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A NOTE ON PASSIVITY OF NONLINEAR RL AND RC CIRCUITS

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Abstract: It is well–known that arbitrary interconnections of passive (possibly nonlinear) resistances (R), inductances (L) and capacitances (C) define passive ports, with port variables the external sources voltages and currents, and storage function the total stored energy. In this brief note we establish some new passivity properties of RC and RL circuits under the more restrictive assumption that the characteristic functions of the capacitances and inductances are nondecreasing. In particular, we prove that for this class of nonlinear RC circuits passivity is preserved even if we take as port variables the sources currents and the voltage *derivatives*. We also show that nonlinear RL circuits define passive ports with port variables the sources current *derivatives* and the voltages. The storage function in both cases is the sum of the resistors co–content, which is a function associated to the power dissipation. The proof of these properties is a direct application of Tellegen's theorem. We illustrate the applicability of these results to generate new Lyapunov functions for nonlinear stabilization problems.

Notation We consider RLC circuits consisting of n_R resistances, n_L inductances and n_C capacitances. We will use the symbols $i_R, v_R, i_L, v_L, i_C, v_C$ to denote the resistors, inductors and capacitors currents and voltages, respectively. We will also use ϕ_L, q_C to denote inductor flux and capacitor charge. Resistors are characterized by a graph $i_R = \hat{i}_R(v_R)$, (or $v_R = \hat{v}_R(i_R)$.) The inductors behavior is described by the relation $v_L = \dot{\phi}_L$, with characteristic function $i_L = \hat{i}_L(\phi_L)$; while the capacitors satisfy $i_C = \dot{q}_C$, with characteristic function $v_C = \hat{v}_C(q_C)$. The circuits are interconnected to the environment through external ports with voltages and currents v_S, i_S. Boldface will be used for vector quantities, with the k–th element identified with a subindex $(\cdot)_k$., these indexes belong to the indexed sets $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{C}$, for the R, L and C elements, respectively. When clear from the context the subindex k will be omitted.

1. INTRODUCTION

Tellegen's theorem states the set of branch voltage vectors that satisfy Kirchoff's voltage law, say $\mathcal{K}_v$, and the set of branch currents that satisfy Kirchoff's current law, say $\mathcal{K}_I$, are *orthogonal* linear spaces—see (Wyatt 1969) for a lucid, compact derivation of this fundamental result. An immediate consequence of Tellegen's theorem is power conservation, that is, the voltages and currents in an RLC circuit satisfy

$$\mathbf{i}_C^\top \mathbf{v}_C + \mathbf{i}_L^\top \mathbf{v}_L + \mathbf{i}_R^\top \mathbf{v}_R = \mathbf{i}_S^\top \mathbf{v}_S$$

where we have adopted the standard sign convention for the sources voltage and current. Integrating this equation, and recalling that the stored energy of capacitors and inductances are given by

$$\mathcal{E}_C[q_C(t)] := \int_0^{q_C(t)} \hat{v}_C(q_C')dq_C'$$

and

$$\mathcal{E}_L[\phi_L(t)] := \int_0^{\phi_L(t)} \hat{i}_L(\phi_L')d\phi_L',$$

respectively, we obtain the well–known energy–balance equation for RLC circuits

$$\begin{aligned}\sum_{k\in\mathcal{C}}\{\mathcal{E}_{kC}[q_{kC}(t)] - \mathcal{E}_{kC}[q_{kC}(0)]\} +\\ \sum_{k\in\mathcal{L}}\{\mathcal{E}_{kL}[\phi_{kL}(t)] - \mathcal{E}_{kL}[\phi_{kL}(0)]\} =\\ \int_0^t \mathbf{v}_S^\top(\tau)\mathbf{i}_S(\tau)d\tau - \int_0^t \mathbf{v}_R^\top(\tau)\mathbf{i}_R(\tau)d\tau\end{aligned} \quad (1)$$

From (1), and the fact that the R, L and C elements are passive if and only if their stored energy is non–negative[1] for all $t \geq 0$, we obtain the following classical result of circuit theory, see e.g., Section 19.3.3 of (Desoer *et al.* 1969).

Proposition 1. Arbitrary interconnections of passive RLC elements, with external port variables $\mathbf{v}_S, \mathbf{i}_S \in I\!R^{n_s}$, defines a passive port with port variables $(\mathbf{i}_S, \mathbf{v}_S)$ and storage function the *total stored energy*

$$\mathcal{E}_{tot}(\mathbf{q}_C, \phi_L) := \sum_{k\in\mathcal{C}} \mathcal{E}_{kC}(q_{kC}) + \sum_{k\in\mathcal{L}} \mathcal{E}_{kL}(\phi_{kL})$$

In this short note we will use the following non–obvious corollary of Tellegen's theorem

$$\mathbf{i}_C^\top \frac{d\mathbf{v}_C}{dt} + \mathbf{i}_L^\top \frac{d\mathbf{v}_L}{dt} + \mathbf{i}_R^\top \frac{d\mathbf{v}_R}{dt} = \mathbf{i}_S^\top \frac{d\mathbf{v}_S}{dt} \quad (2)$$

[1] Notice that, while resistors are passive if and only if its characteristic function $i_R = \hat{i}_R(v_R)$ lies in the first and third quadrants, this restriction is not imposed for L and C elements.

which stems for the fact that $\frac{dv}{dt} \in \mathcal{K}_v$ for all $v \in \mathcal{K}_v$, to establish some new passivity properties of RL and RC circuits.

2. MAIN RESULTS

To present our main results we need to introduce the concepts of *content* and *co–content* of a resistance, which are well–known in circuit theory (Millar 1951, Penfield *et al.* 1970), and have been recently used to study stability of neuromorphic circuits (Wyatt 1969, Shi *et al.* 1999).

Definition 1. The *co–content* of a voltage–controlled resistance is defined as

$$J(v_R) = \int_0^{v_R} \hat{i}_R(v_R')dv_R'$$

while for current–controlled resistances the function

$$G(i_R) = \int_0^{i_R} \hat{v}_R(i_R')di_R'$$

is called the resistors *content*.

Notice that the co–content has units of power, and in particular for linear resistors, where $v_R = Ri_R$, is simply half the dissipated power. Also, remark that for passive resistances it is a positive, nondecreasing function.

Proposition 2. Arbitrary interconnections of passive resistances and capacitances with *nondecreasing characteristic function* $v_C = \hat{v}_C(q_C)$ and external port variables $\mathbf{v}_S, \mathbf{i}_S \in I\!R^{n_s}$ satisfy the inequality

$$\int_0^t \dot{\mathbf{v}}_S^\top(\tau)\mathbf{i}_S(\tau)d\tau \geq J_{tot}[\mathbf{v}_R(t)] - J_{tot}[\mathbf{v}_R(0)]$$

with

$$J_{tot}(\mathbf{v}_R) = \sum_{k\in\mathcal{R}} J_k(v_{kR}) \geq 0$$

Consequently, the circuit defines a passive port with port variables $(\mathbf{i}_S, \frac{d\mathbf{v}_S}{dt})$ and storage function the total resistor *co–content* $J_{tot}(\mathbf{v}_R)$.

Proof. The proof proceeds from (2), where we will consider the resistive and capacitive branch types separately. Replacing the constitutive relations for the capacitances we get

$$\mathbf{i}_C^\top \frac{d\mathbf{v}_C}{dt} = \sum_{k\in\mathcal{C}} \left[\frac{\partial \hat{v}_{kC}}{\partial q_{kC}}(q_{kC})\right] i_{kC}^2 \geq 0 \quad (3)$$

where the non–negativity stems from the assumption that the function $\hat{v}_{kC}(q_{kC})$ is nondecreasing.

Now, for the resistances we obtain

$$\begin{aligned}\mathbf{i}_R^\top \frac{d\mathbf{v}_R}{dt} &= \sum_{k\in\mathcal{R}} \hat{i}_{Rk}(v_{kR}) \frac{dv_{kR}}{dt} \\ &= \sum_{k\in\mathcal{R}} \frac{dJ_k(v_{kR})}{dv_{kR}} \frac{dv_{kR}}{dt} \\ &= \sum_{k\in\mathcal{R}} \frac{dJ_k(v_{kR}(t))}{dt} \\ &= \frac{dJ_{tot}(\mathbf{v}_R(t))}{dt} \end{aligned} \quad (4)$$

The proof is completed replacing (3) and (4) in (2) and integrating from 0 to t. ◁

A similar result can be established for RL circuits, noting that $\frac{di}{dt} \in \mathcal{K}_I$ for all $i \in \mathcal{K}_I$. The proof, which follows *verbatim* from the proof of Proposition 2, replacing (2) by

$$\mathbf{v}_L^\top \frac{d\mathbf{i}_L}{dt} + \mathbf{v}_R^\top \frac{d\mathbf{i}_R}{dt} = \mathbf{v}_S^\top \frac{d\mathbf{i}_S}{dt},$$

is omitted for brevity.

Proposition 3. Arbitrary interconnections of passive resistances and inductances with *nondecreasing characteristic function* $i_L = \hat{i}_L(\phi_L)$ and with external port variables $\mathbf{v}_S, \mathbf{i}_S \in I\!R^{n_s}$ satisfy the inequality

$$\int_0^t \hat{\dot{\mathbf{i}}}{}_S^\top(\tau)\mathbf{v}_S(\tau)d\tau \geq W_{tot}[\mathbf{i}_R(t)] - W_{tot}[\mathbf{i}_R(0)]$$

with

$$W_{tot}(\mathbf{i}_R) = \sum_{k\in\mathcal{R}} W_k(i_{kR}) \geq 0$$

Consequently, the circuit defines a passive port with port variables $(\frac{d\mathbf{i}_S}{dt}, \mathbf{v}_S)$ and storage function the total resistor *co–content* $W_{tot}(\mathbf{i}_R)$.

3. DISCUSSION

(1) The new passivity properties of Propositions 2 and 3 differ from the standard result of Proposition 1 in the following respects. First, while Proposition 1 holds for general RLC circuits, the new properties are valid only for RC or RL systems. Using the fact that passivity is invariant with respect to negative feedback interconnections it is, of course, possible to combine RL and RC circuits and establish passivity of some RLC circuits. However, it is not clear at this point how to derive a (relatively) general statement. Second, the condition of nondecreasing characteristic function required for Propositions 2 and 3 is sufficient, but not necessary for passivity of the elements. Hence, the class of admissible C and L terms are more restrictive.

(2) The derivations above are largely inspired by the proof of the Shrinking Dissipation Theorem for RC circuits of the influential paper (Wyatt 1969). In (Wyatt 1969) it is assumed that the capacitances satisfy the constitutive relation

$$i_C = C(v_C)\frac{dv_C}{dt}$$

where $C(v_C) \geq 0$. Notice that, following the description of capacitances used in the paper, we have that $C(v_C) := \frac{\partial \hat{q}_C(v_C)}{\partial v_C}$, with constitutive function $q_C = \hat{q}_C(v_C)$. Therefore, our new passivity property for RC circuits also holds for this class of elements. Similarly, Proposition 3 holds true for RL systems where the inductances are described by

$$v_L = L(i_L)\frac{di_L}{dt}$$

where $L(i_L) \geq 0$.

(3) The results of Propositions 2 and 3 (a well as Proposition 1) can be easily extended—under suitable assumptions—in several directions, e.g., to consider time–varying elements, or reciprocal multiterminal devices (Wyatt 1969).

4. APPLICATION FOR STABILIZATION

In this section we will apply the new passivity properties described above to the problem of stabilization—via a suitable definition of the external source variables—of the equilibria of nonlinear RL or RC circuits. In particular, we will prove that for RL systems we can use the new passivity property to overcome the dissipation obstacle for energy–balancing stabilization identified in (Ortega *et al.* 2001).

In (Ortega *et al.* 2001) we presented a new method to stabilize nonlinear systems satisfying (1) based on the following observation. It is clear from (1) that if no power is delivered to the circuit, i.e., if $\mathbf{v}_S^\top(t)\mathbf{i}_S(t) = 0$ for all $t \geq 0$, then the energy function $\mathcal{E}_{tot}[x(t)]$, (where we have defined the state $x := (\mathbf{q}_C, \phi_L) \in I\!R^n$), is not increasing and—in the presence of adequate dissipation—the trajectories will evolve towards the minimum of $\mathcal{E}_{tot}(x)$. As this minimum will not necessarily coincide with the desired operating point, say $x_* = (\mathbf{q}_{*C}, \phi_{*L})$, the central idea of (Ortega *et al.* 2001) is to assign to the interconnected system a new energy function $\mathcal{E}_d(x)$ which has a *minimum* at x_*.

A natural approach to shape the energy function is to design a source system such that the energy that it supplies to the circuit can be expressed as a function of the circuit state.[2] Indeed, assume that

$$-\int_s^t \mathbf{v}_S^\top(\tau)\mathbf{i}_S(\tau)d\tau = \mathcal{E}_a[x(t)] - \mathcal{E}_a[x(s)] \qquad (5)$$

holds for some function $\mathcal{E}_a(x)$, and for all $t \geq s$. Then, replacing (5) in (1), we see that the trajectories of the system will converge now to the minimum of the *new energy function*

$$\mathcal{E}_d(x) := \mathcal{E}_{tot}(x) + \mathcal{E}_a(x) \qquad (6)$$

and the desired equilibrium x_* will be stable. Notice that the new energy functions is equal to the difference between the stored and the supplied energies. Therefore, we refer to this particular class of controllers as *energy–balancing*.

As discussed in (Ortega *et al.* 2001), and illustrated in the example below, energy–balancing stabilization is possible only for systems that can be stabilized extracting a finite amount of energy from the source, which in particular implies that the supplied power evaluated at x_* should be equal to zero. For instance, a series RLC circuit is energy–balancing stabilizable (because in steady state there is no current drained from the source), but not a parallel RLC circuit. (For the case of linear time–invariant systems without poles at the origin and transfer matrix $\Sigma(s) = C(sI-A)^{-1}B$ we can give a complete characterization of energy–balancing stabilizable systems. Indeed, in this case, admissible equilibria must satisfy $x_* = A^{-1}Bu_*$ for some (nonzero) constant vector $u_* \in I\!R^m$. We have then that

$$u_*^\top C x_* = 0 \Leftrightarrow \Sigma(0) = 0,$$

In other words, a necessary and sufficient condition for the extracted power to be zero at the equilibrium is that the plant has a blocking zero at zero. This is clearly a quite restricted class of systems.)

It turns out that the new passivity properties established in Propositions 2 and 3 can be used to overcome the dissipation obstacle indicated above, assigning Lyapunov–like functions based on the resistor co–content, instead of the total energy. We will illustrate this point with a simple example.

Example. Consider a voltage–controlled nonlinear series RL circuit whose dynamics are described by

$$\dot{\phi}_L = -\hat{v}_R(i_R) + v_S$$
$$i_R = i_L$$
$$i_L = \hat{i}_L(\phi_L)$$

where $\hat{v}_R(i_R)$ is a first third quadrant function, $\hat{i}_L(\phi_L)$ is nondecreasing, and both functions are zero only at zero. As the voltage across the inductance is zero in steady state, it is clear that, at any equilibrium $\phi_{L*} \neq 0$, the extracted power is nonzero, hence the circuit is not energy–balancing stabilizable—even in the linear case! This fact becomes evident if we look at the differential form of (5)

$$\dot{\mathcal{E}}_a(\phi_L) = \frac{\partial \mathcal{E}_a(\phi_L)}{\partial \phi_L}\left[-\hat{v}_R(\hat{i}_L(\phi_L)) + \hat{v}_S(\phi_L)\right] \equiv -\hat{v}_S(\phi_L)\hat{i}_L(\phi_L),$$

where we have postulated a (control) function $v_S = \hat{v}_S(\phi_L)$. Now, as the term in square brackets equals $\dot{\phi}_L$ it should be zero at ϕ_{L*}, but the extracted power at any nonzero equilibrium, $\hat{v}_S(\phi_{L*})\hat{i}_L(\phi_{L*})$, is nonzero, hence the PDE does not have a solution, and we cannot shape the energy function as desired.

On the other hand, to construct our Lyapunov–like function, we can proceed from the inequality of Proposition 3, which in this particular example takes the form

$$\int_0^t \overset{\frown}{\dot{i}_L}(\tau)\hat{v}_S[i_L(\tau)]d\tau \geq W[i_L(t)] = \int_0^{i_L(t)} \hat{v}_R(i'_L)di'_L \geq 0.$$

We can then try to express the left–hand side as a function of i_L, say $-G_a(i_L)$. (Notice that, to simplify the notation, we postulated now a feedback of i_L as the controlled voltage.) This is tantamount to solving the PDE

$$\dot{G}_a(i_L) = \frac{\partial G_a(i_L)}{\partial i_L}\overset{\frown}{\dot{i}_L} \equiv -\hat{v}_S(i_L)\overset{\frown}{\dot{i}_L}$$

that, for any arbitrary $G_a(i_L)$, has the trivial solution

$$\hat{v}_S(i_L) = -\frac{\partial G_a(i_L)}{\partial i_L}$$

The design is completed selecting a function $G_a(i_L)$ such that

$$\hat{i}_L(\phi_{L*}) = \arg\min\{G(i_L) + G_a(i_L)\}$$

and applying the control $v_S = \hat{v}_S(i_L)$. ◇

Obviously, the stabilization problem described above can be solved with much simpler techniques, but the objective of the exercise was to illustrate—with the simplest example—a procedure that can be generalized to more complicated cases.[3] Also, although it is clear that there are many other applications of the new

[2] For ease of presentation the derivations below are presented with some abuse of notation, for a mathematically precise treatment see (Ortega *et al.* 2001) and the example below.

[3] Provided, of course, that we can solve the PDE!

dissipation inequalities for controller design, for the sake of brevity, we will not elaborate on this topic any further. Results along this line will be reported elsewhere.

5. CONCLUSIONS

We have established some new passivity properties for nonlinear RC and RL circuits, when the R, C and L elements are passive and, furthermore, the constitutive relations of the C and L elements are nondecreasing. It is shown that, in this case, it is possible to "add a differentiation" to the port terminals preserving passivity. In contrast to the well–known passivity of general RLC circuits, the new passivity properties have been derived only for RC or RL circuits. This class of circuits contains, however, some interesting application examples. For instance, in VLSI circuits the inductive effects are negligible, while for electric machines one can usually disregard the capacitive terms. The result is then proven to be useful to overcome the dissipation obstacle in RL circuits that are not stabilizable via energy–balancing. At a more general level, one objective of our paper was to put forth the resistor co–content as a new (Lyapunov–like) building block for controller design.

There are close connections of our result and the well–known Brayton–Moser's description of RLC networks (Brayton *et al.* 1964).[4] In this model RLC circuits are gradient systems with potential function the dissipation function defined by the resistors. It is interesting that the model is defined with respect to an indefinite inner product, which is definite if the circuit contains only RL or RC elements, as considered here. See (Weiss *et al.* 1998) for a recent extension of Brayton–Moser's result and a review of the relevant literature.

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