15th Triennial World Congress, Barcelona, Spain

# ON STABILITY TESTS FOR LINEAR SYSTEMS

**Maurício C. de Oliveira** [*,1] **Robert E. Skelton** [**]

[*] *Department of Telematics, School of Electrical and Computer Engineering, University of Campinas, Av. Albert Einstein 400, Campinas, SP, 13083-970, Brazil.*
[**] *Department of Mechanical and Aerospace Engineering, Mail Code 0411, 9500 Gilman Drive, La Jolla, California, 92093-0411, USA.*

Abstract: This paper is yet another demonstration of the fact that enlarging the design space allows simpler tools to be used for analysis. It shows that several problems in linear systems theory can be solved by combining Lyapunov stability theory with Finsler's Lemma. Using these results, the differential or difference equations that govern the behavior of the system can be seen as constraints. These *dynamic constraints*, which naturally involve the state derivative, are incorporated into the stability analysis conditions through the use of scalar or matrix Lagrange multipliers. No *a priori* use of the system equation is required to analyze stability. One practical consequence of these results is that they do not necessarily require a state space formulation. This has value in mechanical and electrical systems, where the inversion of the *mass* matrix introduces complicating nonlinearities in the parameters. The introduction of multipliers also simplify the derivation of robust stability tests, based on quadratic or parameter-dependent Lyapunov functions.

Keywords: Stability analysis, Stability tests, Linear systems

## 1. A MOTIVATION FROM LYAPUNOV STABILITY

Consider the continuous-time linear time-invariant system described by the differential equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \tag{1}$$

where $x(t) : [0, \infty) \to \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Define the quadratic form $V : \mathbb{R}^n \to \mathbb{R}$ as

$$V(x) := x^T P x, \tag{2}$$

where $P \in \mathbb{S}^n$. The symbol $(\cdot)^T$ denotes transposition and $\mathbb{S}^n$ denotes the space of the square and symmetric real matrices of dimension $n$. If

$$V(x) > 0, \quad \forall x \neq 0,$$

then matrix $P$ is said to be *positive definite*. The symbol $X \succ 0$ ($X \prec 0$) is used to denote that the symmetric matrix $X$ is *positive* (*negative*) *definite*.

The equilibrium point $x = 0$ of the system (1) is said to be (globally) *asymptotically stable* if

$$\lim_{t\to\infty} x(t) = 0, \quad \forall x(0) = x_0, \tag{3}$$

where $x(t)$ denotes a solution to the differential equation (1). If (3) holds, then, by extension, the system (1) is said to be asymptotically stable. A necessary and sufficient condition for the system (1) to be asymptotically stable is that matrix $A$ be Hurwitz, that is, that all eigenvalues of $A$ have negative real parts. According to Lyapunov stability theory, system (1) is asymptotically stable if there exists $V(x(t)) > 0$, $\forall x(t) \neq 0$ such that

$$\dot{V}(x(t)) < 0, \quad \forall \dot{x}(t) = Ax(t), \quad x(t) \neq 0. \tag{4}$$

That is, if there exists $P \succ 0$ such that the time derivative of the quadratic form (2) is negative *along all trajectories* of system (1). Conversely, it is well known that if the linear system (1) is asymptotically stable then there always exists $P \succ 0$ that renders (4) feasible. Notice that in (4), the time derivative $\dot{V}(x(t))$ is a func-

[1] Maurício C. de Oliveira is supported by a grant from FAPESP, "Fundação de Amparo à Pesquisa do Estado de São Paulo", Brazil.

tion of the state $x(t)$ only, which implicitly assumes that the *dynamical constraint* (1) has been previously substituted into (4). This yields the equivalent condition

$$\dot{V}(x(t)) = x(t)^T \left(A^T P + PA\right) x(t) < 0, \quad \forall x(t) \neq 0.$$

Hence, asymptotic stability of (1) can be checked by using the following lemma.

*Lemma 1.* (Lyapunov). The time-invariant linear system is asymptotically stable if, and only if, $\exists P \in \mathbb{S}^n : P \succ 0, \quad A^T P + PA \prec 0.$

One might ask whether would it be possible to characterize the set defined by (4) without substituting (1) into (4)? The aim of this work is to provide an answer to this question. The recurrent idea is to analyze the feasibility of sets of inequalities subject to *dynamic equality constraints*, as (4), from the point of view of *constrained optimization*. By utilizing the well known Finsler's Lemma (Finsler, 1937) it will be possible to characterize existence conditions for this class of problems without explicitly substituting the dynamic constraints. Equivalent conditions will be generated where the dynamic constraints appear weighted by *multipliers*, a standard expedient in the optimization literature. The method is conceptually simple, yet it seems that it has never been used with that purpose in the systems and control literature so far.

The advantage of substituting the dynamic constraints in the stability test conditions is the reduced size of the space on which one must search for a solution. In the context of the problem of Lyapunov stability this reduced space is characterized by the state $x(t)$. In contrast, the space composed of $x(t)$ and $\dot{x}(t)$ can be seen as an *enlarged* space. In this paper it will be shown that the use of larger search spaces for linear systems analysis provides better ways to explore the *structure* of the problems of interest. This will often lead to mathematically more tractable problems. Whereas working in a higher dimensional space requires the introduction of some extra variables to search for — which one might think at first sight as being a disadvantage, — this is frequently accompanied by substantial benefits. The authors believe that the technique that will be introduced in this work has the potential to show new directions to be explored in several areas, such as decentralized control (Siljak, 1990), fixed order dynamic output feedback control (Syrmos *et al.*, 1997), integrating plant and controller design (Grigoriadis *et al.*, 1996), singular descriptor systems (Cobb, 1984). In all these areas, the standard tests based on Lyapunov stability theory can be tough to manipulate. The introduction of a different perspective may reveal easier ways to deal with these difficult problems. Besides, several recent results can be given a broader and more consistent interpretation. For instance, the robust stability analysis results (Geromel *et al.*, 1998; de Oliveira *et al.*, 1999*c*; de Oliveira *et al.*, 1999*a*; Scherer, 2000) and the *extended* controller and filter synthesis procedures (de Oliveira *et al.*, n.d.; de Oliveira *et al.*, 1999*b*; Geromel *et al.*, 1999) can be interpreted and generalized using these new tools.

Due to space limitations, this paper has been shortened to fit the conference format. The interested reader is refered to de Oliveira and Skelton for a complete version with all proofs.

## 2. LYAPUNOV STABILITY CONDITIONS WITH MULTIPLIERS

Consider the set of inequalities with dynamic constraints (4) arising from Lyapunov stability analysis of the linear time-invariant system (1). Define the quadratic form $\dot{V} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$\dot{V}(x(t), \dot{x}(t)) := x(t)^T P \dot{x}(t) + \dot{x}(t)^T P x(t), \qquad (5)$$

which is the time derivative of the quadratic form (2) expressed as a function of $x(t)$ and $\dot{x}(t)$. Do not explicitly substitute $\dot{x}(t)$ in (5) using (1), and build the set

$$\dot{V}(x(t), \dot{x}(t)) < 0, \forall \dot{x}(t) = Ax(t), \, (x(t), \dot{x}(t)) \neq 0. \quad (6)$$

In the sequel, stability will be characterized by using (6) instead of (4). This replacement is possible even though (4) requires only that $x(t) \neq 0$ while (6) requires that $(x(t), \dot{x}(t)) \neq 0$. Utilizing an argument similar to the one found in (Boyd *et al.*, 1994), pp. 62–63, this equivalence between (4) and (6) can be proved by verifying that the set

$$\begin{aligned} &\dot{V}(x(t), \dot{x}(t)) < 0, \\ &\qquad \forall \dot{x}(t) = Ax(t), \quad x(t) = 0, \quad \dot{x}(t) \neq 0 \quad (7) \end{aligned}$$

is empty. But from (5), it is not possible to make $\dot{V}(x(t), \dot{x}(t)) < 0$ with $x(t) = 0$, which shows that (7) is indeed empty. Moreover, $\dot{V}(x(t), \dot{x}(t))$ is never strictly negative for all $(x(t), \dot{x}(t)) \neq 0$ without the presence of the dynamic equality constraint (1).

The advantage of working with (6) instead of (4) is that the set of feasible solutions of (6) can be characterized using the following lemma, which is originally attributed to (Finsler, 1937) (see also (Uhlig, 1979)).

*Lemma 2.* (Finsler). Let $x \in \mathbb{R}^n$, $\mathcal{Q} \in \mathbb{S}^n$ and $\mathcal{B} \in \mathbb{R}^{m \times n}$ such that $\operatorname{rank}(\mathcal{B}) < n$. The following statements are equivalent:

i) $x^T \mathcal{Q} x < 0, \quad \forall \mathcal{B} x = 0, \quad x \neq 0.$
ii) $\mathcal{B}^{\perp T} \mathcal{Q} \mathcal{B}^{\perp} \prec 0.$
iii) $\exists \mu \in \mathbb{R} : \mathcal{Q} - \mu \mathcal{B}^T \mathcal{B} \prec 0.$
iv) $\exists \mathcal{X} \in \mathbb{R}^{n \times m} : \mathcal{Q} + \mathcal{X} \mathcal{B} + \mathcal{B}^T \mathcal{X}^T \prec 0.$

Lemma 2 has many available proofs in the literature as, for instance, in (de Oliveira and Skelton, 2001; Skelton *et al.*, 1997). In Lemma 2, statement *i*) is a *constrained* quadratic form, where the vector $x \in \mathbb{R}^n$ is confined to lie in the null-space of $\mathcal{B}$. In other words, vector $x$ can be parametrized as $x = \mathcal{B}^{\perp} y$, $y \in \mathbb{R}^r$, $r := \operatorname{rank}(\mathcal{B}) < n$, where $\mathcal{B}^{\perp}$ denotes a basis for the null-space of $\mathcal{B}$. Statement *ii*) corresponds to explicitly substituting that information back into *i*), which then provides an *unconstrained* quadratic form in $\mathbb{R}^r$. Finally, items *iii*) and *iv*) give equivalent *unconstrained* quadratic forms in the original $\mathbb{R}^n$, where the constraint is taken into account by introducing multipliers. In *iii*) the multiplier is the scalar $\mu$ while in *iv*) it is the matrix $\mathcal{X}$. In this sense, the quadratic forms given in *iii*) and *iv*) can be identified as *Lagrangian* functions. Reference (Hamburger, 1999) explicitly identifies $\mu$ as a *Lagrange multiplier* and uses constrained optimization theory to prove a version of Lemma 2.

Finsler's Lemma has been previously used in the control literature mainly with the purpose of eliminating design variables in matrix inequalities. In this context, Finsler's Lemma is usually referred to as Elimination Lemma. Most applications move from statement *iv*) to statement *ii*), thus *eliminating* the variable (multiplier) $\mathcal{X}$. Several versions of Lemma 2 are available under different assumptions. A special case of item *iv*) served as the basis for the entire book (Skelton *et al.*, 1997), which shows that at least 20 different control problems can be solved using Finsler's Lemma.

Recalling that the requirement $V(x(t)) > 0$, $\forall x(t) \neq 0$ can be stated as $P \succ 0$, and rewriting (6) in the form

$$\begin{pmatrix} x(t)^T & \dot{x}(t)^T \end{pmatrix} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} < 0,$$
$$\forall \begin{bmatrix} A & -I \end{bmatrix} \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = 0, \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} \neq 0,$$

it becomes clear that Lemma 2 can be applied to (6).

*Theorem 3.* (Linear System Stability). The following statements are equivalent:

i) The linear time-invariant system (1) is asymptotically stable.
ii) $\exists P \in \mathbb{S}^n : P \succ 0, \quad A^T P + PA \prec 0.$
iii) $\exists P \in \mathbb{S}^n, \mu \in \mathbb{R}:$
$$P \succ 0, \quad \begin{bmatrix} -\mu A^T A & \mu A^T + P \\ \mu A + P & -\mu I \end{bmatrix} \prec 0.$$
iv) $\exists P \in \mathbb{S}^n, F, G \in \mathbb{R}^{n \times n}:$
$$P \succ 0, \quad \begin{bmatrix} A^T F^T + FA & A^T G^T - F + P \\ GA - F^T + P & -G - G^T \end{bmatrix} \prec 0.$$

**PROOF.** Item *i*) can be stated as $P \succ 0$ and (6). Lemma 2 can be used with

$$x \leftarrow \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}, \mathcal{Q} \leftarrow \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}, \mathcal{B}^T \leftarrow \begin{bmatrix} A^T \\ -I \end{bmatrix},$$
$$\mathcal{X} \leftarrow \begin{bmatrix} F \\ G \end{bmatrix}, \mathcal{B}^{\perp} \leftarrow \begin{bmatrix} I \\ A \end{bmatrix}.$$

and (6) to generate items *ii*) to *iv*).

It is a nice surprise that Finsler's Lemma has been able to generate item *ii*) of Theorem 3 which is exactly the standard Lyapunov stability condition given in Lemma 1. Items *iii*) and *iv*) are new stability conditions. Since $A$ is a constant given matrix, all three conditions are LMI (Linear Matrix Inequalities) and the feasible sets of conditions *ii*), *iii*) and *iv*) are convex sets (see Boyd *et al.* for details). Notice that the first block of the second inequality in condition *iii*) is $\mu A^T A \succ 0$, which implies that $\mu > 0$ and $A$ is nonsingular. This agrees with the fact that Lyapunov stability requires that no eigenvalues of matrix $A$ should lie on the imaginary axis.

The multipliers $\mu$, $F$ and $G$ represent extra degrees of freedom that can be used, for instance, for robust analysis or controller synthesis. In some cases, not all degrees of freedom introduced by the multipliers are really necessary, and it can be useful to constrain the multipliers. Notice that constraining a multiplier is usually less conservative than constraining the Lyapunov matrix (see de Oliveira *et al.*). Some constraints on the matrix multiplier can be enforced without loss of generality. For instance, $\mathcal{X}$ can always be set to $-(\mu/2)\mathcal{B}^T$ without loss of generality. Besides this "trivial" choice, some more elaborated options might be available. For example, choosing the variables in item *iv*) to be

$$F = F^T = P, \quad G = \varepsilon I,$$

introduces no conservativeness in the sense that there will always exist a sufficiently small $\varepsilon$ that will enable the proof of stability. This behavior is similar to the one exhibited by the stability condition developed in (Geromel *et al.*, 1998). In fact, item *iv*) is a particular case of (Geromel *et al.*, 1998), which has been obtained as an application of the positive-real lemma.

The introduction of *extra variables*, here identified as Lagrange multipliers, is the core of the recent works (Geromel *et al.*, 1998; de Oliveira *et al.*, 1999*c*; de Oliveira *et al.*, 1999*a*), which investigate robust stability conditions using parameter dependent Lyapunov functions. A link with these results is provided by considering that matrix $A$ in system (1) is not precisely known with values that lie on a convex and bounded polyhedron $\mathcal{A}$. This polyhedron is described as the *unknown* convex combination of $N$ given extreme matrices $A_i \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, N$, through

$$\mathcal{A} := \left\{ A(\xi) : A(\xi) = \sum_{i=1}^{N} A_i \xi_i, \quad \xi \in \Xi \right\},$$

where

$$\Xi := \left\{ \xi = (\xi_i) : \sum_{i=1}^{N} \xi_i = 1, \xi_i \geq 0, i = 1, \ldots, N \right\}.$$

If all matrices in $\mathcal{A}$ are Hurwitz then system (1) is said to be robustly stable in $\mathcal{A}$. The following theorem can be derived from Theorem 3 as an extension.

*Theorem 4.* (Robust Stability). If at least one of the following statements is true:

i) $\exists P \in \mathbb{S}^n : P \succ 0, \quad A_i^T P + P A_i \prec 0, \forall i = 1, \ldots, N,$

ii) $\exists F, G \in \mathbb{R}^{n\times n}, P_i \in \mathbb{S}^n, i = 1, \ldots, N :$

$$P_i \succ 0, \quad \begin{bmatrix} A_i^T F^T + F A_i & A_i^T G^T - F + P_i \\ G A_i - F^T + P_i & -G - G^T \end{bmatrix} \prec 0,$$

for all $i = 1, \ldots, N$,

then the linear time-invariant system (1) is robustly stable in $\mathcal{A}$.

Theorem 4 illustrates how the degrees of freedom obtained with the introduction of the Lagrange multipliers can be explored in order to generate less conservative robust stability tests. Notice that although the items *ii*) and *iv*) of Theorem 3 are equivalent statements, their robust stability versions provided in Theorem 4 have different properties. The Lyapunov function used in the robust stability condition *i*) is quadratic (Barmish, 1985) while the one used in item *ii*) is parameter dependent (Feron *et al.*, 1996). Robust versions of all results presented in this paper can be derived using the same reasoning. The methodology described so far can be adapted to cope with stability of discrete-time linear time-invariant systems with no difficulties (see de Oliveira and Skelton).

## 3. HANDLING INPUT/OUTPUT SIGNALS

At this point, a natural question is if the method introduced in this paper can be used to handle systems with inputs and outputs. For instance, consider the linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \quad x(0) = 0, \\ z(t) &= Cx(t) + Dw(t). \end{aligned} \tag{8}$$

In the presence of inputs, there is no sense in talking about stability of system (8) without characterizing the input signal $w(t)$. Thus, assume that the signal $w(t) : [0, \infty) \to \mathbb{R}^m$ is a piecewise continuous function in $\mathcal{L}_2$, that is,

$$\|w\|_{\mathcal{L}_2} := \left( \int_0^\infty w(\tau)^T w(\tau)\, d\tau \right)^{1/2} < \infty.$$

The system (8) will be said to be $\mathcal{L}_2$ stable if the output signal $z(t) \in \mathbb{R}^p$ is also in $\mathcal{L}_2$ for all $w(t) \in \mathcal{L}_2$. This condition can be checked, for instance, by evaluating the $\mathcal{L}_2$ to $\mathcal{L}_2$ gain

$$\gamma_\infty := \sup_{w(t)\in\mathcal{L}_2} \frac{\|z\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}}.$$

For a linear and time-invariant stable system (8) it can be shown that

$$\|H_{wz}(s)\|_\infty := \sup_{\omega\in\mathbb{R}} \|H_{wz}(j\omega)\|_2 = \gamma_\infty,$$

where $H_{wz}(s)$ is the transfer function from the input $w(t)$ to the output $z(t)$, and $\|(\cdot)\|_2$ denotes the maximum singular value of matrix $(\cdot)$.

Now, define the same Lyapunov function $V(x(t)) > 0$, $\forall x(t) \neq 0$, considered in Section 2, and the modified Lyapunov stability conditions

$$\begin{aligned} &\dot{V}(x(t), \dot{x}(t)) < 0, \quad \gamma^2 w(t)^T w(t) \leq z(t)^T z(t), \\ &\qquad \forall (x(t), \dot{x}(t), w(t), z(t)) \text{ satisfying (8)}, \\ &\qquad\qquad (x(t), \dot{x}(t), w(t), z(t)) \neq 0, \end{aligned} \tag{9}$$

defined for a given $\gamma > 0$. These inequalities appear in the stability analysis of system (8) under the feedback

$$w(t) := \Delta(t) z(t), \quad \|D\|_2 < \gamma, \quad \forall \Delta(t) : \|\Delta(t)\|_2 \leq \gamma^{-1}.$$

Following the same steps as in (Boyd *et al.*, 1994), pp. 62–63, the *S*-procedure can be used to generate the equivalent condition [2, 3]

$$\begin{aligned} &\dot{V}(x(t), \dot{x}(t)) < \gamma^2 w(t)^T w(t) - z(t)^T z(t), \\ &\qquad \forall (x(t), \dot{x}(t), w(t), z(t)) \text{ satisfying (8)}, \\ &\qquad\qquad (x(t), \dot{x}(t), w(t), z(t)) \neq 0. \end{aligned} \tag{10}$$

Hence, when the above conditions are satisfied it is possible to conclude that

$$\begin{aligned} 0 < V(x(t)) &= \int_0^t \dot{V}(x(\tau), \dot{x}(\tau))\, d\tau \\ &< \int_0^t \gamma^2 w(\tau)^T w(\tau) - z(\tau)^T z(\tau)\, d\tau, \end{aligned}$$

which is valid for all $t > 0$. In particular, taking $t \to \infty$,

$$\|z\|_{\mathcal{L}_2}^2 < \gamma^2 \|w\|_{\mathcal{L}_2}^2,$$

which implies that $\gamma > \gamma_\infty$. In other words, feasibility of (10) yields an upperbound to $\|H_{wz}(s)\|_\infty$. For the linear system (8), it is known that

$$\gamma_\infty = \inf \gamma : (10).$$

Therefore, if (10) is feasible for some $0 < \gamma < \infty$ then it is possible to conclude that the system (8) is $\mathcal{L}_2$ stable. Moreover, the conditions (10) also guarantee that (8) is *internally* asymptotically stable. When the state space realization of system (8) is minimal, i.e., controllable and observable, both notions of stability coincide. If minimality does not hold, then system (8) might be $\mathcal{L}_2$ stable but not internally asymptotically stable [4], in which case the set (10) is empty.

A generalized version of (10) can be obtained by considering constraints on the input and on the output signals in the form

$$\begin{pmatrix} z(t)^T & w(t)^T \end{pmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} \geq 0,$$

---

[2] In this particular case, the *S*-procedure produces a necessary and sufficient equivalent test. This result can also be seen as a version of Finsler's Lemma where the constraint is a quadratic form (see Boyd *et al.*, pp. 23–24).

[3] As in (Boyd *et al.*, 1994), p. 63, the function $\dot{V}(x(t), \dot{x}(t))$ is homogeneous in $P$ so that the scalar introduced with the *S*-procedure can be set to 1 without loss of generality.

[4] Some uncontrollable or unobservable mode of (8) may not be asymptotically stable.

where $Q \in \mathbb{S}^p$, $R \in \mathbb{S}^m$, $S \in \mathbb{R}^{p\times m}$. After applying the $S$-procedure this constraint yields the inequality

$$\dot{V}(x(t),\dot{x}(t)) < -\begin{pmatrix} z(t)^T & w(t)^T \end{pmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix},$$
$$\forall (x(t),\dot{x}(t),w(t),z(t)) \text{ satisfying (8)},$$
$$(x(t),\dot{x}(t),w(t),z(t)) \neq 0, \quad (11)$$

In de Oliveira and Skelton, the result of the application of Finsler's Lemma on the above condition is given in the form of a complete theorem.

## 4. ANALYSIS OF SYSTEMS DESCRIBED BY TRANSFER FUNCTIONS

So far Finsler's Lemma has been used to generate stability conditions for systems given in state space form. In this section, it will be used on linear time-invariant systems described by transfer functions. For simplicity, consider a second-order SISO system represented by the transfer function

$$H_{wz}(s) = \frac{b(s)}{a(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}. \quad (12)$$

The results to be presented can be generalized to cope with transfer functions of higher order. Asymptotic stability of this transfer function can be analyzed by considering the second order differential equation

$$\ddot{x}(t) + a_1\dot{x}(t) + a_0 x(t) = 0,\ \dot{x}(0) = \dot{x}_0,\ x(0) = x_0. \quad (13)$$

The stability of this equation can be probed by the quadratic Lyapunov function

$$V(x(t)) := x(t)^T P x(t), \quad P := \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \succ 0,$$

and the associated stability conditions

$$\dot{V}(x(t),\dot{x}(t)) < 0, \forall (x(t),\dot{x}(t),\ddot{x}(t)) \neq 0 \text{ satisfying (13)}.$$

Arguments similar to the ones used in Section 2 can be used to show that the above conditions and $P \succ 0$ fully characterize the stability of (13) or, equivalently, of the transfer function (12).

*Theorem 5.* (Stability — Transfer Function). The following statements are equivalent:

i) The linear time-invariant system (12) is asymptotically stable.
ii) $\exists P \in \mathbb{S}^2 : P \succ 0, \quad A^T P + PA \prec 0$, where
$$A := \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix},$$
iii) $\exists P \in \mathbb{S}^2, \mu \in \mathbb{R} : P \succ 0, \quad \mathbf{U}(P) - \mu \mathbf{a}\mathbf{a}^T \prec 0$, where
$$\mathbf{U}(P) := \begin{bmatrix} 0 & p_1 & p_2 \\ p_1 & 2p_2 & p_3 \\ p_2 & p_3 & 0 \end{bmatrix}, \quad \mathbf{a} := \begin{bmatrix} a_0 \\ a_1 \\ 1 \end{bmatrix},$$
iv) $\exists P \in \mathbb{S}^2, \mathbf{f} \in \mathbb{R}^{3\times 1} : P \succ 0,\ \mathbf{U}(P) + \mathbf{f}\mathbf{a}^T + \mathbf{a}\mathbf{f}^T \prec 0$.

Item *ii*) of Theorem 5 recovers exactly the standard Lyapunov stability test that would have been obtained if item *ii*) of Theorem 3 had been applied to the *companion* state space realization

$$\begin{pmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}. \quad (14)$$

On the other hand, items *iii*) and *iv*) are *polynomial* stability conditions. Notice that they differ from the *state-space* conditions *iii*) and *iv*) given by Theorem 3 for (14).

The input/output results of Section 3 can also be generalized to cope with transfer functions. Consider again the simple second-order SISO system (12), and define the dynamic constraints

$$\ddot{x}(t) + a_1\dot{x}(t) + a_0 x(t) = w(t),\ (\dot{x}(0), x(0)) = (0,0),$$
$$b_2\ddot{x}(t) + b_1\dot{x}(t) + b_0 x(t) = z(t). \quad (15)$$

The analog of the integral quadratic performance conditions (11) can be shown to be given by

$$\dot{V}(x(t),\dot{x}(t),\ddot{x}(t)) < -\begin{pmatrix} z(t)^T & w(t)^T \end{pmatrix} \begin{bmatrix} q & s \\ s & r \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix},$$
$$\forall (x(t),\dot{x}(t),\ddot{x}(t),w(t),z(t)) \neq 0 \text{ satisfying (15)},$$

where $q,s,r \in \mathbb{R}$. The form of the dynamic constraint (15) deserves some comments. First, it is based on the *phase-variable* canonical realization (Skelton, 1988), where (12) is realized via

$$H_{wz}(s) = \frac{Z(s)}{W(s)},\ Z(s) = b(s)\xi(s),\ a(s)\xi(s) = W(s).$$

Second, in standard state space methods, the second equation (output equation) of (15) must have the term $\ddot{x}$ substituted from the first equation. This yields the standard phase-variable canonical form

$$\ddot{x}(t) + a_1\dot{x}(t) + a_0 x(t) = w(t), \quad (\dot{x}(0), x(0)) = (0,0),$$
$$c_1\dot{x}(t) + c_0 x(t) = z(t) - b_2 w(t)$$

where $c_i := (b_i - b_2 a_i)$, $i = 0,1$. Finsler's Lemma can handle both forms without further ado.

This approach can be generalized to cope with higher order transfer functions. Extensions to general MIMO systems with $m$ inputs and $p$ outputs are also straightforward by considering

$$H_{wz}(s) = Z(s)W(s)^{-1},$$
$$Z(s) = N(s)\xi(s), \quad D(s)\xi(s) = W(s). \quad (16)$$

This factorization can be obtained by computing $N(s)$ and $D(s)$ as right coprime polynomial factors of $H_{wz}(s)$. Another possible generalization of these results is for systems described by higher order *vector* differential equations as, for instance, *vector second-order systems* in the form

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = Bw(t).$$

A version of Theorems 5 would be able to provide stability conditions that enables one to take into account uncertainties on *all matrices*, including the mass matrix $M$.

The complete version of this paper (de Oliveira and Skelton, 2001) discuss other interesting stability analysis problems that we have to skip due to lack of space.

## 5. CONCLUSION

In this paper Lyapunov stability theory has been combined with Finsler's Lemma providing new stability tests for linear time-invariant systems. In a new procedure, the dynamic differential or difference equations that characterize the system are seen as constraints, which are naturally incorporated into the stability conditions using Finsler's Lemma. In contrast with standard state space methods, where stability analysis is carried in the space of the state vector, the stability tests are generated in the enlarged space containing both the state and its time derivative. This accounts for the flexibility of the method, that does not necessarily rely on state space representations. Stability conditions involving the coefficients of transfer functions representing linear systems are derived using this technique. Systems with inputs and outputs can be treated as well. Alternative new formulations of stability analysis tests with integral quadratic constraints, which contain the bounded-real lemma and the positive-real lemma as special cases, are provided for systems described by transfer functions or in state space. The philosophy behind the generation of these new stability tests can be summarized as follows:

(1) Identify the Lyapunov stability inequalities in the enlarged space.
(2) Identify the dynamic constraints in the enlarged space.
(3) Apply Finsler's Lemma to incorporate the dynamic constraints into the stability conditions.

The dynamic constraints are incorporated into the stability conditions via three main processes: a) evaluating the null space of the dynamic constraints, b) using a scalar Lagrange multiplier or c) using a matrix Lagrange multiplier. These multipliers bring extra degrees of freedom that can be explored to derive robust stability tests. Quadratic stability or parameter-dependent Lyapunov functions can be used to test robust stability.

## 6. REFERENCES


Barmish, B. R. (1985). Necessary and sufficient conditions for quadratic stabilizability of an uncertain system. *JOTA* **46**, 399–408.

Boyd, S. P., L. El Ghaoui, E. Feron and V. Balakrishnan (1994). *Linear Matrix Inequalities in System and Control Theory*. SIAM. Philadelphia, PA.

Cobb, D. (1984). Controllability, observability, and duality in descriptor systems. *IEEE Trans. on Automatic Control* **29**, 1076–1082.

de Oliveira, M. C. and R. E. Skelton (2001). Stability tests for constrained linear systems. In: *Perspectives in Robust Control* (S. O. Reza Moheimani, Ed.). pp. 241–257. Springer-Verlag.

de Oliveira, M. C., J. Bernussou and J. C. Geromel (1999*a*). A new discrete-time robust stability condition. *Sys. & Cont. Letters.* **37**(4), 261–265.

de Oliveira, M. C., J. C. Geromel and J. Bernussou (1999*b*). An LMI optimization approach to multiobjective controller design for discrete-time systems. In: *Proc. of the 38th IEEE Conf. on Dec. and Cont.*. Phoenix, AZ. pp. 3611–3616.

de Oliveira, M. C., J. C. Geromel and J. Bernussou (n.d.). Extended $H_2$ and $H_\infty$ norm characterizations and controller parametrizations for discrete-time systems. Submitted to *Int. J. of Control*.

de Oliveira, M. C., J. C. Geromel and L. Hsu (1999*c*). LMI characterization of structural and robust stability: the discrete-time case. *Lin. Alg. and Its Appl.* **296**(1-3), 27–38.

Feron, E., P. Apkarian and P. Gahinet (1996). Analysis and synthesis of robust control systems via parameter-dependent Lyapunov functions. *IEEE Trans. on Automatic Cont.* **41**(7), 1041–1046.

Finsler, P. (1937). Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratischer Formem. *Comment. Math. Helvetici* **9**, 188–192.

Geromel, J. C., M. C. de Oliveira and J. Bernussou (1999). Robust filtering of discrete-time linear systems with parameter dependent Lyapunov functions. In: *Proc. of the 38th IEEE Conf. on Dec. and Cont.*. Phoenix, AZ. pp. 570–575.

Geromel, J. C., M. C. de Oliveira and L. Hsu (1998). LMI characterization of structural and robust stability. *Lin. Alg. and Its Appl.* **285**(1-3), 69–80.

Grigoriadis, K. M., G. Zhu and R. E. Skelton (1996). Optimal redesign of linear systems. *J. Dyn. Sys. Meas. and Cont.: Trans. ASME* **118**(3), 598–605.

Hamburger, C. (1999). Two extensions to Finsler's recurring theorem. *Applied Mathematics & Optimization* **40**, 183–190.

Scherer, C. W. (2000). Robust mixed control and linear parameter-varying control with full block scalings. In: *Adv. in Lin. Mat. Ineq. Meth. in Contr.* (L. El Gahoui and S.-L. Niculesco, Eds.). pp. 187–207. SIAM. Philadelphia, PA.

Siljak, D. D. (1990). *Decentralized Control of Complex Systems*. Academic Press. London, UK.

Skelton, R. E. (1988). *Dynamics Systems Control: linear systems analysis and synthesis*. John Wiley & Sons, Inc. New York, NY.

Skelton, R. E., T. Iwasaki and K. Grigoriadis (1997). *A Unified Algebraic Approach to Control Design*. Taylor & Francis. London, UK.

Syrmos, V. L., C. T. Abdallah, P. Dorato and K. Grigoriadis (1997). Static output feedback — a survey. *Automatica*.

Uhlig, F. (1979). A recurring theorem about pairs of quadratic forms and extensions: a survey. *Lin. Alg. and Its Appl.* **25**, 219–237.