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SYSTEMS CONCEPTS IN FIANCIAL PRICING THEORY

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Abstract: Pricing theory is concerned with determining a realistic market-related price of an asset that is not yet marketed. There are several approaches to this issue, most of which use systems theory concepts, such as optimization, dynamic recursion, probability, stochastic process, and control. Pricing theory has therefore benefited greatly from systems theory. Systems theory can also benefit from pricing theory, for pricing theory provides critical guidance regarding the proper objective function for control problems involving random cash flow streams. This paper outlines these two complementary aspects of pricing theory and systems theory.

Keywords: finance, systems, control.

1. INTRODUCTION

The theory of modern finance can be divided into two questions: pricing and portfolio section. Pricing is the determination of a fair (or reasonable or predicted) price of an asset. Portfolio design is the determination of the best portfolio given existing prices or estimates of future prices. This paper is primarily concerned with pricing.

Pricing relates to systems theory in two important ways. First, systems theory concepts contribute significantly to the development of pricing theory. Optimization, dynamic recursion, probability, stochastic process, and control all figure importantly in pricing theory. Second, pricing theory can fundamentally change the way important systems and control problems are formulated. Many optimal control problems have objective functions that evaluate stochastic cash flow streams. The objective function for such problems should translate such streams into meaningful economic measures. From a finance viewpoint, the appropriate measure is the price that would be realized if the cash flow stream were sold in the financial market. This price would account for the relation of the particular cash flow stream to all those available. This is the price that systems theorists should most often maximize when treating cash flow problems. Emphasizing these complementary roles of systems theory and pricing theory—each contributing to the other—is the main objective of this paper.

There are several different approaches to pricing. The most ambitious is that of equilibrium analysis, where one attempts to determine a complete spectrum of prices for all assets. This requires a complete model of the economy.

Most practical methods are incremental in natural, focusing on a new asset that is introduced into an existing environment and assuming that prices of existing assets will not change. The new asset's price is determined relative to the existing prices.

The simplest case is when the payoff of the new asset can be expressed as a linear combination of the payoff of existing priced assets. To be consistent, the price of the new asset should be the same linear combination of prices as that of the payoff. The price is determined linearly.

A generalization of the consistency approach is the no-arbitrage approach. In this view a new asset must be priced so that its introduction does not create an arbitrage opportunity; that is, the possibility of attaining positive payoff with no chance of negative payoff and no initial cost. The no-arbitrage condition is widely accepted as a necessary condition for a new price, but this condition, by itself, rarely defines a unique price.

Another important approach is the optimization approach, which determines the threshold price for an investor to include the asset in an optimal portfolio. This price is the zero-level price, the price at which the investor will include the asset at the zero level.

Consideration of multiple periods introduces important new considerations because of the possibility of control. If the asset is a project, for example, it can be cancelled if its initial phases are unsuccessful, it can be expanded if there is good success, or phases of the project can be delayed. In practice there are several possible control actions, and the availability of these actions must be considered when assessing value.

A most important set of actions are market participation actions. The recognition of these market actions led to the major break-through in thinking embodied in the Black–Scholes equation for pricing stock options, and these considerations should be blended with the other project control considerations.

2. LINEARITY AND ARBITRAGE

One tenet of pricing theory is that prices should be free of arbitrage opportunities. Such opportunities may briefly exist in the real world, but it can be argued that they are quickly resolved. Pricing theory deals with an idealized version of financial markets, where arbitrage is not possible, there are no transactions fees, securities are infinitely divisible, and shorting is exactly the opposite of buying.

A consequence of the no-arbitrage requirement is the rule of linear pricing. Suppose two securities i and j with payoffs y_i and y_j (at the end of a period) have corresponding prices p_i and p_j. Then a security with payoff $\alpha y_i + \beta y_j$ must have price $\alpha p_i + \beta p_j$. This must be true, for otherwise the new security could be constructed at a cost of $\alpha p_i + \beta p_j$ and if this were less (or greater) than the market price, one could construct the security and sell (or buy) it at a profit.

Under fairly broad assumptions, there is a range of prices for an asset such that introduction of the asset at a price in that range will not introduce arbitrage. It is logical, therefore, to require that the price we assign should lie in this range.

As an example, consider a coin flip asset. The coin pays 3 units on heads and 0 units on tails. The range of prices that preclude arbitrage is the open interval (0, 3). Unfortunately, the no-arbitrage condition does not provide a specific value.

3. THE SYSTEMS VIEW OF CAPM

Perhaps the most well-known pricing formula is the Capital Asset Pricing Model (CAPM). It is used by investors seeking good portfolios, as a way to price new companies, as a basis for measuring the performance of mutual funds, and as a way to price new ventures of all sorts. Its primary lesson is that risk is not equivalent to volatility, but rather to market correlation.

The CAPM is derived from Markowitz optimal portfolio design, and hence CAPM can be considered to be in the family of optimization approaches, although as we shall see it is best regarded as a consistency result; that is, as a convenient formula for carrying out linear pricing.

To set the framework, assets are represented by their payoffs which are random variables representing amounts of cash to be delivered at the end of the period (say one year). We suppose there is universe of n such assets with corresponding payoffs $y_1, y_2, \ldots, y_n$. These assets have corresponding prices $p_1, p_2, \cdots p_n$.

A portfolio of these assets is a combination of the individual payoffs, $\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n$, with cost $\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n$. The total return of this portfolio is the payoff divided by the cost.

In the Markowitz framework (Markowitz, 1952), it is assumed that an investor measures performance in terms of expected value and standard deviation of the return. By varying the α_i's is possible to achieve many different combinations of expected value and standard deviation of portfolio return. These can be plotted in a figure to produce a feasible region as shown in Fig. 1(a).

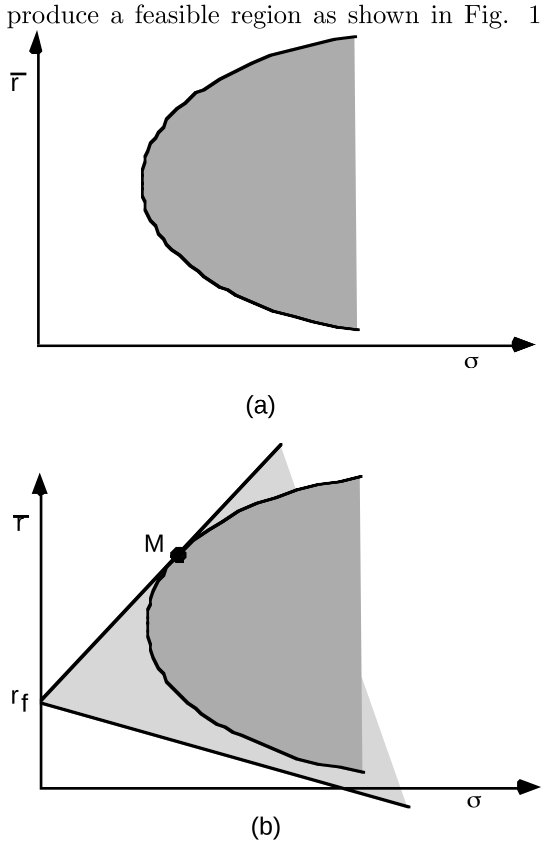

Fig. 1. Feasible region. (a) With risky assets. (b) Inclusion of a riskfree asset.

Investors will wish to be on the upper left hand portion of this curve, termed the efficient frontier.

The picture changes somewhat if a riskfree asset is added to the collection. This asset has price 1 and certain payoff $R = 1 + r_f$. This asset can be combined with other assets in various combinations to form portfolios that broaden the feasible region. Because the riskfree asset has zero variance and zero correlation with other assets, a fifty-fifty mix of the riskfree asset with a feasible asset produces a portfolio with an expected return halfway between the expected returns of the two components and a standard deviation that is half that of the risky component. This argument can be generalized to arbitrary weighted combinations with the result that any straight line from the riskfree point to a point in the original feasible region can be added to the feasible region, producing the triangular-shaped region shown in Fig. 1(b). There is now a single efficient portfolio M of risky assets with payoff at the point where a line from the riskfree rate r_f is tangent to the old feasible region. The payoff of this efficient payoff is denoted y_M and termed the Markowitz portfolio. Investors will seek to be somewhere on the efficient frontier defined by the line through the riskfree point and the point corresponding to y_M.

We are not so interested in the Markowitz problem for the sake of determining a portfolio but rather as a basis for pricing.

The necessary conditions for determining the efficient point can be reduced to n equations for the n unknowns $\alpha_1, \alpha_2, \ldots \alpha_n$. These n equations involve the n prices. If we assume that we know the α_i's we can regard these n equations as equations for the p_i's. This is the essence of the optimization approach to pricing. The necessary conditions are turned around. Instead of equations for the α_i's given the prices, the same conditions are regarded as equations for the p_i's given the optimal α_i's. For the Markowitz problem the necessary conditions when turned around yield the CAPM formulas (Sharpe, 1964) for the p_i's

$$p_i = \frac{1}{R}\Big[\mathrm{E}[y_i] - \mathrm{cov}(y_i, y_M)(\overline{y}_M - p_M R)/\sigma_M^2\Big], \quad (1)$$

where y_M is payoff of the Markowitz portfolio, p_M is its price (usually normalized to $p_M = 1$), and σ_M^2 is the variance of y_M.

The important lesson from this formula is that risk is measured by the coefficient $\beta_{i,M}$ defined as

$$\beta_{i,M} = \mathrm{cov}(y_i, y_M)/\sigma_M^2,$$

which is a normalized version of the covariance of the asset's payoff with the efficient portfolio. It is not the variance of the asset, but only its covariance with the efficient risky asset that matters. If the asset has zero covariance with the efficient portfolio, then no matter what its variance, its price is simply $p_i = \mathrm{E}[y_i]/R$.

3.1 The CAPM and Other Assets

The formula (1) is linear in y_i. Hence it can be applied to any linear combination of the y_i's to obtain the corresponding price. A general $y = \sum_{i=1}^{n} a_i y_i$ can be priced consistently with the other assets by substituting y for y_i in (1).

Sharpe (1964) observed that if all investors select portfolios on the efficient frontier, then all investors will purchase the same bundle of risky assets, augmented by various amounts of the riskfree asset. And, if everyone purchases the same bundle of risky assets, this bundle must be proportional to the market portfolio. Under this assumption, y_M becomes the market portfolio. Therefore, in practice it is not necessary to compute y_M, for it can be observed as the amounts of each security outstanding.

The pricing formula (1) is a tautology; it simply recovers the prices p_i that were originally used to determine y_M. It provides no new information. It is a convenient way to carry out linear pricing—to find the price of any marketed asset or combination of marketed assets by using only its expected value and covariance with the Markowitz portfolio. It is quite convenient, since given a payoff y it is not necessary to explicitly determine the linear combination of $y_1, y_2, \ldots y_n$ equal to y and then combine the p_i's correspondingly. The pricing formula does all that automatically once y_M is known.

In practice, the CAPM (usually with y_M taken as the market portfolio) is used to price new assets, assets whose payoffs are not in the span of the original payoffs. There is no actual justification for this, since the formula was developed only for prices of the original y_i's and their linear combinations. Nevertheless for an asset with payoff y, a CAPM price is defined as

$$p_y = \frac{1}{R}\big[\mathrm{E}[y] - \mathrm{cov}(y, y_M)(\overline{y}_M - p_M R)/\sigma_M^2\big]. \quad (2)$$

Let us apply this formula to the coin flip example. If we assume that the coin flip asset is not part of the original set of marketed assets, then its payoff is uncorrelated with the Markowitz portfolio. Hence the price is

$$p = \frac{1}{R}\mathrm{E}(y).$$

If, because the time to payoff is short, we take $R = 1$, then the CAPM price of the coin flip is the expected value, 1.5 units.

The practice of applying CAPM to assets outside its original region of derivation is so common, it is natural to ask just what price it yields. We answer this in the next section.

4. PROJECTION PRICING

From a systems perspective, it is natural to regard the pricing problem in a vector space. We regard $y_1, y_2, \ldots, y_n$ and the new payoff y as $n+1$ vectors in an inner product space. The first n vectors define a linear subspace M of this space. The inner product is $(y_i|y_j) = \mathrm{E}(y_i y_j)$. The norm (or distance) of a vector is $\|y\| = \sqrt{(y|y)} = \sqrt{\mathrm{E}[y^2]}$. We say y is orthogonal to M if $(y|m) = 0$ for all $m \in M$.

A natural way to price the payoff y is by projection. Specifically, we project the payoff y onto the subspace M, obtaining a vector $m_0 \in M$ and then assign a price to y equal to the price of m_0 which can be found by linear pricing from the n marketed assets. This approach is based on the classical projection theorem, which is a familiar tool of systems theory (Luenberger, 1969).

Theorem 1 *Let $\mathcal{H}$ be a Hilbert space and M a closed subspace of $\mathcal{H}$. Let $y \in \mathcal{H}$. Then there is an $m_0 \in M$ such that $\|y - m_0\| \leq \|y - m\|$ for all $m \in M$. Furthermore, $y - m_0$ is orthogonal to M. Conversely, if there is an $m_0 \in M$ such that $y - m_0$ is orthogonal to M, then $\|y - m_0\| \leq \|y - m\|$ for all $m \in M$.*

This theorem is illustrated in Fig. 2.

Once the projection m_0 of a payoff y onto the subspace M of marketed payoffs is found, the price p_{m_0} of m_0 can found by linear pricing in M. We the set $p_y = p_{m_0}$ to define the projection price of y.

This procedure may seem somewhat arbitrary, but it has the advantage of simplicity and uniqueness. One of the most important properties of this projection method, however, is that it duplicates the CAPM. The CAPM price (2) with y_M equal to the Markowitz portfolio is equal to the projection price. See (Luenberger, 2001).

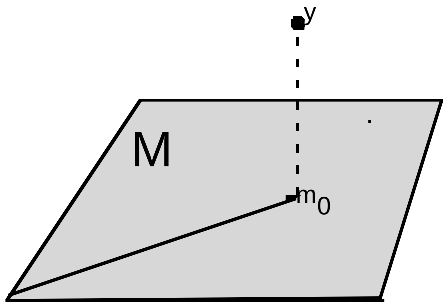

Fig. 2. The projection theorem.

One difference between the projection price and the Markowitz (or equivalently CAPM) price is that the efficient portfolio y_M used in the Markowitz formula (1) may not exist. Yet, the projection price itself always exists, as guaranteed by the projection theorem. The projection theorem therefore suggests that there may be an alternative formula that does not share the potential nonexistence difficulty of the standard Markowitz or CAPM formula.

5. CORRELATION PRICING

One way to calculate the projection price is to use a dual formulation of the projection theorem.

Theorem 2 *Let $\mathcal{H}$ be a Hilbert Space and M a closed subspace of $\mathcal{H}$. Let $y \in H$. Then there is an $m' \in M$ with $\|m'\| = 1$ such that $(y|m') \geq (y|m)$ for all $m \in M$ with $\|m\| = 1$. Furthermore, if the projection m_0 of y is nonzero, then m' is unique and a positive multiple of m_0.*

The theorem is illustrated in Fig. 3.

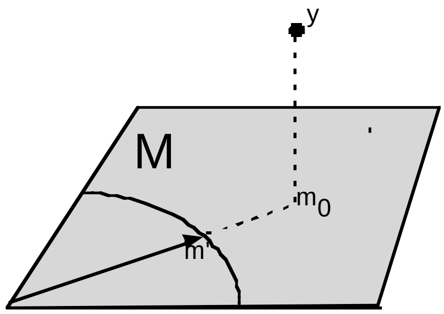

Fig. 3. The dual projection theorem.

An application of this theorem to pricing is to find the projection of y onto M by finding the asset in M making the smallest angle with y and then projecting y onto that. This leads to the alternative price formula

$$p = \frac{1}{R}\left[\mathrm{E}[y] - \mathrm{cov}(y, y_M)[\overline{y}_M - p_M R)/\sigma_m^2\right] \quad (3)$$

where now y_M is an asset most correlated with y. Such a y_M is defined only to within a scale factor and an additive constant.

This alternative formula (Luenberger, 2002a), termed the Correlation Pricing Formula (CPF), is a very practical form of the projection pricing. It can be viewed as a rigorous version of the common practice of pricing assets by finding similar assets that are already priced. For example, in pricing a house, one looks at "comparables", similar houses in the same neighborhood that have sold recently, in order to judge the price.

Note, for example, that if $y \in M$ then we may take $y_M = y$ and the formula (3) reduces to $p = p_y$.

Let us apply this formula to the coin flip problem. A most-correlated asset is the riskfree asset, and the covariance of this with the coin flip is zero. Hence, the formula reduces to

$$p = \frac{1}{R}\mathrm{E}[y] = 1.5,$$

as in the CAPM.

This form of the projection pricing formula has the advantage that it always exists (unlike the standard CAPM). But its greatest advantage is that it is not necessary to relate the asset to the market as a whole, but rather to a most-correlated marketed asset.

6. ZERO-LEVEL PRICING

The projection price has the disadvantage that it seems somewhat arbitrary, in that it is not directly related to market activity or to individual choice. There is an indirect relation, since the projection price is identical to the CAPM price which is based on mean–variance investor choice. However, the mean–variance framework itself seems somewhat arbitrary. One might expect a sophisticated theory of pricing to be based on a sophisticated view of investor choice. This leads us to consider an expected utility version of the investor's portfolio problem.

We employ the same framework as in the earlier sections with respect to the nature of the available assets. Now, however, we consider an investor who has a utility function U for final wealth and seeks to maximize the expected value of this function. The investor's problem is

$$\begin{array}{lll} \max_{\alpha_1,\alpha_2,\ldots\alpha_n} & \mathrm{E}[U(y_0)] & (4) \\ \text{subject to} & \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n = y_0 & \\ & \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n = W. & \end{array}$$

The necessary conditions for this problem are easily found to be

$$\mathrm{E}[U'(y_0)y_i] = \lambda p_i \quad \text{for } i = 1, 2, \ldots n \qquad (5)$$

for some $\lambda > 0$. Note that this is a linear pricing formula since

$$p_i = \mathrm{E}[U'(y_0)y_i]/\lambda.$$

If there is a riskfree asset with return R among the marketed assets (as we shall always assume), then the necessary condition (5) for it implies

$$\mathrm{E}[U'(y_0)]R = \lambda. \qquad (6)$$

Hence we may write (5) as

$$p_i = \frac{\mathrm{E}[U'(y_0)y_i]}{R\,\mathrm{E}[U'(y_0)]} \quad \text{for } i = 1, 2, \ldots n. \qquad (7)$$

Now suppose a new asset with payoff x outside M is introduced. Suppose also that a price p_x is assigned to this payoff. The investor may wish to modify his or her portfolio to take advantage of this new opportunity. The degree of modification will, of course, depend on the price p_x. If it low, the investor will want to include a large fraction of this asset in the portfolio; if it is high, the investor will want to short the asset. In fact, some price assignments may lead to arbitrage possibilities. However, under rather mild conditions, there is a unique zero-level price p_x^0 such that the investor will find that it is not advantageous to include the new asset in the portfolio in either a positive or negative amount. Zero is the optimal level. This price is called the zero-level price. See (Luenberger, 2002b).

This price is easily computed to be

$$p_x^0 = \frac{\mathrm{E}[U'(y_0)x]}{R\,\mathrm{E}[U'(y_0)} \qquad (8)$$

since it is the same optimization problem as before, and has the same optimal portfolio y_0.

It may turn out that different investors have different zero-level prices. This is fine when considering a particular investor's situation. The zero-level price defines a threshold for that investor. Only if the price is lower than that threshold should the investor actually consider purchasing a (positive) amount of the asset. Price nonuniqueness, however, is not desirable as a price assignment methodology.

6.1 Universal Zero-Level Prices

In some situations the zero-level price of an asset is actually independent of investor's utility and wealth. If this is the case, the zero-level price is said to be a universal zero-level price. One obvious case is when the asset is already marketed; that is, its payoff is in the subspace M. The zero-level price must then agree with the market price, or an arbitrage opportunity would exist. Hence, the zero-level price must be fixed at the linearly defined price, independent of U and W.

As another example, suppose x is the coin flip payoff. We have

$$p = \frac{\mathrm{E}[U'(y_0)x]}{R\,\mathrm{E}[U'(y_0)]}.$$

Since the payoff of the coin flip is independent of y_0 the numerator can be written as $\mathrm{E}[U'(y_0)x] = \mathrm{E}[U'(y_0)]\mathrm{E}[x]$. Hence again

$$p = \frac{1}{R}\mathrm{E}[y] = 1.5,$$

equal to the CAPM or projection price.

This example generalizes. If the payoff of the asset to be priced is independent of all marketed assets, the zero-level price is universal.

Another case is where all asset returns are distributed according to a joint normal distribution. In this case, y can be written as $y = y_L + y_N$ where y_L is a linear combination of $y_1, y_2, \dots y_n$ and y_N is independent of the y_i's. Hence, we may write

$$p = p_L + \frac{1}{R}\mathrm{E}[y_N],$$

where p_L is the price of y_L (which is found by linear pricing from the prices of the y_i's). This p is independent of U and W, and is thus a universal zero-level price.

It can be shown that in these cases, where the zero-level price is universal, that the universal zero-level price is equal to the projection price, as it is for the coin flip example.

7. SIMPLE CONTROL

Suppose that x represents the payoff of a project that we can manage. The nature of x may then depend on the management (or control) actions that we take. These actions may shape the probability distribution of x, increasing the mean, narrowing the variance, or chopping off regions. The payoff x is really a random function $x(u)$, where u is a control action. We wish to determine the optimal u.

This kind of problem occurs often in practice, and engineers deal with it frequently. It is an integral part of systems theory. The standard way to treat the problem is to define an objective function, say $J(x(u))$ and maximize its expected value. That is, select u to maximize $\mathrm{E}[J(x(u))]$.

This approach, although common, ignores the nature of the market in which the project is embedded. An alternative, is to recognize that for each u there is a zero-level price $p^0(u)$ associated with $x(u)$. This price accounts for both the market and the utility function of the individual. That is, the problem becomes maximization of $p^0(u)$ with respect to u. The problem can then be formulated as maximization of the zero-level price. The zero-level price is appropriate for projects that are small or medium-sized relative to the size of the optimizing unit. Typically, this means that projects of up to 15% of a firm's revenue or an individual's wealth can be treated this way. Really large projects require additional considerations.

If the project payoff happens to be independent of the market for every value of u, the zero-level price is the discounted expected value. Hence, the problem becomes maximize $\mathrm{E}[x(u)]/R$ with respect to u, which is a common formulation of the problem. If the project is not independent of the market, the correct zero-level price should be used as the objective.

8. MULTIPERIOD DERIVATIVE PRICING

The concepts presented so far can be extended to multiperiod situations. As before, the simplest case is that of pricing an asset that is within the span of the original assets. Such an asset is priced by linearity.

As simple as that sounds, the application of linear pricing in a multiperiod setting is subtle, and its full resolution represents a most important advance in finance, as embodied in the Black–Scholes equation for derivative securities (Black and Scholes, 1973). The reason that the issue is subtle is that the subspace of payoffs generated by n assets is not merely n-dimensional as it is in the single-period setting; it is much larger.

This expansion of dimension can be illustrated by a simple finite-state model. Suppose there are two assets. One is a stock that at the end of each period either goes *up* or *down*, each with probability .5. If it goes *up* the return is 2 if it goes *down* the return is 1/2. There is also a riskfree asset (a bond) with total return of 1 each period. The total return of the stock over two periods takes on three possible values: 4, 1, 1/4, corresponding to either: *two ups*, an *up and a down*, or *two downs*. The three possible conditions of ups and downs define three possible states of the system after two periods. The riskfree asset always has a return of 1, no matter what the state: that is, its state returns are 1, 1, 1.

Suppose another asset has a payoff that depends on these three states. Perhaps it is an option on the stock and pays 9, 0, 0. According to the standard viewpoint we can price this uniquely only if it can be expressed as a linear combination of the stock and the bond. In the example given, this cannot be done, for there is no way to combine (4, 1, 1/4) and (1, 1, 1) to obtain (9, 0, 0).

However, additional state outcomes can be generated by the stock and bond by changing the weights at the end of each period. For example, there is a strategy for attaining (9, 0, 0). Specifically, at time 0 buy 2 units of the stock and short 1 unit of the bond, for a total cost of 1 unit. At the end of the first period there are two possible states corresponding to an *up* or *down* movement of the stock. The portfolio is worth 3 or 0, respectively. If we are in the *up* state with 3 units, we readjust the holdings to 6 units of the stock and -3 units of the bond. The cost of this new portfolio is 3, so since we already have 3 in this state, this is a no-cost adjustment. If we are in the *down* state with 0 units. We take no further positions. At the end of the second period, we will have a total of 9 units if we are in the *up up* state, and 0 units in the *up down* or *down down* state. The final payoff by state is therefore 9, 0, 0, which is what we sought. The opportunity for trading expanded the span of the the marketed assets. We conclude that the price of the payoff (9, 0, 0) is 1 since that was the initial cost of the strategy that attained it.

As mentioned earlier, the most remarkable result of this viewpoint is the theory of option pricing initiated by the Black–Scholes theory. In continuous time, when assets are governed by Ito processes, it is possible to replicate any function of the marketed payoffs by suitable trading (control) at every instant.

In the Black–Scholes framework a stock price $x(t)$ is governed by the Ito process

$$\mathrm{d}x = \mu x \,\mathrm{d}t + \sigma x \,\mathrm{d}z, \tag{9}$$

where z is a standardized Wiener process (with mean zero and variance 1). There is also a bond with price y satisfying

$$\mathrm{d}y = ry \,\mathrm{d}t.$$

A derivative of x has a payoff at time T that is an explicit function of $x(T)$; say $F(x(T)$. For example, it may be the payoff of an option on $x(T)$. The value of the derivative at time $t \leq T$ is defined to be $V(x(t), t)$. It must satisfy the boundary condition $V(x(T), T) = F(x(T)$ and the Black–Scholes equation

$$rV(x,t) = V_t(x,t) + V_x(x,t)rx(t) + \tfrac{1}{2}V_{xx}(x,t)x(t)^2\sigma^2. \tag{10}$$

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8. FULL CONTROL

The theory of derivatives represents a major step in the theory of pricing. The theory has been applied in many areas, including interest rate derivatives, credit risk securities, and real options. However, it cannot price assets that are multiperiod but not derivative of a priced asset.

Consider, for example, a project within a technical firm to develop a new product. The project's payoff will depend on consumer acceptance of the product, its price, and on the general market conditions. This payoff is not a derivative of a security, and hence the Black–Scholes methodology is not applicable. An alternative is to use projection pricing.

Suppose there are n assets with prices that follow processes of the form

$$x_i(k+1) = a_i x_i(k) + w_i(k) x_i(k)$$

for $i = 1, 2, \ldots n$. There is also a process defined by the state x_0 that satisfies

$$x_0(k+1) = f_0(x_0(k), w_0(k), u(k))$$

where $u(k)$ is a control. The $w_i(k)$'s are random, and $w_i(k)$ is uncorrelated with $w_j(l)$ for all j and all $l \neq k$.

The x_i's for $i = 1, 2, \ldots n$ are prices of marketed assets and they can be combined into a portfolio that can be modified each period. The special variable x_0 is not a price. It can be observed but not traded. Its value at $t = T$ determines a payoff $F(x_0(T))$. For example, x_0 may represent the progress of a project that is subject to control u.

At each period k, let y_M denote the payoff of the Markowitz portfolio. We wish to maximize the projection price of the project by selection of a suitable control strategy. We may do this by combining dynamic programming with dynamic projection pricing.

The recursive process is

$$V(x_0(k),k) = \frac{1}{R}\max_{u(k)}\big[\mathrm{E}[V(x_0(k+1),k+1)]$$

$$-\mathrm{cov}(V(x_0(k+1),k+1),y_M)(\overline{y}_M - p_M R)/\sigma_M^2\big], \tag{11}$$

where $V(x(T),T) = F(x(T))$.

If the project is uncorrelated with all assets in the market, then the value $V(x_0(0),0)$ computed by (11) is the maximum discounted expected value of the project payoff. In general, however, the result is the maximum projection price. In the case of normal distributions and suitable linearity, the value will be the maximum universal zero-level price.

The recursion (11) is only a one-dimensional recursion, even though implicitly there are $n+1$ variables.

This approach can be extended to projects with higher-order dynamics and to marketed assets of more complex form.

The approach generalizes to continuous-time as well, where even stronger results apply. The basis for this extension is a generalization of the Black–Scholes equation that yields the universal zero-level price even when the payoff to be priced is not a derivative. In this framework, there are n marketed assets following Ito processes of the form

$$\mathrm{d}x_i = \mu_i x_i\,\mathrm{d}t + \sigma_i x_i\,\mathrm{d}z_i$$

and a special process, that can be observed but cannot be traded, governed by

$$\mathrm{d}x_0 = \mu_0 x_0\,\mathrm{d}t + \sigma_0 x_0\,\mathrm{d}z_0.$$

The payoff is a function $F(x_0(T))$. It can be shown that the zero-level price of this payoff follows an extended Black–Scholes equation (Luenberger, 2002c).

This general approach melds modern finance with control theory, so that control theorists can formulate important problems in a manner consistent with finance concepts.

9. FUTURE DIRECTIONS

A great deal remains to be done in pricing theory, and especially its interplay with systems methodology. The heavy computational requirements of complex models within the framework of a high-dimensional market present an important challenge that may be met with continued cooperation between the finance and systems fields.

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