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MIXED SENSITIVITY DESIGN: AN AEROSPACE CASE STUDY

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Abstract. The mixed sensitivity design method for multivariable linear control systems proposed in the companion paper (Kwakernaak, 2001) is applied to the design of a stability augmentation system for the longitudinal motion of a fighter airplane previously studied by Safonov and Chiang (1988). Both H_∞ and H_2 mixed sensitivity design are considered. They lead to quite similar results. *Copyright © 2002 IFAC*

Keywords. Mixed sensitivity design, stability augmentation system, H_∞ optimization, H_2 optimization.

1. INTRODUCTION

In this paper the H_∞ and H_2 mixed sensitivity design methods of the companion paper (Kwakernaak, 2001) are applied to the design of a system for the control of the longitudinal dynamics of a fighter aircraft (Chiang and Safonov, 1988, Safonov and Chiang, 1988).

2. DESCRIPTION

Safonov and Chiang (Chiang and Safonov, 1988, Safonov and Chiang, 1988). discuss the design of a stability augmentation system for the longitudinal motion of a fighter aircraft. Trimmed at 25000 ft and 0.9 Mach the longitudinal dynamics are unstable with two right-half plane phugoid modes. In state space form the dynamics may be modeled as

$$\dot{x} = Ax + Bu, \quad y = Cx$$

where

$$A = \begin{bmatrix} -0.0226 & -36.6170 & -18.8970 & -32.0900 & 3.2509 & -0.7626 \\ 0.0001 & -1.8997 & 0.9831 & -0.0007 & -0.1708 & -0.00050 \\ 0.0123 & 11.7200 & -2.6316 & 0.0009 & -31.6040 & 22.3960 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -30 & 0 \\ 0 & 0 & 0 & 0 & 0 & -30 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 30 & 0 \\ 0 & 30 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The two input variables are the input of the elevon and canard actuators, respectively. The two output variables successively are the angle of attack and the attitude angle. The state variables are the forward speed, angle of attack, pitch rate, attitude angle, actuator no. 1, and actuator no. 2.

The design specifications are not very comprehensive and consist of

(1) −40 dB/dec roll-off of the singular value of the complementary sensitivity function T and at least −20 dB at 100 rad/s.

(2) Minimization of the sensitivity function S as much as possible.

Exploratory analysis

The open-loop plant transfer matrix is 2×2, with poles at $0.6899 \pm j\,0.2484$, −5.6757, −0.2580, −30.00, and −30.00. The first pole pair represents the unstable phugoid mode. The two poles at −30.00 derive from the actuator dynamics. The plant has a single zero at −0.0209.

The magnitude of the unstable pole pair is 0.7332 rad/s. This implies a minimal bandwidth of 0.7332 rad/s. Inspection of the Bode diagram of the inverse transfer matrix P^{-1} indicates that starting at frequencies corresponding to the minimum bandwidth the entries of P^{-1} start to increase significantly. This indicates that bandwidths greater than 1 rad/s may only be achieved at the expense of large control inputs.

The open-loop plant zero at −0.0209 does not constitute a special design handicap.

3. ROBUST CONTROL TOOLBOX H_∞ DESIGN

The H_∞ design approach presented by Chiang and Safonov (1988) consists of maximizing the value of γ such that

$$\left\| \begin{bmatrix} \gamma W_1 S \\ W_2 T \end{bmatrix} \right\|_\infty \leq 1$$

The weighting matrices W_1 and W_2 are chosen as

$$W_1(s)=\begin{bmatrix}\frac{s+100}{100s+1} & 0\\ 0 & \frac{s+100}{100s+1}\end{bmatrix},$$

$$W_3(s)=\begin{bmatrix}\frac{s^2}{1000} & 0\\ 0 & \frac{s^2(0.0005s+1)}{1000}\end{bmatrix}$$

Computation shows that the largest possible value of γ is about 17. Given this value of γ, Chiang and Safonov's approach amounts to minimizing the mixed sensitivity criterion

$$\left\|\begin{bmatrix}\gamma W_1 S\\ W_2 PU\end{bmatrix}\right\|_\infty$$

$P(s)=C(sI-A)^{-1}B$ is the plant transfer matrix, and S and U are the sensitivity function and the input sensitivity function, respectively.

Comparison with the standard mixed sensitivity criterion shows that the shaping matrix V is the identity matrix. As a result, the polynomial matrix M equals the denominator matrix D of the plant (see the companion paper for the notation). It follows from the properties of the mixed sensitivity problem that the stable open-loop poles and the left-half plane mirror images of the unstable open-loop poles are all closed-loop poles.

The compensator has order 8. Six of the closed-loop poles are –5.676, -0.2580, –30.00, –30.00, and $-0.6899 \pm j\ 0.2484$. The first four correspond to the stable open-loop poles and the latter two are the mirror images of the unstable open-loop poles, as predicted.

The remaining closed-loop poles are –2000, –561.7, –103.4, $-22.76 \pm j\ 18.39$, $-23.94 \pm j\ 20.53$, and –0.0209.

Further computation reveals that the poles and zeros of the compensator are

Poles: –2000, –559.1, $-75.67 \pm j15.47$, –50.56, –0.0209, –0.01, –0.01

Zeros: –30.00, –30.00, –5.676, -0.2580, $-0.6016 \pm j\ 0.3199$.

The compensator pole at –0.0209 corresponds to the open-loop plant zero. This pole-zero pair cancels between the compensator and the plant, which is why the zero also appears as a closed-loop pole.

The compensator zeros at –5.676 and –0.2580, –30 and –30 correspond to open-loop poles. These pole-zero pairs also cancel between the compensator and the plant and thus appear as closed-poles.

The singular value plots of the sensitivity functions S and T are displayed in Fig. 4. Fig. 5 shows the step responses corresponding to the entries of the matrix T.

Assessment

Fig. 4 shows that the singular value of the complementary sensitivity function satisfies the design specifications. The singular values of the sensitivity functions S and T peak at 2.6 dB and 1.1 dB, respectively.

As seen in Fig. 5, the step response of T is nicely decoupled. The closed-loop poles at $-22.76 \pm j\ 18.39$ and $-23.94 \pm j\ 20.53$ dominate the step response. The step response also exhibits a trailing behavior (best visible in the (2, 2) entry) that corresponds to the closed-loop pole pair at $-0.6899 \pm j\ 0.2484$.

The open-loop pole pair at $0.6899 \pm j\ 0.2484$ and closed-loop pair at $-0.6899 \pm j\ 0.2484$ result in all-pass factor in the closed-loop sensitivity functions. This factor limits the effective bandwidth although it hardly shows in the magnitude plots of the sensitivity functions.

4. H_∞ MIXED SENSITIVITY DESIGN

The results discussed in the preceding section show that if the shaping filter V is not introduced in the mixed sensitivity criterion then the resulting design has a serious handicap. Stable open-loop poles are canceled against compensator zeros, and unstable open-loop poles reappear at their mirror images as closed-loop poles. If the latter closed-loop poles are located inside a semi-circle in the left-half plane with center at the origin and radius equals to the desired bandwidth then they effectively reduce the bandwidth.

The mixed sensitivity approach described in the companion paper allows removing this handicap by partial pole assignment. The offending poles are simply moved to locations outside the semi-circle. We use the disturbance modeling technique described in the companion paper to do this.

Choice of V

Following the Robust Control Toolbox we enforce integrating action by choosing the weighting function W_1 as

$$W_1(s)=\begin{bmatrix}\frac{1+\tau_1 s}{s} & 0\\ 0 & \frac{1+\tau_2 s}{s}\end{bmatrix} \tag{1}$$

Note that some "fudge factors" have been removed. The time constants τ_1 and τ_2 control the width of the band over which the low-frequency disturbances are especially heavily penalized.

We resort to block diagram substitution to place the non-stabilizable block W_1 inside the feedback loop as shown in Fig. 3 of the companion paper. Next, we re-model the disturbances *v* by distributing them over different sources v_1, v_2 and *w* as in Fig. 1.

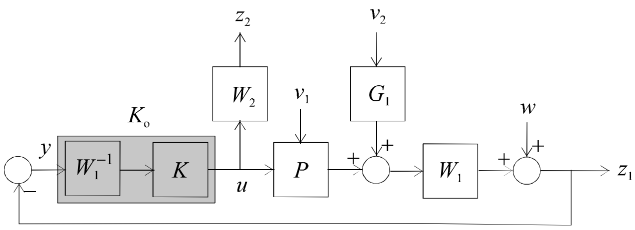

Fig. 1 Modified disturbance model

The signal v_1 drives the disturbances that are internal to the plant. We shall see how this signal is used in re-assigning the open-loop poles to their closed-loop locations. If the dynamics of the plant is represented by $\dot{x}_p = A_p x_p + B_p u$, $y_p = C_p x_p$, then we include the internal disturbances by modifying the state differential equation to

$$\dot{x}_p = A_p x_p + B_p u + G_1 v_1 \qquad (2)$$

The signals v_2 and *w* are extra driving signals. The gains G_1 and G_2 are to be determined. Writing the equations augmented plant of Fig. 1 in the form

$$\dot{x} = Ax + Bu + Gv,\ \ y = Cx + w$$

where $v = (v_1, v_2)^T$ we see that the contribution of the disturbances to the output *y* is

$$w + C(sI - A)^{-1} Gv = w + Q(s)v$$

where $Q(s) = C(sI - A)^{-1} G$. If v and w are both white noise with unit intensities then the spectral density matrix of the disturbance at the output is

$$I + Q(s)Q^T(-s) \qquad (3)$$

Spectral co-factorization of this rational matrix yields the shaping filter $V = D_q^{-1} M$ for the standard mixed sensitivity problem. Representing Q in left polynomial matrix fraction form as $Q = D_q^{-1} N_q$ we rewrite (3) as

$$D_q^{-1}(s)\Big[N_q(s)N_q^T(-s) + D_q(s)D_q^T(-s)\Big]D_q^{-T}(-s)$$

It follows that $M(s)M^T(-s) = N_q(s)N_q^T(-s) + D_q(s)D_q^T(-s)$. Thus, the roots of *M* are the left-half plane roots of the para-Hermitian polynomial matrix $N_q(s)N_q^T(-s) + D_q(s)D_q^T(-s)$. We only use this fact to compute the roots of *M*. It is not necessary to perform the spectral factorization.

We expect the roots of *M* to be closed-loop poles, and arrange them to be dominant. Our starting point is the design of the Robust Control Toolbox. As noted, the only characteristic of the design that possibly needs improvement is the closed-loop pole pair at $-0.6899 \pm j\ 0.2484$, which is the mirror image of the open-loop pole pair at $0.6899 \pm j\ 0.2484$. To reassign this closed-loop pole pair to a more suitable location it is necessary to excite the open-loop mode corresponding to the pole pair $0.6899 \pm j\ 0.2484$ in the internal disturbance model. To this end we choose

$$G_1 = \alpha\big[\,f_1 \quad f_2\,\big] \qquad (4)$$

The vectors f_1 and f_2 form a basis of the invariant subspace of the matrix A_f corresponding to the pole pair $0.6899 \pm j\ 0.2484$, and α is a real number to be determined. In practice, f_1 and f_2 are chosen as the real and imaginary parts of the eigenvector of A_f corresponding to either of the eigenvalues $0.6899 \pm j\ 0.2484$ (as computed by MATLAB).

If any of the other open-loop poles also needs to be moved then the corresponding modes should also be excited. In the present case there seems to be no need for this because the two remaining open-loop poles are stable. In the H_∞ design they will turn out to be canceled by compensator zeros.

Obviously, also the two open-loop poles at 0 of the constant disturbance model need to be shifted. The corresponding modes are excited if there is internal excitation. The gain G_2 provides additional flexibility if needed but it turns out that setting $G_2 = 0$ gives satisfactory results.

A sensible strategy to determine the parameter α is to consider the root loci of *M* as α varies. Before we can do this we need to choose the parameters τ_1 and τ_2 in the weighting function (1). After some experimenting $\tau_1 = \tau_2 = 1/20$ turned out to be suitable values. Using the Polynomial Toolbox for MATLAB it may be found that as α increases from 0 the root loci of *M* start at 0, 0, $-0.6899 \pm j\ 0.2484$. Asympotically, two of the roots go to -20 while the two remaining roots approach infinity. For $\alpha = \sqrt{10^9}$ the roots are $-3.889 \pm j3.716$ and $-18.65 \pm j9.929$. The location of the first pole pair agrees with what military specifications suggest for the dominant poles (Stevens and Lewis, 1992).

The weighting matrix W_2

We still need to choose the weighting matrix W_2. V is biproper. If W_2 is a constant matrix then we may therefore expect the compensator transfer function and, hence, also the sensitivity function U, to have zero high-frequency roll-off. As a consequence, T has the required roll-off of 40 dB/decade (because the plant roll-off is 40 dB/decade). To improve this to a possible 60 dB/decade, we let

$$W_2(s) = \begin{bmatrix} c_1(1+r_1 s) & 0 \\ 0 & c_2(1+r_2 s) \end{bmatrix} \tag{5}$$

The constants c_1 and c_2 are overall weighting factors. The time constants r_1 and r_2 control the high-frequency roll-off of *K*, *U* and *T*.

Computation

Because the weighting function W_2 may be non-proper we write it as $W_2(s) = W_{2o}W_{21}(s)$ where

$$W_{2o} = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \tag{6}$$

By further block diagram substitution we convert the configuration of Fig. 1 to that of Fig. 2. The generalized plant is strictly proper and satisfies all the other requirements for the application of the standard state space algorithm for H_∞-optimization. For this we use the function *hinfsyn* from the Mu-Analysis and Synthesis Toolbox for MATLAB. Once the compensator K_o has been computed K follows as $K = W_{21}^{-1}K_oW_1$.

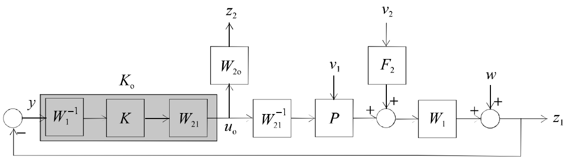

Fig. 2 Block diagram for computation

Results

For the time being we set $r_1 = r_2 = 0$. With the choice of the disturbance model previously discussed a not quite but almost acceptable result is obtained for $c_1 = c_2 = 2\times10^{-4}$. The dominant closed poles are $-3.889 \pm j3.716$ as planned. The singular value of the complementary sensitivity function *T* rolls off at the required rate, but at 100 rad/s it exceeds that of the Robust Toolbox design. Increasing c_1 and c_2 compromises the dominance of the pair $-3.889 \pm j3.716$. Setting $r_1 = r_2 = 1/10$, however, makes *T* roll-off at 60 dB/decade and reduces its singular value at 100 rad/s to the specified value. The peak values of the singular values of the sensitivity functions *S* and *T* are 7.3 dB and 4.9 dB, respectively.

The compensator poles are –30848, –84.50, -35.91, $-31.23 \pm j62.38$, $-12.36 \pm j22.89$, -0.02076, 0, 0, -19.31, -20.00. The pole at -0.02076 cancels the open-loop plant zero at the same location. The large size of the first pole indicated that the order of the compensator may be reduced by 1.

The compensator zeros are -30.00, -30.00, -7.235, -5.676, –2.568, -0.2580, –20.00, –20.00. The zeros at -30.00, -30.00, -5.676, -0.2580 cancel against open-loop plant poles at the same locations.

The compensator has two pole-zero pairs at -10 that cancel internally. This pair corresponds to the time constants $r_1 = r_2 = 0.1$ in the weighting function W_2. This cancellation is caused by the block diagram substitution technique. The two pole-zero pairs at or near –20 may probably also be cancelled.

The closed-loop poles are –30848, $-3.889 \pm j3.716$, $-18.65 \pm j9.929$ (these are the pre-assigned poles), -30.00, -30.00, -0.2580, -0.02093, -5.676 (these result from pole-zero cancellations between the plant and the compensator), and $-22.35 \pm j\,42.43$, –50.86, –34.34, $-14.34 \pm j12.74$, -21.82, –20.05. The latter two poles originate from the two near-canceling pole-zero pairs in the compensator.

5. H_2 MIXED SENSITIVITY DESIGN

We present an H_2 mixed sensitivity design that is based on the dominant pole placement approach outlined in the companion paper. Similarly to the H_∞ approach we augment the plant as in Fig. 2. The block W_1 has a transfer function of the form (1) and ensures integrating action. The blocks W_{2o} and W_{21} implement the weighting function W_2 according to (5)–(6) and serve to control the high-frequency characteristics of the design. For the purposes of pole placement we use the noise model (2) for the plant. The gains G_1 and G_2 are to be chosen so that the observer poles assume suitable dominant positions.

Assume that the augmented plant of Fig. 2 (consisting of the series connection of *P* and W_1) has the state space representation $\dot{x} = Ax + Bu + Gv$, $-y = Cx + w$, where $v = (v_1, v_2)^T$ and $G = [G_1 \quad G_2]$. Then in the H_2 case, the observer poles are the eigenvalues of $A - KC$, where the observer gain *K* follows by solving the appropriate Riccati equation. If *v* and *w* have unit intensity matrices then the observer poles are the left-half pane zeros of the rational matrix

$$I + C(sI - A)^{-1}GG^T(-sI - A^T)^{-1}C^T$$

Interestingly, this corresponds precisely to the expression (3). Hence, we may follow the same procedure as for the H_∞ case to determine the dominant closed-loop poles. Moreover we see that we may determine the dominant positions by solving a Riccati equation as an alternative to the polynomial matrix approach followed for the H_∞ case.

Thus, the first step is to solve the observer part of the LQG problem. To this end we choose G_1 according to (4) with $\alpha = \sqrt{10^9}$. We let $G_2 = 0$. The intensity of the state noise and the measurement noise are both unit matrices. By solving the observer Riccati equation we find that the observer poles are $-3.889 \pm j3.7156$, $-18.65 \pm j9.929$, -0.2580, -5.676, -30.00, -30.00. The latter four poles are open-loop poles that are left in place.

The second step in the design is to solve the regulator part of the LQG problem. By varying the number ρ in the criterion

$$\lim_{t\to\infty} E\left(z^T(t)z(t) + \rho u^T(t)u(t)\right)$$

we find that for $\rho = \sqrt{10^{-7}}$ the regulator poles are -0.0209, $-11.82 \pm j10.07$, -20.00, -28.71, $-32.75 \pm j53.99$, -64.95. The first closed-loop pole coincides with the plant zero. The remaining poles are non-dominant.

The compensator poles are -91.32, $-36.77 \pm j70.65$, $-20.44 \pm j16.03$, -23.50, -20.00, -0.0209, 0, 0. The pole at -0.0209 cancels the plant pole at the same location.

The compensator zeros are -30.00, -30.00, -20.00, -20.00, -5.676, -2.763, and -0.2580. The zeros at -30.00, -30.00, -5.676, -0.2580 cancel against open-loop poles at the same locations.

The peak value of the singular value of the sensitivity function S of the resulting design is 5.2 dB and that of the complementary sensitivity function T is 3.8 dB. At 100 rad/s the singular value of T very nearly equals that of the *Robust Toolbox* design. It rolls off at 60 dB/decade.

The step responses and the singular value plots for this design are remarkably similar to those for the H_∞ design.

6. DISCUSSION AND CONCLUSIONS

Comparison of the Safonov and Chiang design with the two mixed sensitivity designs shows that the sensitivity functions of the S&C design peak less. This is the trade-off for the much steeper roll-off of the complementary sensitivity function beyond the cut-off frequency.

For the mixed sensitivity designs the step response matrix corresponding to the complementary sensitivity matrix T shows much more interaction and overshoot than that for the S&C design. The step response matrix would not be acceptable as the closed-loop step response matrix, which is precisely T in the one-degree-of freedom configuration of Fig. 3. A much better step response matrix may be achieved by a suitable two-degree-of-freedom arrangement.

It is interesting to note how close the H_∞ and the H_2 mixed sensitivity designs are. Given the much greater computational complexity of the H_∞ algorithm the choice of the method is not difficult difficult.

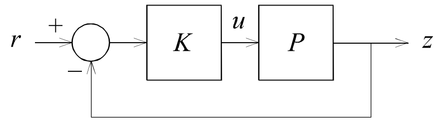

Fig. 3 One-degree-of-freedom configuration

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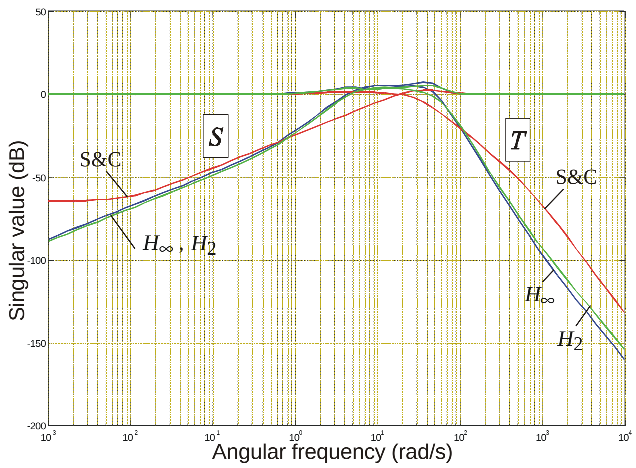

Fig. 4 Singular values of S and T

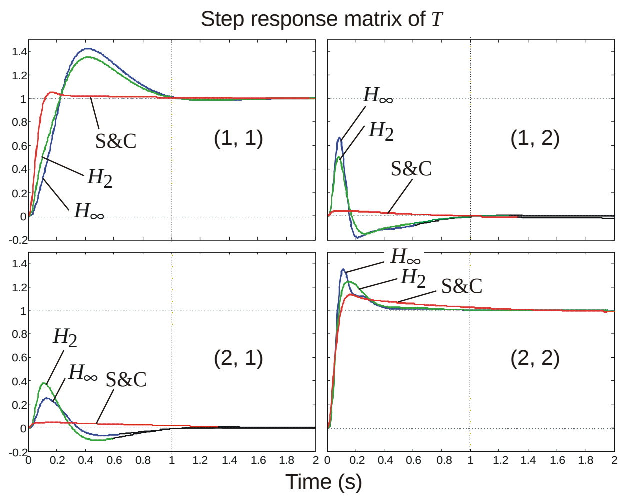

Fig. 5 Step responses of T