WHEN DOES THE H^∞ FIXED-LAG SMOOTHING PERFORMANCE SATURATES?[*]

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Abstract: A notable difference between the H^2 and H^∞ smoothing is that the achievable performance in the latter problem might "saturate" as the function of the smoothing lag in the sense that there might exist a finite smoothing lag for which the achievable performance level is the same as for the infinite smoothing lag. In this paper necessary and sufficient conditions under which such a saturation takes place are derived. In particular, it is shown that the H^∞ performance saturates only if the H^∞ norm of the optimal error system is achieved at the infinite frequency, i.e., if the worst case disturbance is "infinitely fast" and thus in a sense unpredictable.   *Copyright © 2002 IFAC*

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1   INTRODUCTION

Let $G_1(s)$ and $G_2(s)$ be proper transfer matrices with equal input dimensions. The (continuous-time) H^∞ fixed-lag smoothing problem is the problem of the design of a stable and proper transfer matrix $K(s)$ guaranteeing

$$\|e^{-sh}G_1(s) - K(s)G_2(s)\|_\infty < \gamma \qquad (1)$$

for given positive scalars γ and h. The latter is usually referred to as *the smoothing lag*. The fixed-lag smoothing formulation of a general estimation problem reflects the situation where some delay or latency between the measurement and the generation of estimation can be tolerated (e.g., in numerous signal processing applications) (Anderson, 1999).

One of the most important issues in the fixed-lag smoothing is to understand and quantify *how the smoothing lag affects the achievable performance.* In the H^2 setting this issue is now well-understood, see (Anderson and Moore, 1979) and the references therein. On the other hand, early H^∞ solutions (Grimble, 1991; Theodor and Shaked, 1994; Colaneri *et al.*, 1998) fall short of providing an insight into the effect of h on the achievable γ. Recently, an alternative solution was proposed in (Mirkin, 2001), where it was

shown that both the smoother structure and the performance improvement due to h are similar to those in the H^2 case. Yet it was also shown that there exists a remarkable difference between the H^2 and H^∞ solutions: whereas in the former case the performance improves monotonically with h, in the latter case the achievable γ might "saturate" after some finite smoothing lag and any further increase of h has no effect on the achievable H^∞ performance[1]. To illustrate the point, consider the following simple example from (Mirkin, 2001):

Example 1. Consider the smoothing problem for

$$G_1 = \begin{bmatrix} \frac{aq}{s-a} & 0 \end{bmatrix} \quad \text{and} \quad G_2 = \begin{bmatrix} \frac{aq}{s-a} & 1 \end{bmatrix}$$

and $a > 0$. In this case the optimal achievable H^∞ performance, $\gamma_o(h)$, is

$$\gamma_o(h) = \begin{cases} \frac{(1-\eta)e^{ah/\eta}+(1+\eta)e^{-ah/\eta}}{2} & \text{if } h \le h_\infty, \\ \gamma_\infty = \sqrt{1-\eta^2} & \text{otherwise,} \end{cases}$$

where $h_\infty \doteq \frac{1}{2a}\eta\ln\frac{1+\eta}{1-\eta}$ and $\eta \doteq 1/\sqrt{q^2+1} < 1$. It is seen that $\gamma_o(h)$ saturates after $h = h_\infty$ and any further increase of the smoothing lag does not affect $\gamma_o(h)$. Moreover, one can show that $\gamma_o(h)$ is differentiable at every h, including h_∞.

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[1]Similar phenomenon in the context of the preview tracking was pointed out by Kojima and Ishijima (1999), though by numerical simulations only.

The saturation property, however, is not intrinsic as seen by another example:

Example 2. Let us consider now the smoothing problem for

$$G_1 = \frac{2a}{s+a} \quad \text{and} \quad G_2 = \frac{s-a}{s+a}$$

and $a > 0$. Using the standard interpolation arguments (Francis, 1987) one can show that

$$\gamma_o(h) = e^{-ah},$$

which is achievable with the optimal smoother $K = \frac{s+a}{s-a}(\frac{2a}{s+a}e^{-sh} - e^{-ah}) \in H^\infty$. Thus, in this case $\gamma_o(h) \to \gamma_\infty = 0$ only as $h \to \infty$.

The purpose of this paper is to clarify the saturation phenomenon described above. In particular, the case of strictly proper $G_1(s)$ is studied and necessary and sufficient conditions under which the optimal H^∞ smoothing performance saturates as a function of the smoothing lag are derived. To this end, the solvability conditions of (Mirkin, 2001) are revised. It is shown that if $G_1(s) \not\equiv 0$, then $\gamma_o(h)$ does not saturate iff $\lim_{h\to\infty} \gamma_o(h) = 0$ (this is what happens in Example 2). This condition can be interpreted as the impossibility to "predict" the worst-case disturbance, which in this case becomes arbitrarily fast, using a finite preview.

2   SOLVABILITY CONDITIONS

In this section the solvability conditions derived in (Mirkin, 2001) for the H^∞ fixed-lag smoothing problem are presented in a slightly more general formulation. We assume that G_1 and G_2 are as follows:

$$\left[\begin{array}{c} G_1(s) \\ G_2(s) \end{array} \right] = \left[\begin{array}{c|c} A & B \\ \hline C_1 & 0 \\ C_2 & D_2 \end{array} \right] \qquad (2)$$

and suppose that

(A1): (C_2, A) is detectable;

(A2): $\left[\begin{array}{cc} A - j\omega I & B \\ C_2 & D_2 \end{array} \right]$ has full row rank $\forall \omega \in \mathbb{R}$;

(A3): $D_2 D_2' > 0$.

As argued in (Mirkin, 2001), the solvability of (1) can be accounted for by the *inverse* of the H^2 and H^∞ filtering Riccati solutions. The main reason is that the solution to the Riccati equation associated with the smoothing problem is typically discontinuous as a function of both γ and h, while its kernel is independent of both of these variables. Moreover, the non-invertible part can be excluded from the analysis. To this end, let

$$\left[\begin{array}{cc} \hat{A} & \hat{B} \end{array} \right] \doteq \left[\begin{array}{cc} A & B \end{array} \right] - BD_2'(D_2 D_2')^{-1} \left[\begin{array}{cc} C_2 & D_2 \end{array} \right]$$

and

$$\left[\begin{array}{cc} \hat{C}_2 & \hat{D}_2 \end{array} \right] \doteq (D_2 D_2')^{-1/2} \left[\begin{array}{cc} C_2 & D_2 \end{array} \right].$$

Then the stabilizing solutions to both H^2 and H^∞ filtering Riccati equations are invertible iff $(\hat{A}, \hat{B})$ has no stable uncontrollable modes (which, in turn, is equivalent to the absence of stable invariant zeros of G_2 in realization (2)). If this condition does not hold, then there exists a unitary matrix $U = \left[\begin{array}{cc} U_1 & U_2 \end{array} \right]$ such that

$$U'\hat{A}U = \left[\begin{array}{cc} \hat{A}_c & ? \\ 0 & \hat{A}_s \end{array} \right], \qquad U'\hat{B} = \left[\begin{array}{c} \hat{B}_c \\ 0 \end{array} \right],$$

$\hat{A}_s$ is Hurwitz, and the pair $(\bar{A}, \bar{B})$ has no uncontrollable modes in the left half-plane (here "?" stands for an irrelevant block). Then the solvability of smoothing problems for (2) and for

$$\left[\begin{array}{c} \tilde{G}_1(s) \\ \tilde{G}_2(s) \end{array} \right] = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C}_1 & 0 \\ \tilde{C}_2 & \tilde{D}_2 \end{array} \right] \doteq \left[\begin{array}{c|c} \hat{A}_c & \hat{B}_c \\ \hline C_1 U_1 & 0 \\ \hat{C}_2 U_1 & \hat{D}_2 \end{array} \right] \quad (2')$$

are equivalent. The transformation of the problem data from (2) to (2′) simplifies the further analysis considerably. Besides guaranteeing the solvability of the inverse Riccati equations, it also normalizes $\tilde{D}_2$ and makes $\tilde{B}$ and $\tilde{D}_2$ orthogonal.

Now, define two inverse filtering Riccati equations associated with (2′): the H^2 (Kalman filtering) equation

$$-\tilde{Y}_\kappa \tilde{A} - \tilde{A}' \tilde{Y}_\kappa - \tilde{Y}_\kappa \tilde{B} \tilde{B}' \tilde{Y}_\kappa + \tilde{C}_2' \tilde{C}_2 = 0 \qquad (3)$$

and the H^∞ equation:

$$-\tilde{Y}_\gamma \tilde{A} - \tilde{A}' \tilde{Y}_\gamma - \tilde{Y}_\gamma \tilde{B} \tilde{B}' \tilde{Y}_\gamma + \tilde{C}_2' \tilde{C}_2 = \frac{1}{\gamma^2} \tilde{C}_1' \tilde{C}_1. \quad (4)$$

The solutions to (3) and (4) are said to be stabilizing if the matrices $\tilde{A}_\kappa \doteq -(\tilde{A} + \tilde{B}\tilde{B}'\tilde{Y}_\kappa)$ and $\tilde{A}_\gamma \doteq -(\tilde{A} + \tilde{B}\tilde{B}'\tilde{Y}_\gamma)$, respectively, are Hurwitz. Define also the quantity

$$\gamma_\infty \doteq \|\tilde{C}_1(sI - \tilde{A}_\kappa)^{-1}\tilde{B}\|_\infty,$$

which is the achievable H^∞ performance for the case of $h \to \infty$. Alternatively, γ_∞ is the largest γ for which the Hamiltonian matrix associated with (4) has imaginary-axis eigenvalues. Finally, we need also the solution $W_c \geq 0$ to the Lyapunov equation

$$\tilde{A}_\kappa W_c + W_c \tilde{A}_\kappa' + \tilde{B}\tilde{B}' = 0. \qquad (5)$$

We are now in the position to formulate the following result, which is essentially from (Mirkin, 2001, Theorem 2):

Lemma 1. *Let $\gamma > \gamma_\infty$. Then the Riccati equations* (3) *and* (4) *have stabilizing solutions $\tilde{Y}_\kappa > 0$ and $\tilde{Y}_\gamma \leq \tilde{Y}_\kappa$, the matrices*

$$\tilde{Q}_\kappa \doteq \tilde{Y}_\kappa^{-1} - W_c \geq 0$$

and

$$\tilde{Q}_\gamma \doteq \left(I - (\tilde{Y}_\kappa - \tilde{Y}_\gamma)W_c\right)^{-1}(\tilde{Y}_\kappa - \tilde{Y}_\gamma) \geq 0$$

exist, and for a given h the H^∞ fixed-lag smoothing problem is solvable iff

$$\|\tilde{C}_\gamma e^{\tilde{A}_\kappa h}\tilde{B}_\kappa\| < 1 \qquad (6)$$

for any matrices $\tilde{B}_\kappa$ and $\tilde{C}_\gamma$ satisfying $\tilde{B}_\kappa\tilde{B}_\kappa' = \tilde{Q}_\kappa$ and $\tilde{C}_\gamma'\tilde{C}_\gamma = \tilde{Q}_\gamma$, respectively.

Remark 2.1. One can easily show that $\|\tilde{C}_\gamma\tilde{B}_\kappa\| < 1$ iff $\tilde{Y}_\gamma > 0$, so the latter is the necessary and sufficient condition for the solvability of the H^∞ filtering ($h = 0$) problem.

3 LIMITING PERFORMANCE

Denote by $\gamma_o(h)$ the maximal γ for which condition (6) fails for a given h. In other words, $\gamma_o(h)$ is the optimal achievable H^∞ smoothing performance for a given smoothing lag. Obviously, $\gamma_o(h)$ is monotonically non-increasing and also $\lim_{h\to\infty}\gamma_o(h) = \gamma_\infty$. Our purpose is to characterize all cases when $\gamma_o(h)$ reaches γ_∞ for a *finite* smoothing lag h.

Theorem 1. *Let $\gamma_o(0) > \gamma_\infty$ (i.e., smoothing outperforms filtering). Then*

  i) *there exists a finite smoothing lag h_∞ such that $\gamma_o(h) = \gamma_\infty$ for all $h > h_\infty$ iff $\gamma_\infty \neq 0$; in this case $\tilde{C}_\infty \doteq \lim_{\gamma\to\gamma_\infty}\tilde{C}_\gamma$ exists and h_∞ is the maximal solution to $\|\tilde{C}_\infty e^{\tilde{A}_\kappa h}\tilde{B}_\kappa\| = 1$;*

  ii) *if $\gamma_\infty = 0$, then $\gamma_o(h) > 0$ for every finite h unless $\tilde{C}_1 = 0$.*

The results of Theorem 1 have an interesting interpretation. Indeed, any H^∞ estimation problem can be roughly thought of as a "prediction" of the worst-case disturbance. At the same time, the case $\gamma_\infty = 0$ is actually the only possibility when the optimal H^∞ norm is achieved at the infinite frequency, i.e., the worst case disturbance in this case might be arbitrarily fast. Thus, any finite preview (smoothing lag) does not suffice to "predict" infinitely fast worst-case disturbance. On the other hand, nonzero γ_∞ implies that the worst-case disturbance is band limited and therefore can be "predicted" with a finite preview.

The rest of this section is devoted to the proof of Theorem 1.

3.1 Preliminary: Riccati equation for $\tilde{Q}_\gamma$

Since the only term in (6) which depends on γ is $\tilde{C}_\gamma$, we start the proof with studying $\tilde{Q}_\gamma = \tilde{C}_\gamma'\tilde{C}_\gamma$. To this end, define the Hamiltonian matrix

$$\bar{H}_\gamma \doteq \begin{bmatrix} \tilde{A}_\kappa' + \frac{1}{\gamma^2}\tilde{C}_1'\tilde{C}_1 W_c & -\frac{1}{\gamma^2}\tilde{C}_1'\tilde{C}_1 \\ \frac{1}{\gamma^2}W_c\tilde{C}_1'\tilde{C}_1 W_c & -\tilde{A}_\kappa - \frac{1}{\gamma^2}W_c\tilde{C}_1'\tilde{C}_1 \end{bmatrix}.$$

We have:

Proposition 1. *The matrix $\tilde{Q}_\gamma$ satisfies the following Riccati equation:*

$$\begin{bmatrix} I & \tilde{Q}_\gamma \end{bmatrix}\bar{H}_\gamma = \bar{A}_\gamma\begin{bmatrix} I & \tilde{Q}_\gamma \end{bmatrix} \qquad (7)$$

for $\bar{A}_\gamma = (I - (\tilde{Y}_\kappa - \tilde{Y}_\gamma)W_c)^{-1}\tilde{A}_\gamma(I - (\tilde{Y}_\kappa - \tilde{Y}_\gamma)W_c)$. Moreover, if $\gamma_\infty \neq 0$, then $\lim_{h\to h_\infty}\tilde{Q}_\gamma$ exists and is positive semi-definite.

Proof. It is a standard result from the Riccati theory (Lancaster and Rodman, 1995) that equation (4) can equivalently be written as

$$\begin{bmatrix} I & \tilde{Y}_\gamma \end{bmatrix}\tilde{H}_\gamma = \tilde{A}_\gamma\begin{bmatrix} I & \tilde{Y}_\gamma \end{bmatrix},$$

where

$$\tilde{H}_\gamma \doteq \begin{bmatrix} -\tilde{A}' & \frac{1}{\gamma^2}\tilde{C}_1'\tilde{C}_1 - \tilde{C}_2'\tilde{C}_2 \\ -\tilde{B}\tilde{B}' & \tilde{A} \end{bmatrix}$$

is the Hamiltonian matrix. Introduce the matrix

$$T \doteq \begin{bmatrix} I & \tilde{Y}_\kappa \\ W_c & W_c\tilde{Y}_\kappa - I \end{bmatrix} = \begin{bmatrix} I & 0 \\ W_c & I \end{bmatrix}\begin{bmatrix} I & \tilde{Y}_\kappa \\ 0 & -I \end{bmatrix}.$$

Since

$$\begin{bmatrix} I & \tilde{Y}_\gamma \end{bmatrix}T^{-1} = \begin{bmatrix} I - (\tilde{Y}_\kappa - \tilde{Y}_\gamma)W_c & \tilde{Y}_\kappa - \tilde{Y}_\gamma \end{bmatrix},$$

$\tilde{Q}_\gamma$ (when exists) is given by $\tilde{Q}_\gamma = M_1^{-1}M_2$ for any M_1 and M_2 satisfying

$$\begin{bmatrix} M_1 & M_2 \end{bmatrix}T\tilde{H}_\gamma T^{-1} = \bar{A}_\gamma\begin{bmatrix} M_1 & M_2 \end{bmatrix},$$

where $\bar{A}_\gamma$ is Hurwitz. Now, detectability of the pair $(\tilde{A}_\kappa' + \frac{1}{\gamma^2}\tilde{C}_1'\tilde{C}_1 W_c, \frac{1}{\gamma}\tilde{C}_1 W_c)$ implies that M_1 is nonsingular for all $\gamma > \gamma_\infty$. Hence, the first claim follows by noticing that $T\tilde{H}_\gamma T^{-1} = \bar{H}_\gamma$.

Finally, the second claim follows by applying the arguments of Scherer (1990) to eq. (7). $\square$

The second claim of Proposition 1 proves actually the existence of the matrix $\tilde{C}_\infty$ in Theorem 1.

3.2 $\gamma_\infty \neq 0$

Assume first that $\gamma_\infty \neq 0$. Our goal in this subsection is to prove that in this case h_∞ is finite. To this end, the following technical result is required:

Proposition 2. $\|\tilde{C}_\gamma e^{\tilde{A}_\kappa h}\tilde{B}_\kappa\|$ *is monotonically non-increasing function of γ.*

Proof. To prove the Proposition it is sufficient to show that $\tilde{Q}_\gamma$ is monotonically non-increasing function of γ in the sense that $\tilde{Q}_{\gamma_1} \geq \tilde{Q}_{\gamma_2}$ whenever $\gamma_1 < \gamma_2$. To this end, let $Q_\alpha \doteq \frac{d}{d\alpha}\tilde{Q}_\gamma$, where $\alpha \doteq \frac{1}{\gamma^2}$. Differentiating the Riccati equation associated with (7) one gets:

$$\bar{A}_\gamma Q_\alpha + Q_\alpha\bar{A}_\gamma'$$
$$+ (I + \tilde{Q}_\gamma W_c)C_1'C_1(I + W_c\tilde{Q}_\gamma) = 0,$$

which, together with the stability of $\bar{A}_\gamma$, yields that $Q_\alpha \geq 0$. Thus, $\tilde{Q}_\gamma$ is non-decreasing function of α and, hence, non-increasing function of γ. $\square$

Proposition 2 is intuitively clear. In fact, it establishes that the smaller is γ, the larger smoothing lag might be required to satisfy (6). Taking into account the existence of $\tilde{C}_\infty$, one can see that $\|\tilde{C}_\infty e^{\tilde{A}_\kappa h}\tilde{B}_\kappa\| \geq \|\tilde{C}_\gamma e^{\tilde{A}_\kappa h}\tilde{B}_\kappa\|$ for all $h \geq 0$ and $\gamma \geq \gamma_\infty$. On the other hand, since $\|\tilde{C}_\infty \tilde{B}_\kappa\| > 1$ (by the assumption of Theorem 1) and $\lim_{h\to\infty}\|\tilde{C}_\infty e^{\tilde{A}_\kappa h}\tilde{B}_\kappa\| = 0$, the continuity of $\|\tilde{C}_\infty e^{\tilde{A}_\kappa h}\tilde{B}_\kappa\|$ as a function of h implies that there must exist a finite h for which $\|\tilde{C}_\infty e^{\tilde{A}_\kappa h}\tilde{B}_\kappa\| < 1$. This leads to the "if" claim of statement *i)*.

3.3 $\gamma_\infty = 0$

In this case all terms of the Hamiltonian matrix $\bar{H}_\gamma$ become unbounded. Hence, the boundedness of $\lim_{\gamma\to\infty}\tilde{C}_\gamma$ can no longer be guaranteed (in fact, it is generically unbounded). Fortunately, the analysis can be simplified by noticing that $\gamma_\infty = 0 \iff G_\gamma(s) \doteq \tilde{C}_1(sI - \tilde{A}_\kappa)^{-1}\tilde{B} \equiv 0$. Since W_c is the controllability Gramian of G_γ, the latter equality is equivalent to $\tilde{C}_1 W_c = 0$. Hence, (7) can be rewritten as

$$\begin{bmatrix} I & \tilde{Q}_\gamma \end{bmatrix}\begin{bmatrix} \tilde{A}'_\kappa & -\frac{1}{\gamma^2}\tilde{C}'_1\tilde{C}_1 \\ 0 & -\tilde{A}_\kappa \end{bmatrix} = \bar{A}_\gamma\begin{bmatrix} I & \tilde{Q}_\gamma \end{bmatrix},$$

which, in turn, yields that $\tilde{Q}_\gamma = \frac{1}{\gamma^2}\tilde{Q}_1$, where $\tilde{Q}_1 \geq 0$ satisfies the Lyapunov equation

$$\tilde{Q}_1\tilde{A}_\kappa + \tilde{A}'_\kappa\tilde{Q}_1 + \tilde{C}'_1\tilde{C}_1 = 0$$

(it is seen now that unless $\tilde{C}_1 = 0$, $\tilde{Q}_\gamma$ is indeed unbounded).

Thus, if $\gamma_\infty = 0$, then

$$\|\tilde{C}_\gamma e^{\tilde{A}_\kappa h}\tilde{B}_\kappa\| = \frac{1}{\gamma}\|\tilde{C}_0 e^{\tilde{A}_\kappa h}\tilde{B}_\kappa\|,$$

where $\tilde{C}_0$ is any matrix satisfying $\tilde{C}'_0\tilde{C}_0 = \tilde{Q}_1$, and there exists a finite h for which $\gamma_\infty = 0$ is achievable iff $\tilde{C}_0 e^{\tilde{A}_\kappa h}\tilde{B}_\kappa \equiv 0$. The latter implies that $\gamma_o(0) = 0$ as well, which contradicts the assumption of Theorem 1 and therefore proves the "only if" claim of statement *i)*.

To analyze when $\tilde{C}_0 e^{\tilde{A}_\kappa h}\tilde{B}_\kappa \equiv 0$, let us introduce the system

$$G_\alpha(s) \doteq \left[\begin{array}{c|c} \tilde{A}'_\kappa & \tilde{C}'_1 \\ \hline \tilde{C}_2\tilde{Y}_\kappa^{-1} & 0 \end{array}\right].$$

It is seen that $\tilde{Q}_1$ is its controllability Gramian. Below, we show that $\tilde{Q}_\kappa$ is the observability Gramian of G_α. To see this, note that (3) can be rewritten as

$$\tilde{Y}_\kappa\tilde{A}_\kappa + \tilde{A}'_\kappa\tilde{Y}_\kappa + \tilde{Y}_\kappa\tilde{B}\tilde{B}'\tilde{Y}_\kappa + \tilde{C}'_2\tilde{C}_2 = 0$$

or, since $\tilde{Y}_\kappa$ is invertible, as

$$\tilde{A}_\kappa\tilde{Y}_\kappa^{-1} + \tilde{Y}_\kappa^{-1}\tilde{A}'_\kappa + \tilde{B}\tilde{B}' + \tilde{Y}_\kappa^{-1}\tilde{C}'_2\tilde{C}_2\tilde{Y}_\kappa^{-1} = 0.$$

Extracting (5) from this equation one gets:

$$\tilde{A}_\kappa\tilde{Q}_\kappa + \tilde{Q}_\kappa\tilde{A}'_\kappa + \tilde{Y}_\kappa^{-1}\tilde{C}'_2\tilde{C}_2\tilde{Y}_\kappa^{-1} = 0,$$

from which it follows that $\tilde{Q}_\kappa$ is the controllability Gramian of G_α. The latter, in turn, implies that $\tilde{C}_0 e^{\tilde{A}_\kappa h}\tilde{B}_\kappa \equiv 0$ iff $G_\alpha(s) \equiv 0$ which, in turn, is equivalent to $\tilde{C}_1\tilde{Q}_\kappa = \tilde{C}_1(\tilde{Y}_\kappa^{-1} - W_c) = 0$. Yet $\tilde{C}_1 W_c = 0$ since $\gamma_\infty = 0$. Therefore,

$$\tilde{C}_0 e^{\tilde{A}_\kappa h}\tilde{B}_\kappa \equiv 0 \iff \tilde{C}_1 = 0.$$

This completes the proof of statement *ii)* of Thm. 1.

4 CONCLUDING REMARKS

In this paper the saturation of the achievable H^∞ smoothing performance as a function of the smoothing lag has been studied. By "saturation" we mean the existence of a finite smoothing lag with which γ_∞ can be achieved (here γ_∞ stands for the H^∞ performance achievable with the *infinite* smoothing lag) and no further increase of the smoothing lag affects the H^∞ performance. It has been shown that the saturation phenomenon takes place iff $\gamma_\infty \neq 0$. This condition can be interpreted as a "predictability" of the (band limited) worst-case disturbance using a finite preview.

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