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NONLINEAR FUNCTIONAL CHARACTERIZATIONS OF UNCERTAINTY IN MODEL VALIDATION [1]

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Abstract: Model validation provides a useful means of assessing the ability of a model to account for a specific experimental observation, and has application to modeling, identification and fault detection. In this paper we consider a new approach to the linear fractional transformation (LFT) model validation problem by deploying quadratic functionals, and more generally *nonlinear functionals*, to specify noise and dynamical perturbation sets. Sufficient conditions for invalidation of such models are provided in terms of semidefinite programming problems.

Keywords: Model validation, Nonlinear functionals, Semidefinite programming

1. INTRODUCTION

Models for robust control design contain bounded uncertainties (perturbations and unknown noise or signals) with explicitly specified bounds. Robust control models are specified as sets, where the sizes of the perturbation and noise/disturbance bounds specify the boundary of the sets. In this context, model validation for robust control models can be stated as follows: Given a robust control model, is there a perturbation and noise/disturbance signal from the assumed sets which makes the model consistent with the experimental observation? No assumptions are made about the nature of the physical system. Rather, measurements are taken, and the assumption that the model describes the system is directly tested. The quantitative information obtained is a lower bound on the mismatch between the model and the corresponding physical system. A model cannot be validated by an experiment, only invalidated or *not* invalidated.

The robust control model validation problem was first considered in the frequency domain, using structured LFT models, by Smith and Doyle (1992). Poolla et al. (1994) considered the discrete time-domain case, with a more restricted class of perturbation model structures. Zhou and Kimura (1993) addressed the issue of identifying certain system parameters in this framework. Chen and Wang (1996) and Toker and Chen (1996) extended the time-domain validation approach to certain LFT structures and shown that the model validation problem can be reformulated as a biaffine matrix inequality which they have shown to be $\mathcal{NP}$ hard. Structured LFT frameworks for model validation are also considered in Chen (1997); Rangan and Poolla (1998); Xu et al. (1999).

More recent work in Smith and Dullerud (1996) and Rangan and Poolla (1996) has focused on sampled-data framework and the emphasis has again been structures in which the norm-bounded perturbations affect the residual data/model mismatch linearly. The sampled-data results share the same complexity properties as the aforementioned time-domain results.

The contribution of this paper is that we consider model validation in a more general setting than previous work. Our approach considers an LFT formulation, and use nonlinear functional constraints—in particular, multinomial functionals—to provide a wider set of possibilities for expressing the characteristics

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of the input disturbances and dynamical perturbations. This general approach to modeling perturbations and signals is motivated by Yakubovich (1971) and Megretski and Rantzer (1997). We also make use of the recent work in Parrilo (2000) on non-negativity of multinomial functionals, and we derive sufficient conditions for invalidation in terms of semidefinite programming problems.

2. THE MODEL VALIDATION PROBLEM

The linear fractional framework to be considered is described by the equations,

$$\begin{aligned} z &= P_{11}v + P_{12}w + P_{13}u \\ y &= P_{21}v + P_{22}w + P_{23}u \\ v &= \Delta z, \end{aligned} \tag{1}$$

where systems P and Δ are discrete-time and causal. We consider the signals in the above as finite data records, of length T. P is taken to be a *linear* system, and Δ to be a more general mapping residing in a prespecified class. Despite its simple structure this paradigm is remarkably general when Δ is used to account for both static nonlinearities and unmodeled dynamics.

The input w is an unknown exogenous signal, taken from a known bounded set, $w \in \mathscr{X}_w \subset \mathbb{R}^{n_w T}$. The other input, u, is known or measured and might correspond to the excitation input in an identification experiment. In the model validation problem, the output signal, y, is also considered to be known or measured. Measurement noise is modeled as a component of w, and in any experimentally based problem P_{22} can be assumed to be left invertible. The function Δ is constrained in terms of the ordered pair (v,z), by $(z,v) \in \mathscr{X}_\Delta \subset \mathbb{R}^{n_z T} \times \mathbb{R}^{n_v T}$.

Model validation is the data based assessment of this model. Given measurements of the input u, and output y, we wish to determine whether or not there exist $(v,z) \in \mathscr{X}_\Delta$ and $w \in \mathscr{X}_w$, consistent with (1). If no such (z,v) and w exist, then the particular datum (u,y) *invalidates* the model. This is formally stated as follows.

Problem 1. (Model Validation). Let P be a matrix, and $\mathscr{X}_w$ and $\mathscr{X}_\Delta$ be appropriately defined subsets of Euclidean space. Given measurements $(u,y) \in \mathbb{R}^{n_u T} \times \mathbb{R}^{n_y T}$, do there exist internal signals $(v,z) \in \mathscr{X}_\Delta$, and a disturbance signal $w \in \mathscr{X}_w$, such that

$$\begin{bmatrix} z \\ y \end{bmatrix} = P \begin{bmatrix} v \\ w \\ u \end{bmatrix}. \tag{2}$$

If no triple $(w,v,z) \in \mathscr{X}_w \times \mathscr{X}_\Delta$ satisfying the conditions of Problem 1 exists then the model is *invalidated* by the datum. Strictly speaking model validation methods can only invalidate models. This also motivates us to search for sufficient conditions for invalidation as they are the only conditions which lead to definitive statements about model quality. If there exists a triple $(w,v,z) \in \mathscr{X}_w \times \mathscr{X}_\Delta$ satisfying the validation conditions then we say that the datum *corroborates* the model. Note that this is not as strong a statement as invalidation as the elements of the triple (w,v,z) do not necessarily match physical signals in the system, and subsequent experiments may invalidate the model. Our ability to answer this question in some form will clearly depend on the characterization of the sets $\mathscr{X}_w$ and $\mathscr{X}_\Delta$, an issue we now consider.

2.1 *Model sets*

The object we will use to specify our model sets is the concept of a quadratic functional.

Definition 2. A mapping $F : \mathbb{R}^n \to \mathbb{R}$ is a *quadratic functional* if there exist a matrix $A \in \mathbb{R}^{n\times n}$, a vector $b \in \mathbb{R}^n$, and a scalar $c \in \mathbb{R}$ such that

$$F(x) = x^* A x + b^* x + c \quad \text{for all } x \text{ in } \mathbb{R}^n.$$

We will define the signal sets $\mathscr{X}_w$ and $\mathscr{X}_\Delta$ by intersecting sets of the form

$$\{x \in \mathbb{R}^n : \; F(x) \geq 0\},$$

given a quadratic functional F. A special instance of the constraint defining the set is when the vector b and scalar c are zero, in which case it is known as an integral quadratic constraint (IQC) set; IQCs are a very useful tool for analyzing feedback systems, see for instance Megretski and Rantzer (1997). Constraints involving quadratic functionals can be used to describe many dynamical properties of a perturbation Δ, and a disturbance signal w. Using this concept we will explicitly define our models sets by

$$\begin{aligned} \mathscr{X}_w &:= \{w \in \mathbb{R}^{n_w T} : \; W_0(w) > 0, W_1(w) \geq 0, \ldots \\ &\qquad\qquad \ldots, W_d(w) \geq 0\} \\ \mathscr{X}_\Delta &:= \{(z,v) \in \mathbb{R}^{n_z T} \times \mathbb{R}^{n_v T} : Q_1(z,v) \geq 0, \ldots \\ &\qquad\qquad \ldots, Q_r(z,v) \geq 0\}, \end{aligned}$$

for prespecified quadratic functionals W_i and Q_i. Note that the constraint given by W_0 is strict, whereas the others are non-strict. This makes little difference from a practical perspective, and is convenient technically. The use of such constraints, particularly IQCs, for modeling signals and and perturbations is a well-developed area, and in the next two subsections we provide some basic illustrative examples from the literature in this area for tutorial purposes.

Specification of the $\mathscr{X}_\Delta$ Model Sets In most of the model validation frameworks studied to date, the uncertainty sets, $\mathscr{X}_\Delta$, can be cast as quadratic forms, Q_i.

Example 1: Discrete-time norm-bounded LTV operators. Here we consider the case where the operator Δ is linear and causal, and satisfies the ℓ_2 induced norm bound constraint of $\|\Delta\|_{\ell_2 \to \ell_2} \leq \gamma$, for some specified $\gamma > 0$.

Given two finite data vectors $z = (z(0), z(1), \ldots, z(T-1))$ and $v = (v(0), v(1), \ldots, v(T-1))$, we would like to characterize when there exists such a causal $\Delta : \ell_2 \to \ell_2$, satisfying $v = \Gamma\Delta\Gamma^* z$, where Γ is the truncation operator mapping $\ell_2^n \to \mathbb{R}^{nT}$ in the obvious way. It is shown in Poolla et al. (1994) that a necessary and sufficient condition is that $\|\Pi_i v\|_2 \leq \gamma\|\Pi_i w\|_2$, for $1 \leq i \leq T$, where Π_i is the truncation map from sequences of length T to sequences of length i. Defining the quadratic functional (form)

$$Q_i(z,v) = \begin{bmatrix} z \\ v \end{bmatrix}^* \begin{bmatrix} \gamma^2 \Pi_i^* \Pi_i & 0 \\ 0 & -\Pi_i^* \Pi_i \end{bmatrix} \begin{bmatrix} z \\ v \end{bmatrix},$$

and we see that a Δ exists exactly when $Q_i(z,v) \geq 0$ holds for all $1 \leq i \leq T$. □

The results discussed here extend easily to the sampled-data case. See Smith and Dullerud (1996); Rangan and Poolla (1996) for details.

Example 2: Block diagonal operators. A common type of perturbation Δ is one that has a block diagonal structure, and is associated with the structured singular value in the time-invariant case; see e.g., Packard and Doyle (1993). The given data sequences z and v are partitioned into *spatial* channels $z_1, \ldots, z_m$ and $v_1, \ldots, v_m$, and we are looking to determine when there exists a causal perturbation of the form

$$\Delta = \operatorname{diag}(\Delta_1, \cdots, \Delta_m),$$

so that $v_i = \Gamma\Delta_i\Gamma^* z_i$ for $1 \leq i \leq m$ and $\|\Delta\|_{\ell_2 \to \ell_2} \leq \gamma$. If we are considering LTV operators, it is clear that such a perturbation exists exactly when each pair (z_i, v_i) satisfies the constraints of Example 1; namely, $Q_i(z_k, v_k) \geq 0$ for all $1 \leq i \leq T$ and $1 \leq k \leq m$. □

Example 3: Sector bounded memoryless nonlinearities. We now consider the case where Δ is not an unknown operator, but a specific memoryless nonlinearity, $v = \Delta z = \psi(z)$.

In the general case some part of z will be unknown and the effect of the nonlinearity will have to be addressed. An optimization problem based on the explicit knowledge of ψ, will usually be complex and non-convex. A reasonable alternative is to replace $v = \psi(z)$, by an IQC based constraint. This is potentially conservative but will still give sufficient conditions for invalidation.

For example, consider ψ to be a memoryless nonlinearity in the sector $[\alpha, \beta]$. Then,

$$\alpha z(k)^2 \leq \psi(z(t))\, z(k) \leq \beta z(k)^2$$

for all $z(k) \in \mathbb{R}$ and for all $k \geq 0$. Observing that the operator,

$$\psi(z) - \frac{(\alpha+\beta)}{2} z$$

is in the sector $[-(\beta-\alpha)/2, (\beta-\alpha)/2]$, shows that the above conditions are clearly captured by the quadratic constraints,

$$\begin{bmatrix} z(k) \\ v(k) \end{bmatrix}^* \begin{bmatrix} -2\alpha\beta & \alpha+\beta \\ \alpha+\beta & -2 \end{bmatrix} \begin{bmatrix} z(k) \\ v(k) \end{bmatrix} \geq 0,$$

for all $z(k) \in \mathbb{R}$ and for all $k \geq 0$. In some problem structures z, and therefore v, may be calculated from other measured signals. In such cases the nonlinearity poses no additional difficulties, and the model validation problem may be treated by the more standard linear methods; see Smith and Dullerud (1999) for examples. □

A large number of useful model set specifications (including all of the $\mathscr{H}_\infty$ model validation work to date) can be expressed in terms of quadratic functionals. The common framework makes it particularly easy to apply operator set descriptions to the component blocks, Δ_i. In particular, Megretski and Rantzer (1997) give a detailed description of IQCs including a list of IQCs for a wide range of nonlinear and uncertain operators.

Specification of the $\mathscr{X}_w$ Model Sets A similar approach is taken for the definition of the exogenous signal set in terms of functionals, $W_i(w)$.

Example 4: Norm bounded signals. The signal set $\mathscr{X}_w$ is from an open norm bounded ball. Thus this set is totally specified in terms of the single functional $W_0(w) = \beta^2 - w^* w > 0$. □

Example 5: White noise signals. Given a scalar data sequence $w(0), \ldots, w(T-1)$, we define its circular autocorrelation via

$$r_w(i) := \sum_{t=0}^{T-1} w(t + i \bmod T) w(t).$$

A standard characterization of a sequence w being "white", is to say that the values of $r_w(i)$ at nonzero values of i, are small compared with $r_w(0)$; see e.g., Paganini (1995). More precisely, we say that the sequence w is white up to accuracy γ if

$$|r_w(i)| \leq \gamma r_w(0), \qquad \text{for } 1 \leq i \leq T-1. \tag{3}$$

Let Z be the cyclic shift matrix on sequences of length T, and then observe that

$$\gamma r_w(0) - r_w(k) = w^* \left(\gamma I - \frac{1}{2}(Z^k)^* - \frac{1}{2} Z^k \right) w.$$

For $1 \leq i \leq T-1$ define $W_i(w)$ to be the quadratic form on the right-hand side above, and thus the constraints in (3) are simply given by inequalities $W_i(w) \geq 0$, and similar inequalities constructed from the constraints $\gamma r_w(0) + r_w(k) \geq 0$. □

3. A GENERAL FORMULATION AND SOLUTION

Our goal in this section is to provide a readily computable way to approach the model validation question. The approach is motivated by the work in Yakubovich (1971). We begin by considering the equation (2) and rearrange to get

$$\begin{bmatrix} -P_{13} & 0 \\ -P_{23} & I \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} P_{12} & P_{11} & -I \\ P_{22} & P_{21} & 0 \end{bmatrix} \begin{bmatrix} w \\ v \\ z \end{bmatrix}.$$

Thus we see that the variables which satisfy (2) can be parametrized by a vector variable ξ via

$$\begin{bmatrix} w(\xi) \\ v(\xi) \\ z(\xi) \end{bmatrix} = x(\xi) = x_0 + R\xi, \qquad (4)$$

for some matrix R and some particular solution x_0. *We remark that if the parametrization in (4) is empty, then the model is immediately invalidated for all sets $\mathscr{X}_\Delta$ and $\mathscr{X}_w$*. Observe that if F is a quadratic functional in the variable x, then $G(\xi) := F(x(\xi))$ is a quadratic functional in ξ.

We now state the following important result, which is a corollary to (Fradkov and Yakubovich, 1973, Thm 2.1):

Theorem 3. Suppose $H_0(\xi), \ldots, H_n(\xi)$ are *quadratic functionals* in the real vector variable ξ. If there exist scalars $\tau_i \geq 0$ such that the inequality

$$H_0(\xi) + \tau_1 H_1(\xi) + \cdots + \tau_n H_n(\xi) \leq 0 \qquad (5)$$

holds for all ξ, then there does not exist a vector ξ such that the inequalities

$$H_0(\xi) > 0, H_1(\xi) \geq 0, \ldots, H_n(\xi) \geq 0 \qquad (6)$$

are simultaneously satisfied. Furthermore, if $n = 1$ and there exists ξ such that $H_1(\xi) > 0$, then if there does not exist a solution to (6), then there exist $\tau_i \geq 0$ satisfying (5).

Returning to our model validation problem, we recall that our sets $\mathscr{X}_w$ and $\mathscr{X}_\Delta$ are defined in terms of the quadratic functionals W_i and Q_i respectively. Using the parametrization of signals w, v, z from (4) define the quadratic functionals $G_i(\xi)$ by

$$G_i(\xi) = \begin{cases} W_i(w(\xi)), & 0 \leq i \leq d \\ Q_{i-r}(v(\xi), z(\xi)), & d < i \leq d + r. \end{cases} \qquad (7)$$

We can now state the main result of the section.

Theorem 4. Given a datum (u,y) and the quadratic functionals defined in (7). If there exist scalars $\tau_i \geq 0$ such that

$$G_0(\xi) + \tau_1 G_1(\xi) + \cdots + \tau_{d+r} G_{d+r}(\xi) \leq 0, \qquad (8)$$

for all ξ, then the model in Problem 1 is invalidated. Furthermore, if $n = 1$ and there exists ξ such that $G_1(\xi) > 0$, then the model is invalidated if and only if the condition in (8) can be satisfied.

The theorem states that if multipliers can be found such that (8) holds, then the model specified is not consistent with the datum (u,y). This result is exact when there are only two functionals, and a value of the parameter ξ can be found that makes G_1 positive. In this case the model being not invalidated is equivalent to it being corroborated. In other words, there exists $w \in \mathscr{X}_w$, and $\Delta \in \mathscr{X}_\Delta$ accounting for the observed datum.

4. SOLUTION VIA LMI OPTIMIZATION METHODS

The test in Theorem 3 is readily converted to a linear matrix inequality feasibility problem via the following lemma.

Lemma 5. Suppose $G_i(\xi)$ are quadratic functionals defined by $G_i(\xi) := \xi^* A_i \xi + b_i^* \xi + c_i$, and τ_i are non-negative scalars. Then (8) is satisfied if and only if the matrix inequality

$$M_0 + \tau_1 M_1 + \cdots \tau_{d+r} M_{d+r} \leq 0$$

holds for some $\tau_i \geq 0$, where the matrices

$$M_i := \begin{bmatrix} A_i & b_i^* \\ b_i & c_i \end{bmatrix}.$$

Recall that we typically defined $Q_i(z,v)$ in terms of γ, an assumed norm bound on $\mathscr{X}_\Delta$, and $W(w)$ was defined in terms of the norm bound β. It is frequently useful to determine the smallest γ and β such that the model is invalidated. This can be posed as an LMI optimization problem.

5. GENERALIZATION TO MULTINOMIAL FUNCTIONALS

In the preceding sections we have emphasized quadratic functionals to describe our noise and uncertainty sets. Our goal is now to show that the model validation framework presented can be extended to a more general class of nonlinear functionals, yet a computable test is still possible using a combination of the S-procedure and recent work in Parrilo (2000).

Definition 6. A function $F:\mathbb{R}^n \to \mathbb{R}$ is a *multinomial functional* if, for some $q \in \mathbb{N}_0$, there exist scalars $c_1,\dots,c_q \in \mathbb{R}$ and vectors $\alpha_1,\dots\alpha_q \in \mathbb{N}_0^n$, such that

$$F(x) = \sum_{i=1}^{q} c_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \cdots x_n^{\alpha_{in}}.$$

The *order* of the functional is $d_F := \max_i\{\alpha_{i1}+\cdots+\alpha_{in}\}$. Furthermore, if $d_F = \alpha_{i1}+\cdots+\alpha_{in}$, for each i, F is called a *multinomial form*.

So far in the paper we have restricted our use of functionals to the case where the order $d_F \leq 2$, and our goal is now to outline how our framework might be used in this more general case. Clearly if we allow ourselves to use multinomial functionals for defining the perturbation set $\mathscr{X}_\Delta$ and disturbance set $\mathscr{X}_w$, we still have that the model validation problem has a solution if and only if the positivity constraints on the $G_i(\xi)$ are satisfied. As in Theorem 4, it is therefore clear that the model is invalidated if there exist scalars $\tau_i \geq 0$ such that $G(\xi,\tau) \leq 0$, for all ξ, where

$$S(\xi,\tau) := G_0(\xi) + \tau_1 G_1(\xi) + \cdots \tau_{d+r} G_{d+r}(\xi). \quad (9)$$

Before proceeding to discuss computation of the above condition, we provide an important motivating example.

Example 6: LTI Uncertainty. We now consider characterizing when, given data sequences z and v, is it possible to find a causal linear time-invariant perturbation Δ such that both $\|\Delta\|_{\ell_2\to\ell_2} \leq \gamma$ and $v = \Gamma\Delta\Gamma^* z$ holds. This condition can be expressed very compactly in terms of a matrix inequality. First, define the notation

$$\mathscr{T}(v) = \begin{bmatrix} v(0) & 0 & 0 & \cdots & 0 \\ \vdots & v(0) & 0 & \cdots & 0 \\ v(T-1) & \cdots & v(0) & \cdots & 0 \\ v(T) & v(T-1) & \cdots & \ddots & v(0) \end{bmatrix}.$$

Then the classic Caratheodory interpolation theorem states that an LTI operator Δ exists if and only if the matrix inequality, $M(\xi) \geq 0$, where,

$$M(\xi) := \gamma^2 \mathscr{T}(z(\xi))^* \mathscr{T}(z(\xi)) - \mathscr{T}(v(\xi))^* \mathscr{T}(v(\xi)).$$

See, for example, Foias and Frazho (1990).

A symmetric matrix is positive semidefinite exactly when all its principal minors are non-negative. Let $Q_i(\xi)$ denote the principal minors of $M(\xi)$; namely,

$$Q_i(\xi) = \det(\Upsilon_i^* M(\xi) \Upsilon_i) \quad \text{for } 1 \leq i \leq T,$$

where $\Upsilon_i : \mathbb{R}^T \to \mathbb{R}^i$ projects $\mathbb{R}^T$ onto $\mathbb{R}^i$ in the usual way. Each Q_i is a multinomial functional in the variable ξ. Thus we have that an LTI operator exists mapping the initial sequence $z(\xi)$ to $v(\xi)$ if and only if the following multinomial inequalities are satisfied

$$Q_i(\xi) \geq 0, \quad \text{for each } 1 \leq i \leq T.$$

We remark that the matrix condition $M(\xi) \geq 0$ can be expressed as an infinite number of quadratic functionals: observe that each scalar entry in $M(\xi)$ is a quadratic functional of ξ and thus the matrix inequality is equivalent to

$$F_x(\xi) := x^* M(\xi) x \geq 0,$$

for all vectors $x \in \mathbb{R}^T$. □

In general, computing the multiplier condition in (9) is difficult. However, the recent work in Parrilo (2000) provides an LMI approach, and we now give a brief exposition in our current model validation context to do this. Returning to (9), it is easy to show that there exist (non-unique) matrices $J_0,\dots,J_{d+r}$ such that

$$S(\xi,\tau) = p^*(\xi) \underbrace{\left(J_0 + \tau_1 J_1 + \cdots + \tau_{d+r} J_{d+r}\right)}_{J(\tau)} p(\xi),$$

where $p(\xi)$ is a vector-valued function with entries of the form $p_i(\xi) = \xi_1^{\beta_{1i}} \xi_2^{\beta_{2i}} \cdots \xi_n^{\beta_{ni}}$. Note that $p(\xi)$ can be chosen so that $\beta_{1i}+\cdots+\beta_{mi} \leq \frac{d_{Gmax}+1}{2}$, where $d_{Gmax} = \max_i d_{G_i}$. This representation is studied in a control context in Bose and Li (1968).

Now, notice that the matrices U which satisfy $p^*(\xi) U p(\xi) = 0$, form a finite dimensional *subspace*; let the matrices $N_1,\dots,N_k$ denote a basis for this subspace. Hence, we have for all scalars λ_i that

$$S(\xi,\tau) = p^*(\xi) \underbrace{\left(J(\tau) + \lambda_1 N_1 + \cdots + \lambda_k N_k\right)}_{J(\tau,\lambda)} p(\xi).$$

Furthermore, if $V(\tau)$ is a matrix such that $S(\xi,\tau) = p^*(\xi) V(\tau) p(\xi)$, then there exists values of λ_i such that $V(\tau) = J(\tau,\lambda)$. From this discussion we obtain the following result, which provides a sufficient condition for model invalidation in terms of an LMI feasibility problem based on the matrices defined so far.

Theorem 7. Given the definitions of the matrices J_i and N_i above. If there exist values of $\lambda_i \in \mathbb{R}$ and $\tau_i \geq 0$ satisfying the matrix inequality,

$$J_0 + \tau_1 J_1 + \cdots \tau_{d+r} J_{d+r} + \lambda_1 N_1 + \cdots \lambda_k N_k \leq 0,$$

then the model is invalidated.

Thus this theorem provides a sufficient condition in terms of an LMI for invalidating a model, when the model sets are specified by multinomial functionals. Unfortunately, the conditions in (9) is not equivalent to the above inequality, with the latter only implying the former. However, recent numerical experiments in Parrilo and Sturmfels (2001), indicate that perhaps it is a rare situation, and that satisfiability of these conditions is typically equivalent. This would indicate that for typical examples, the conservatism in our model invalidation condition would only be that introduced by the use of the *S*-procedure condition in (9).

6. CONCLUSIONS

The model validation framework has been significantly generalized by using multinomial functional descriptions to characterize perturbations and nonlinearities. The *S*-procedure can then be used to develop sufficient conditions for invalidation conditions, leading to semidefinite programming methods for computation.

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