate obtained from an earlier fixed interval estimation problem (Rao *et al.*, 2001). We call this Rolling Horizon Estimation and note that it requires that N previous final state estimates be stored for future use (De Doná *et al.*, 2002). Of course, in the absence of constraints, this rolling horizon strategy will again lead to the Kalman filter provided $P_{k|k}^{-1}$ is chosen as the appropriate estimation covariance.

In subsection 10.6 we will review a specific application of Duality ideas to a practical problem.

8. DECISION FEEDBACK EQUALIZERS REVISITED

To give a further illustration of constrained estimation, we will reinterpret the solution shown in Figure 1(b) and 3. To better link with contemporary literature, assume that $G(q)$ in (1) has a finite impulse response (FIR), i.e., $G'(q) = \Sigma_{i=1}^{\ell} g_i q^{-i}$, and that the scalar $\{\omega_k\}$ is drawn from a finite alphabet. We can readily formulate the model in state space form by defining

$$x_k \triangleq \begin{bmatrix} \omega_{k-1} \\ \omega_{k-2} \\ \vdots \\ \omega_{k-\ell-d} \end{bmatrix} \tag{49}$$

Then, the FIR model can be written as

$$x_{k+1} = Ax_k + B\omega_k \tag{50}$$

$$y_k = Cx_k + \nu_k \tag{51}$$

where $\{\omega_k\}$, $\{\nu_k\}$ represent the input data stream and the measurement noise sequence respectively.

Here, A,B,C take the following special forms:

$$A = \begin{bmatrix} 0 & \dots & \dots & 0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{bmatrix} \; ; \; B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \end{bmatrix} \tag{52}$$

$$C = [\underbrace{0 \dots 0}_{d-1} \; \underbrace{g_o \dots g_\ell}_{\ell+1}] \tag{53}$$

As before, we begin by taking $\{\omega_k\}$, $\{\nu_k\}$ as Gaussian noise sequences of variance Q and R respectively. Taking $\{\bar{\omega}_k = 0\}$ and optimizing (34) or (37), without constraints, leads to the standard Kalman Filter, i.e.,

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + P_{k|k-1}C^{\mathrm{T}} (R + CP_{k|k-1}C^{\mathrm{T}})^{-1}(y_k - C\hat{x}_{k|k-1}) \tag{54}$$

$$P_{k|k} = P_{k|k-1} - P_{k|k-1}C^{\mathrm{T}} (R + CP_{k|k-1}C^{\mathrm{T}})^{-1}CP_{k|k-1} \tag{55}$$

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} \tag{56}$$

$$P_{k+1|k} = AP_{k|k}A^{\mathrm{T}} + BQB^{\mathrm{T}} \tag{57}$$

If we define P as the solution of the associated steady state Riccati equation, then $P_{0|0}$ in (34) is defined via (55) with $P_{k|k-1}$ replaced by P. In this case (54) takes the time variant form:

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + J(y_k - C\hat{x}_{k|k-1}) \tag{58}$$

where

$$J = PC^{\mathrm{T}}(R + CPC^{\mathrm{T}})^{-1} = \begin{bmatrix} J_1 \\ \vdots \\ J_{\ell+d} \end{bmatrix} \tag{59}$$

A simple special case arises when $R \to 0$ (vanishing measurement noise) and when the zeros of the channel model are all stable. In this case, it is readily seen that the steady state solution, $P_o = [P_{ij}]$ to the Riccati equations becomes:

$$P_{ij} = \left\{ \begin{array}{ll} Q & \text{for} \quad i = j \le d \\ 0 & \text{otherwise} \end{array} \right\} \tag{60}$$

so that

$$J_i = \left\{ \begin{array}{ll} \dfrac{1}{g_o} & \text{for} \quad i = d \\ 0 & \text{otherwise} \end{array} \right\} \tag{61}$$

Substitution into (49) leads to the intuitively appealing result:-

$$\hat{\omega}_{k|N} = 0 \quad \text{for} \quad k = N, \dots, N-d+1 \tag{62}$$

$$\hat{\omega}_{N-d|N} = \frac{1}{g_o}\left[y_N - g_1\hat{\omega}_{N-d-1|N-1} \cdots - g_\ell\hat{\omega}_{N-d-\ell|N-1}\right] \tag{63}$$

$$\hat{\omega}_{N-d-j|N} = \hat{\omega}_{N-d-j|N-1} \; ; \; j = 1, \dots, \ell \tag{64}$$

Perhaps, not unexpectedly, the above solution corresponds exactly to the inverse described in Figure 3 where the N/L block is replaced by a unity gain.

Next, we consider the case where $\{\omega_k\}$ is known to be drawn from a finite alphabet. In this case, we can proceed by optimizing (34) with $P_{o|o}$ chosen as above and subject to (35) and an additional set of finite set constraints, e.g.

$$\{\hat{\omega}_k\} \in \Omega \tag{65}$$

where Ω is defined in (iii) of section 3.

Due to the discrete nature of the constraint (65), this is a non-convex problem. However, we can proceed to solve the problem using Dynamic Programming. Indeed, this strategy is well known and is usually called the Viterbi algorithm (Viterbi and Omura, 1979). An interesting special case is when we take the optimization horizon as $N = 1$. In this case, one can solve the constrained optimization problem in a closed form, leading to

$$\hat{\omega}_{N-d|N} = q_{\Omega}\{\Theta\} \quad (66)$$

where

$$\Theta = \frac{1}{g_o}[y_N - g_1\hat{\omega}_{N-d-1|N-1} \cdots - g_{\ell}\hat{\omega}_{N-d-\ell|N-1}] \quad (67)$$

where $q_{\Omega}\{\cdot\}$ is the nearest neighbour mapping to the set Ω e.g. q_{Ω} is the sign function in the case that $\Omega = \{-1, +1\}$ – see (Quevedo *et al.*, 2002).

Interestingly, we see that we have rederived the well known Decision Feedback Equalizer solution (Qureshi, 1985). By changing the Optimization Horizon one can generate a spectrum of constrained estimates between Decision Feedback and the full Viterbi algorithm (Forney Jr., 1972) – See also (Eyuboglu and Quereshi, 1988)(Forney Jr. *et al.*, 1984), (Duel-Hallen and Heegard, 1989) and especially (Williamson *et al.*, 1992) and the related work (Quevedo *et al.*, 2002).

9. SOME RECENT RESEARCH RESULTS

There exists a vast literature on the topics of constrained control and estimation. Some of the topics addressed in the recent literature include:

- stability of receding horizon controllers (Mayne *et al.*, 2000), (De Doná *et al.*, 2001)
- stability of anti-windup control schemes (Kapoor *et al.*, 1996)
- connections between anti-windup (in the sense of "constraint horizon one") and receding horizon controllers (De Doná and Goodwin, 2000)
- stability of decision feedback equalizers (Kennedy *et al.*, 2000)
- stability of rolling horizon estimators (Robertson *et al.*, 1996)
- extension to nonlinear systems (Mayne *et al.*, 2000), (Mayne, 2000), (Muske *et al.*, 1993), etc.

A particular observation which may have practical importance (Bemporad *et al.*, 2002), (Seron *et al.*, 2000*b*), (Seron *et al.*, 2000*a*) is that constrained linear control problems with interval constraints of the type discussed in section 5 lead to a *finite set* of linear feedback control laws each of which holds in a particular partition of state space. This gives a finitely paramaterized form of the state feedback control law of (11). This explicit form of the map $K_N(\cdot)$ in (11) allows one to implement constrained Receding Horizon Control by an on-line table look up. Indeed, the "constraint horizon one" results of (De Doná *et al.*, 2000) can be thought of as a special case of these more general structures. For finite alphabet constraints, an explicitly parameterized solution is also possible (Quevedo *et al.*, 2002). These results allow the full range of closed-loop behaviour to be verified prior to implementation. This may enhance the practical appeal of these methods in time-critical applications as well as enhancing the understanding of the nature of constrained control polices.

10. SOME APPLICATIONS

We illustrate the application of constrained inverse problems to several real world questions.

10.1 *Noise Shaped Wordlength Reduction in CD Production*

When the output of a 44.1 kHz 24-bit digital mixing console is fed to the master stage in the production of a CD, the wordlength must be reduced to 16 bits. In essence this is a constrained inverse problem due to the quantized nature of the output. A typical scheme used in practice is shown in Figure 4 (McGrath, 2002).

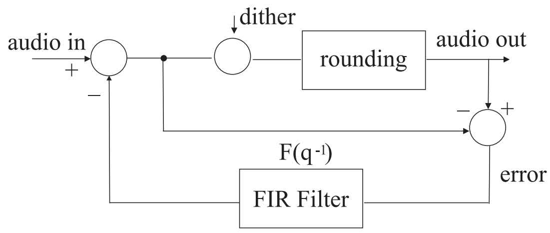

Fig. 4. Noise shaped wordlength reduction

We see that this feedback scheme is a two-degree-of-freedom form of the scheme shown in Figure 2. The dither signal in Figure 4 is used to ensure that the quantization errors are 'decorrelated' from the audio signal. The purpose to the FIR filter is to force the quantization noise to fit a particular profile. Further constraints usually appear due to the fact that it is desirable that the filter have a simple 2's arithmetic implementation. The least mean square output noise occurs when $F(q^{-1})$ is chosen as zero. However, there is a 'frequency dimension' to this problem since the human ear is less sensitive to high frequency 'noise'. Thus the design goal is to adjust the transfer function, $F(q^{-1})$ so as to shape the frequency content of the errors. Taking a linear view of this problem, we see that the transfer function connecting the quantization noise to the audio output is

$$S(q^{-1}) = 1 - F(q^{-1}) \tag{68}$$

A natural question that might be asked here would be; “Are there any limitations on the extent to which the quantization noise can be reduced?” Of course, we recognize $S(q^{-1})$ as a standard feedback sensitivity function which must satisfy a Bode type integral; i.e. $\int_0^{\pi} log|S(e^{i\theta})|d\theta = 0$. We thus know that reduction of the quantization noise level in one frequency band (usually low frequencies) must be accompanied by an increase in quantization noise level in other frequency bands (usually high frequencies). A typical Bode plot for $S(q^{-1})$ is shown in Figure 5. Figure 4 gives a “constraint horizon one” solution. One can also readily conceive of other schemes, based on optimization, which use larger constraint horizons.

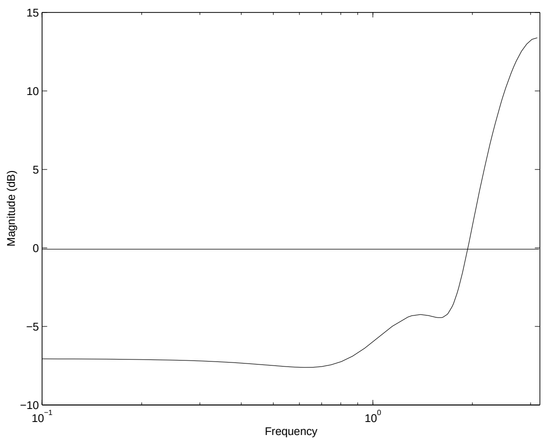

Fig. 5. Bode plot

10.2 *Active Noise Cancelling Headsets*

Most of the research and theory in noise cancelling uses an adaptive feed-forward system. (i.e. you usually have 2 measuring transducers, one to measure the noise input to the system, and one to measure the residual/error noise at the cancellation point). However, the noise cancellation done in most cheap headphones (as used on some airlines etc...) does not use feed-forward. Instead, a simple feed-back loop is used, with a single microphone inside the headset, close to the listener’s ear entrance. A simplified view is shown in Figure 6.

This can again be converted into a constrained inversion problem. Again using the standard Bode theory, it is easy to understand the fundamental limits that apply in this system. For example;

(a) Group delay (from speaker to microphone) is the single biggest problem, because it reduces the maximum bandwidth that the noise reduction will work over (DC-1.5 kHz is typical).
(b) If you want to reduce the noise in one band (DC-1.5kHz) you must also suffer an increase in the noise outside of this band.

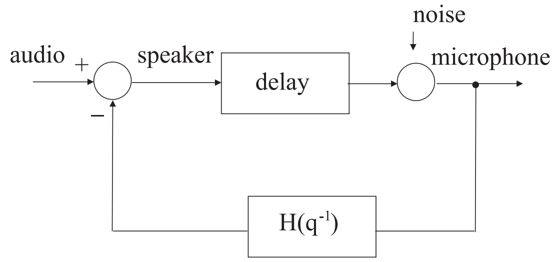

Fig. 6. Noise cancellation

10.3 *Rudder Roll Stabilization (RRS)*

Besides controlling the heading of a ship, it is also desirable to reduce the rolling motion produced by the waves so as to prevent cargo damage and improve crew efficiency and passenger comfort. Conventional methods for ship roll stabilization include water tanks, stabilizing fins, and bilge keels; however, RRS is attractive because no extra equipment needs to be added to the ship and it performs similarly to other methods (Fossen, 1994).

From the control design point of view, the main limiting factor of the RRS problem is the highly non linear behavior of the mechanical devices that command the rudder. These devices essentially impose constraints on the maximum slew rate and excursion attainable for the rudder movement. This, makes, in some cases, traditional linear control design inadequate since the generated control signals are typically too large to be realistic (van Amerongen *et al.*, 1990). MPC is clearly a candidate to solve this constrained inverse problem (Goodwin *et al.*, 2000).

In order to describe the motion of a ship, six independent coordinates are necessary. The first three coordinates and their time derivatives correspond to the position and translational motion while the other three coordinates and their time derivatives correspond to orientation and rotational motion description. For marine vehicles, the six different motion components are called: *surge, sway, heave, roll, pitch,* and *yaw*. Accordingly, the most generally used notation for theses quantities are: x, y, z, ϕ, θ, and ψ respectively, while their time derivatives are denoted u, v, w, p, q, and r respectively. Figure 7 shows the coordinate definitions and the most generally adopted reference frame.

Special features of this control problem include:

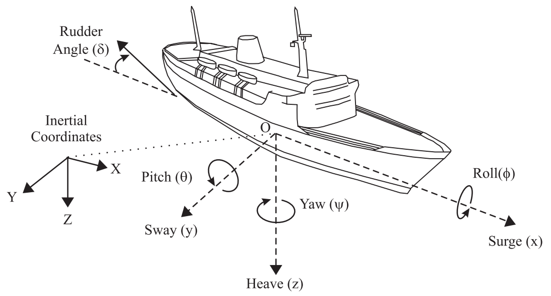

Fig. 7. Magnitudes and conventions for ship motion description

1 The system has two outputs (namely roll and yaw), but only one manipulated variable (rudder angle).
2 The system is ill-conditioned. Specifically, it is nonminimum phase (i.e. the roll response initially goes in the wrong direction). This leads to a form of ill-conditioning in the associated inverse control problem since the model contains a singularity in a region where it cannot be cancelled. It is known that this kind of ill-conditioning is central to linear control performance constraints (see (Seron *et al.*, 1997)).
3 The system is marginally stable (i.e. the yaw will integrate in response to a rudder displacement).
4 The input (rudder) is subject to significant constraints on both amplitude and slew rate.

Hence this problem captures many of the features associated with typical ill-conditioned inverse problems incorporating constraints.

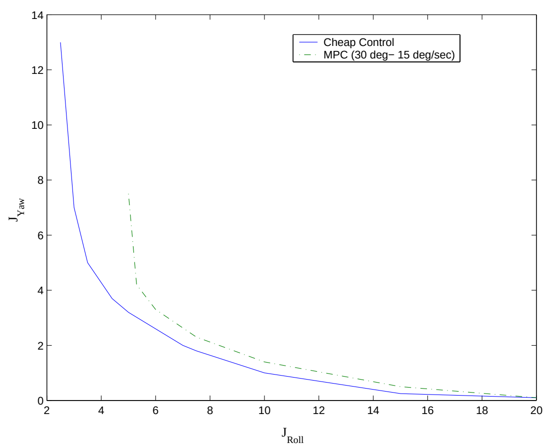

Fig. 8. Preliminary design results for RRS

Preliminary design results are shown in Figure 8. The curve marked *Cheap Control* shows the tradeoff between reducing roll (horizontal axis) and the inevitable impact on yaw (vertical axis). This curve shows the greater the roll reduction, the higher is the cost in terms of yaw perturbations. Also, note that, no matter how large the control effort, there is a limit to how much the roll can be reduced – this is a consequence of the nonminimum phase character of the system which means that exact inversion is not possible. The curve marked *MPC* applies when constraints (rudder, amplitude and slew rate) are added to the problem. We see that the presence of constraints impacts in a negative way on the performance tradeoff and that the best achievable roll reduction is degraded relative to the unconstrained case (Goodwin *et al.*, 2000).

10.4 *Cross Directional Control of Shape in Rolling Mills*

Typically, cross directional control of web forming processes is performed with an identical array of actuators and a corresponding identical array of sensors. Industrial applications are typically under-actuated since, actuator movements are constrained and thus are generally unable to remove all steady state spatial frequency modes of disturbances. Such problems occur in a variety of applications including paper machines, (Dumont, 1986), (Smook, 1982), (Stewart *et al.*, 1998), flat metal rolling (Goodwin *et al.*, 1990), (Carney *et al.*, 1990), (Edwards *et al.*, 1995) and plastic film extrusion (Heath, 1996), (Levy and Carley, 1989). One of the characteristic features of this kind of problem is that each actuator influences a broad cross directional area which typically exceeds the area “seen” by a single corresponding sensor.

Various cross directional control strategies have been described in the literature. For example, Model Predictive Control (MPC) (Heath, 1996), (Garcia *et al.*, 1989) and its many derivatives has been implemented on a verity of web forming processes (Featherstone *et al.*, 2000).

Key issues (Heath, 1996), (Duncan and Bryant, 1997), (Stewart, 2000), (Heaven *et al.*, 1994) (Bergh and MacGregor, 1987), (Goodwin *et al.*, 2001*a*) in this problem are (i) the high degree of coupling between the actuators responses; i.e. the system is almost singular and some form of regularization is needed when evaluating the control via inversion and (ii) the actuators each have very low authority; i.e. input amplitude constraints play a central role in the solution.

10.5 *Control over Communication Networks*

A topic that brings together many elements of signal processing, telecommunications and control is that of control over telecommunication networks. This is likely to become of increasing practical importance in the future since it allows industries to benefit from performing control functions via a single wireless or hard-wired channel. Already, significant research has been carried out in this general area, see (Edited, 2001), (Hristu and Morgansen, 1999), (Ray, 1998), (Nilsson *et al.*, 1998), (Wong and Brockett, 1997), (Wong and Brockett, 1999), (Walsh *et al.*, 1999), (Nair and Evans, 2000), (Tatikonda *et al.*, 1998), (Hristu, 2000).

Typical constrained inverse questions relate to

(i) Design of the coder/decoder to optimize the performance in the face of quantization and channel bandwidth constraints (Gray and Neuhoff, 1998), (Gray, 1996) (Elia and Mitter, 2001).
(ii) Understanding the fundamental design issues associated with control over a communication channel, including an understanding of the *price* one pays for various constraints, e.g. quantization, band limited transmission and communication delays.

10.6 *Filter Design with Quantized Coefficients*

Consider the problem of optimal (FIR) filter design when the filter coefficients are required to belong to a finite (quantized) set. This problem typically arises in high speed signal processing applications so as to facilitate implementation (McGrath, 2002). Say that the signal model is as in (32), (33) and we are interested in a particular (scalar) combination of the states:

$$z(k) = f^{\mathrm{T}} x(k) \tag{69}$$

We assume that z(k) is estimated by a linear FIR filter of the form:

$$\hat{z}(k) = \sum_{j=0}^{k-1} h(k-j) y(j) + g^{\mathrm{T}} \hat{x}(0) \tag{70}$$

where $\{h(k)\}$ is the filter pulse response. We assume that the initial state $x(0)$ of the system satisfies:

$$E\{(x(0) - \hat{x}(0))(x(0) - \hat{x}(0))^{\mathrm{T}}\} = P_o \tag{71}$$

Substituting (70) into (69) and using (50), (51)

$$\tilde{z}(k) = z(k) - \hat{z}(k) \tag{72}$$

$$= f^{\mathrm{T}}(k) - \sum_{j=0}^{k-1} h(k-j)(Cx(j) + \nu(j)) - g^{\mathrm{T}} \hat{x}(0) \tag{73}$$

We introduce Lagrangian adjoint variables satisfying:

$$\lambda(j-1) = A^{\mathrm{T}} \lambda(j) + C^{\mathrm{T}} u(j) \; ; \; \lambda(k-1) = -f \tag{74}$$

where the input, $\{u(k)\}$ represents the reverse time filter coefficients, i.e.

$$u(j) = h(k-j) \tag{75}$$

Substituting (74) into (72) and summing gives:

$$\tilde{z}(k) = -x(0)^{\mathrm{T}} \lambda(-1) - g^{\mathrm{T}} \hat{x}(0) + \sum_{j=0}^{k-1} \lambda(j)^{\mathrm{T}} \omega(j) + u(j) \nu(j) \tag{76}$$

Setting $g = -\lambda(-1)$; squaring and taking mathematical expectation gives

$$E\{\tilde{z}(k)^2\} = \lambda(-1)^{\mathrm{T}} P_o \lambda(-1) + \sum_{j=0}^{k-1} \lambda(j)^{\mathrm{T}} Q \lambda(j) + u(j)^{\mathrm{T}} R u(j) \tag{77}$$

Hence the optimal FIR filter can be found by solving the reverse-time Quadratic Regulator problem (77), (74). To restrict the FIR coefficients to a finite (quantized)set, we simply proceed as discussed in section 5 and 8 by adding appropriate finite set constraints to the 'input' $\{u(k)\}$.

11. CONCLUSIONS

This paper has given a brief introduction to the topic of inverse problems with constraints. We have argued that this is a central problem in signal processing, telecommunication and control. An outline of some of the available tools for solving these problems has been given as well as some indication of recent research results. Many open problems exist in this area and it is suggested that this could be a fertile area for future research in systems and control theory. Indeed, it may not be overly optimistic to predict a sequence of major breakthroughs which parallel the excitement felt in the 1960's and 1980's with the advent of linear quadratic theory and the linear infinity norm problem respectively.

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13. REFERENCES

Astrom, K.J. (1970). *Introduction to Stochastic Control Theory.* Academic Press. New York.

Bazaraa, M.S. and C.M. Shetty (1979). *Nonlinear Programming: Theory and Algorithms*. John Wiley & Sons. New York.

Bemporad, A., M. Morari, V. Dua and P. Pistikopoulos (2002). The explicit linear quadratic regulator for constrained systems. *Automatica* **38**, 3–20.

Bergh, L.G. and J.F. MacGregor (1987). Spatial control of sheet and film forming processes. *The Canadian Journal of Chemical Engineering* **65**, 148–155.

Bernstein, D. and A. Michel (1995). A chronological bibliography on saturating actuators. *Int. Journal of Robust and Nonlinear Control* **5**, 375–380.

Bertero, M. and P. Boccaci (1998). *Introduction to Inverse Problems in Imaging*. Institute of Physics Publishing. Bristol.

Bode, H. (1945). *Network Analysis and Feedback Identifier Design*. D. van Nostrand. New York.

Braslavsky, J., R.H. Middleton and J. Freudenberg (1999). Cheap control performance of a class of non-right invertible nonlinear systems. In: *Proc. 38th IEEE Conference on Decision and Control*. Phoenix, Arizona, USA.

Carney, B.M., W.J. Edwards and G.C. Goodwin (1990). Achieving functional control of camber in rolling mils. In: *Proceedings of the 4th Australian Conference of Control Engineering*.

De Doná, J.A. and G.C. Goodwin (2000). Elucidation of the state-space regions wherein model predictive control and anti-windup strategies achieve identical control policies. In: *Proceedings of the American Control Conference*. Vol. 3. pp. 1924–1928.

De Doná, J.A., G.C. Goodwin and A. Jutan (2002). A rolling horizon state estimator with constraint horizon one. In: *Proceedings of IFAC World Congress*.

De Doná, J.A., G.C. Goodwin and M.M. Seron (2000). Anti-windup and model predictive control: reflections and connections. *European Journal of Control, Special Issue Anti-windup Control* **6**(5), 467–477.

De Doná, J.A., M. Seron, D.Q. Mayne and G.C. Goodwin (2001). Enlarged terminal sets guaranteeing stability of receding horizon control. *To appear, System and Control Letters*.

Ding, Zhi and Ye Li (1998). *Blind Equalization and Identification*. Marcel Dekker. Switzerland.

Duel-Hallen, A. and C. Heegard (1989). Delayed decision-feedback sequence estimation. *IEEE Transactions on Communications* **37**, 428–436.

Dumont, G.A. (1986). Application of advanced control methods in the pulp paper industry - a survey. *Automatica* **22**, 143–153.

Duncan, S.R. and G.F. Bryant (1997). The spatial bandwidth of cross-directional control systems for web processes. *Automatica* **33**, 139–153.

Edited (2001). *IEEE Control Systems Magazine, Special Issue on Networks and Control*.

Edwards, W.J., G. Gómez and G.C. Goodwin (1995). A review of thickness control in reversing cold rolling mills. In: *Control '95*. Melbourne, Australia. pp. 129–134.

Elia, N. and S.K. Mitter (2001). Stabilization of linear systems with limited information. *Automatica* **46**(9), 1384–1400.

Eyuboglu, V.M. and S.U. Quereshi (1988). Reduced-state sequence estimation and decision feedback. *IEEE Trans. on Communications* **36**, 13–20.

Featherstone, A.P., J.G. Van Antwerp and R.D. Braatz (2000). *Identification and control of sheet and film processes*. Springer Verlag. London.

Forney Jr., G.D. (1972). Maximum likelihood sequences in the presence of intersymbol interference. *IEEE Inf. Theory* **IT-18**, 363–378.

Forney Jr., G.D., R.G. Gallager, G.R. Lang, F.M. Longstaff, F.M. and V.U. Quereshi (1984). Efficient modulation for band limited channels. *IEEE J. Selected Areas Commun* **SAC-2**, 632–647.

Fossen, T.I. (1994). *Guidance and Control of Ocean Marine Vehicles*. John Wiley and Sons Ltd.. New York.

Garcia, C.E., D.M. Prett and M. Morari (1989). Model predictive control: Theory and practice - a survey. *Automatica* **25**(3), 335–348.

Goodwin, G.C. and J.S. Welsh (2001). *Model Identification and Adaptive Control: From Windsurfing to Telecommunications, G.C. Goodwin Ed.*. Chap. Singularity issues in closed loop identification. Springer Verlag. London.

Goodwin, G.C., B. Carney and W.J. Edwards (1990). Analysis of thermal camber control in rolling mills. In: *Proceedings of IFAC World Congress*. Tallin USSR.

Goodwin, G.C., G.V. Johnston and O.J. Rojas (2001*a*). A spatial frequency anti-windup strategy for cross directional control problems. *University of Newcastle, School of Elect. Engineering and Comp. Science, Tech Report No. EE01034*.

Goodwin, G.C., S.F. Graebe and M. Salgado (2001*b*). *Control System Design*. Prentice Hall.

Goodwin, G.C., T. Perez, M.M. Seron and C.-Y. Tzeng (2000). On fundamental limitations for rudder roll stabilization of ships. In: *Proc. 39th Conference on Decision and Control*. Sydney Australia.

Gray, R.M. (1996). *Delta Sigma Data Converters*. Chap. Quantization noise in DS A/D converters, Chapter in S. Norsworthy, R. Schreier and G. Themes Eds. IEEE Press.

Gray, R.M. and D.L. Neuhoff (1998). Quantization. *IEEE Trans. on Information Theory* **44**(6), 2325–2383.

Hanus, R., M. Kinnaert and J. Henrotle (1987). Conditioning technique, a general anti-windup and bumpless transfer method. *Automatica* **23**(6), 729–739.

Heath, W.P. (1996). Orthogonal functions for cross directional control of web forming processes. *Automatica* **32**, 183–198.

Heaven, E.M., I.M. Jonnsson, T.M. Kean, M.A. Manness and R.N. Vyse (1994). Recent advances in cross machine profile control. *IEEE Control System Magazine* **14**(5), 35–46.

Hippe, P. and C. Wurmthaler (1997). Controller and plant windup protection in mimo loops with saturation. In: *Proc. 4th European Control Conference*. Brussels, Belgium.

Hristu, D. (2000). Stabililzation of communication channels with communication constraints. In: *Proceedings of the American Control Conference*. Chicago. pp. 2342–2346.

Hristu, D. and K. Morgansen (1999). Limited communication control. *Systems and Control Letters* **37**(4), 193–205.

Iglesias, P.A. (2001). An analogue of bode's integral for stable non linear systems: Relations to entropy. In: *Proc. 40th IEEE Conference on Decision and Control*. Orlando, Florida. pp. 3419–3420.

Kapoor, N., A. Teel and P. Daotidis (1996). On anti-integrator windup and global asymptotic stability. In: *Proc. 13th IFAC World Congress*. Vol. D. pp. 67–72.

Kennedy, R.A., A. Dhing and S. Hasnie (2000). *Quantizer Optimization: Applications of Passivity in Telecommunications, in Model Identification and Adaptive Control*. Springer Verlag.

Kothare, M., P.J. Campo, M. Morari and C. Nett (1994). A unified framework for the study of anti-windup designs. *Automatica* **30**(12), 1869–1883.

Kwakernaak, H. and R. Sivan (1972). *Linear Optimal Control Systems*. Wiley Interscience. New York.

Lane, S.N., J.H. Chandler and K.S. Richards (1994). Developments in monitoring and modelling small-scale river bed topography. *Earth Surf. Processes Landforms* **19**, 349–369.

Levy, S. and J.F. Carley (1989). *Plastics Extrusion Technology Handbook*. 2nd edition ed.. Industrial Press. New York.

Lozier, J. (1956). A steady-state approach to the theory of saturable servo systems. *IEEE Transactions on Automatic Control* pp. 19–39.

Mayne, D. (2000). *Nonlinear Model Predictive Control, Chapter in Nonlinear Model Predictive Control: Challenges and Opportunities*. pp.23-44. Birkhauser Verlag.

Mayne, D., C. Rao J. Rawlings and P. Scokaert (2000). Constrained model predictive control: Stability and optimality. *Automatica*. to appear.

McGrath, D. (2002). *Private Communication*.

Michalska, H. and D. Mayne (1993). Robust receding horizon control of constrained nonlinear systems. *IEEE Transactions on Automatic Control* **38**, 1623–1632.

Michalska, H. and D. Mayne (1995). Moving horizon observers and observer-based control. *IEEE Transactions on Automatic Control* **40**(6), 995–1006.

Middleton, R.H. and J.H. Braslavsky (2000). On the relationship between logarithmic sensitivity integrals and limiting optimal control problems. In: *Proceedings of 39th Conference on Decision and Control*. Sydney, Australia.

Moore, I.D. and R.B. Grayson (1991). Terrain-based catchment partitioning and runoff prediction using vector elevation data. *Water Resour. Res.* **27**(6), 1177–1191.

Moore, I.D., R.B. Grayson, R.B. and A.R. Ladson (1991). Digital terrain modelling: A review of hydrological, geomorphological and biological applications. *Hydrol. Processes* **4**, 3–30.

Morari, M. and J. Lee (1997). Model predictive control: Past, present and future. In: *Proc. Joint 6th International Symposium on Process Systems Engineering (PSE'97) and 30th European Symposium on Computer Aided Process Systems Engineering (ESCAPE7)*. Trondheim, Norway.

Muske, K.R., J.B. Rawlings and J.H. Lee (1993). Receding horizon recursive state estimation. In: *Proc. 1993 American Control Conference*. Vol. 3. pp. 900–804.

Nair, G.N. and R.J. Evans (2000). Stabilization with data-rate-limited feedback: Tightest attainable bounds. *Systems and Control Letters* **41**, 49–56.

Nilsson, J., B. Bernhardsson and B. Wittenmark (1998). Stochastic analysis of control of real time systems with random delays. *Automatica* **34**(1), 57–64.

Pérez, T. and G.C. Goodwin (2001). Stochastic output feedback model predictive control. In: *Proceedings of the American Control Conference*. Arlington, USA.

Pérez, T., G.C. Goodwin and M.M. Seron (2001). Cheap control performance limitations of input constrained linear systems. In: *IFAC World Congress*. Barcellona.

Qin, S. and T. Badgwell (1997). An overview of industrial model predictive control technology. *Chemical Process Control-V, CACHE, AIChE* pp. 232–256.

Quevedo, D.E., J.A. De Dona and G.C. Goodwin (2002). Receding horizon linear quadratic control with finite constraint sets. In: *IFAC World Congress*. Barcellona.

Qui, L. and E.J. Davison (1993). Performance limitations for nonminimujm phase systems in the servomechanism problem. *Automatica* **29**(2), 337–349.

Qureshi, S.U.H. (1985). Adaptive equalization. *Proc. IEEE* **73**, 1349.

Rao, C.V., J.B. Rawlings and J.H. Lee (2001). Constrained linear state estimation - a moving horizon approach. *To appear, IEEE Trans. on Automatic Control*.

Rawlings, J., E. Meadows and K. Muske (1994). Nonlinear model predictive control: a tutorial and survey. In: *Proceedings Adchem '94, Kyoto, Japan.*

Ray, A. (1998). Distributed data communications networks for real time process control. *Chem. Engineering Comm.* **65**, 139–154.

Richalet, J., A. Rault, J. Testud and J. Papon (1976). Algorithmic control of industrial process. In: *Proc. 4th IFAC Symposium on Identification and System Parameter Estimation*. Vol. 3. pp. 1119–1167.

Robertson, D.G., J.H. Lee and J.B. Rawlings (1996). A moving horizon-based approach for least squares state estimation. *AIChEJ* **42**(8), 2209–2224.

Rojas, O. and G.C. Goodwin (2002). A simple anti-windup strategy for state constrained linear control. In: *Proceedings IFAC World Congress*. Barcelona.

Sabatier, P.C. (Ed) (1987). *Basic Methods of Tomography and Inverse Problems*. Institute of Physics Publishing. Bristol.

Seron, M., G.C. Goodwin and J.A. De Doná (2000*a*). Geometry of model predictive control for constrained linear systems. *Technical Report EE0031, Department of Electrical and Computer Engineering, The University of Newcastle, Australia. Submitted for publication.*

Seron, M., J. Braslavsky and G.C. Goodwin (1997). *Fundamental Limitations in Filtering and Control*. Springer Verlag.

Seron, M., J.A. De Doná and Graham C. Goodwin (2000*b*). Global analytical model predictive control with input constraints. In: *Proc. 39th Conference on Decision and Control,*. Vol. 3. Sydney Australia.. pp. 1924–1928.

Smith, O. (1958). *Feedback Control Systems*. McGraw Hill. New York.

Smook, G.A. (1982). *Handbook for Pulp and Paper Technologies*. Tappi. Atlanta.

Stewart, G. (2000). Two dimensional loop shaping controller design for paper machine cross-directional processes. *Ph.D. Thesis, The University of British Columbia, Vancouver, Canada*.

Stewart, G.E., D.M. Gorinevsky and G.A. Dumont (1998). Robust gmv cross directional control of paper machines. In: *Proceedings of the American Control Conference*. Philadelphia. pp. 3002–3007.

Tatikonda, S., A. Sahai and S.K. Mitter (1998). Control of LQG systems under communication constraints. In: *Proc. 37th IEEE Conference on Decision and Control*. Tampa, Florida. pp. 1165–1170.

van Amerongen, J., P. van der Klugt and H. van Nauta Lemke (1990). Rudder roll stabilization for ships. *Automatica* **26**, 679–690.

Viterbi, A. and J.K. Omura (1979). *Principles of Digital Communication and Coding*. McGraw Hill. New York.

Walsh, G., O. Beldiman and L.G. Bushnell (1999). Error encoding algorithms for networked control systems. In: *Proc. 38th IEEE Conference on Decision and Control*. pp. 4933–4938.

Welsh, J.S. and G.C. Goodwin (2002). Finite sample properties of indirect non-parametric closed loop identification. *To appear, IEEE Trans. on Auto. Control*.

Williamson, D., R.A. Kennedy and G.W. Pulford (1992). Block decision feedback equalization. *IEEE Transactions on Communication* **40**, 255–264.

Wong, W.S. and R.W. Brockett (1997). Systems with finite communications bandwidth constraints, part i: State estimation problems. *IEEE Transactions on Automatic Control* **42**, 1294–1299.

Wong, W.S. and R.W. Brockett (1999). Systems with finite communications bandwidth constraints, part ii: Stabilization with limited information feedback. *IEEE Transactions on Automatic Control* **44**, 1049–1053.

Zhou, K., J.C. Doyle and K. Glover (1998). *Robust and Optimal Control*. Prentice Hall. Upper Saddle River, NJ.