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FUZZY-ARITHMETIC-BASED LYAPUNOV FUNCTION FOR DESIGN OF FUZZY CONTROLLERS

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Abstract: A novel approach to design fuzzy controllers using fuzzy-arithmetic-based Lyapunov function that gives a linguistic description on the plant and the control objective is presented in this paper. An inverted pendulum system is used as a benchmark dynamic nonlinear plant for evaluating the proposed method. It is shown that a set of stable fuzzy control rules can be derived from perception-based information systematically, rather than heuristically. Based on Lyapunov's approach, conditions to ensure the stability of a pendulum-cart system are given, and these conditions are then used to verify the perception-based information for balancing a pendulum. Based on these perceptions and standard-fuzzy-arithmetic-based Lyapunov function, a set of traditional fuzzy control rules can be derived. On the other hand, a singleton fuzzy controller can be devised by using constrained-fuzzy-arithmetic-based Lyapunov's function. Further more the stability of the fuzzy controllers can be guaranteed by means of fuzzy version of Lyapunov stability analysis. The results obtained are illustrated with a design of stable fuzzy controllers for an autonomous pole balancing mobile robot. *Copyright © 2002 IFAC*

Keywords: Fuzzy control, Lyapunov function, stability, fuzzy sets, robot control.

1. INTRODUCTION

The advantages of fuzzy control often become most apparent for very complex problems where we have an *intuitive* idea about how to achieve high performance control. How to make use of the intuitive knowledge, or *perceptions*, to design a stable fuzzy control system is still a challenging problem. However, existing results for stability analysis of fuzzy control systems typically require that the plant model be deterministic, satisfy some continuity constraints, and sometimes require the plant to be linear or "linear-analytic." Classical Lyapunov synthesis suggests the design of a controller that should guarantee $\dot{V}(x) < 0$ for a Lyapunov function $V(x)$. Fuzzy Lyapunov synthesis (Magaliot and Langholz, 1999a, b) follows the same idea but the linguistic description (perception-based information) of the plant and control objective is utilized by means of computing with words (CW) (Zadeh, 1996, 1999). The basic assumption of fuzzy Lyapunov synthesis is that, for a Lyapunov function $V(x)$, if the linguistic value of $\dot{V}(x)$ is *Negative*, then $\dot{V}(x) < 0$, so the stability can be guaranteed. As an example, for $\dot{V}(x) = Negative \cdot Negative + Negative \cdot u$, we may choose $u = Positive \;\; Big$ to make $\dot{V}(x) = Negative$. But this is again a *heuristic* method! An important point addressed here is that $\dot{V}(x)$ might not be *Negative* unless there exists a set of suitable linguistic variables and their arithmetic operations to guarantee this. On the other hand, for the fuzzy Lyapunov synthesis proposed by Magaliot and Langholz (1999a, b), only the sign of the fuzzy linguistic value, such as "*Negative*" or "*Positive*" is used. Its magnitude is not considered. This means it ignores the changes in states. It could be considered as a very crude estimator of the derivative. Hence, the information from the perceptions could be very limited. Also, it seems difficult to derive more fuzzy rules as there are only a few of linguistic terms, such as *Negative* and *Positive*, are utilised. The number of fuzzy rules is therefore limited.

To solve the above problems, a fuzzy Lyapunov synthesis approach in connection with fuzzy numbers and their arithmetic operations is investigated in our previous study (Zhou and Ruan, 2001). However, the standard fuzzy arithmetic dose not take into account all the information available, and the obtained results are more imprecise than necessary or, in some cases, even incorrect. On the other hand, the perception-based information used for fuzzy controller design is not always reliable. To overcome the above deficiencies, the constrained fuzzy arithmetic (Klir, 1997) is firstly introduced for "word" manipulation of Lyapunov function. Then Lyapunov's indirect method is employed to verify the perception-based information to make sure it is reliable in terms of Lyapunov's stability.

In the following section, a brief introduction of both standard and constrained fuzzy arithmetic is given. In Section 3, an inverted pendulum balancing system is used as a benchmark to demonstrate a systematic method to design a fuzzy controller from perception-based information using standard-fuzzy-arithmetic-based Lyapunov function. In Section 4 , a deficiency of the standard fuzzy arithmetic in fuzzy controller design is identified, and the constrained-fuzzy-arithmetic-based Lyapunov function is proposed. The practical implementation of fuzzy control to the pole-balancing mobile robot is given in Section 5 to verify the proposed method. This is followed by some discussions and concluding remarks.

2. BASICS OF FUZZY ARITHMETIC

In this paper, the discussion is based on the triangular fuzzy numbers (TFN) as shown in Fig. 1. We can represent this type of TFN by a 3-tuple *A* = <*a*, *b*, *c*>, where a -cut is ${}^{a}A = < a + (b-a)a, c-(c-b)a >$. In Fig. 1, *PB* = <2, 3, 4>, *PM* = <1, 2, 3>, *PS* = <0, 1, 2>, *ZE* = <-1, 0, 1>, *NS* = <-2, -1, 0>, *NM* = <-3, -2, -1>, *NB* = <-4, -3, -2>.

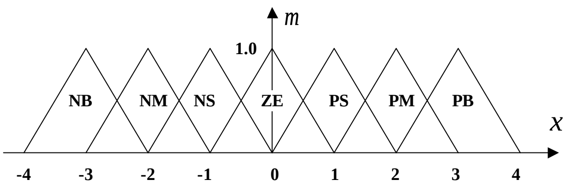

Fig. 1. A linguistic variable with seven terms.

There are two common ways of defining fuzzy arithmetic operations (Klir, 1997). One is based on the a -cut representation and another is on the extension principle of fuzzy set theory. Employing the a -cut representation, arithmetic operations on fuzzy intervals are defined in terms of the well-established arithmetic operations on closed intervals of real numbers. Let A and B denote fuzzy sets, and let $* \in \{+,-,\cdot,/\}$, which denotes any of the four basic arithmetic operations. Then, we define a fuzzy set on $\Re$, $A*B$ by the following equation

$$ {}^{a}(A*B) = \{x*y \mid < x, y > \in {}^{a}A \times {}^{a}B\} \quad (1)$$

where ${}^{a}A$ and ${}^{a}B$ are the a -cuts of fuzzy sets *A* and *B*, $a \in (0,1]$; when the operation is division of *A* and *B*, it is required that $0 \notin {}^{a}B$ for any $a \in (0,1]$. Employing the extension principle, the arithmetic operations on fuzzy sets *A* and *B* are defined by

$$ m_{A*B}(z) = \sup_{z=x*y} \min(m_A(x), m_B(y)) \quad (2)$$

for all $z \in \Re$. As an example, $NB + PM \approx NS$ where $NS' = NB + PM$ = <-3, -1, 1>, and *NS* = <-2, -1, 0>.

Results obtained by the standard fuzzy arithmetic suffer from greater impression than justifiable in all computations that involve the *requisite equality constraint* (Klir, 1997). However, the equality constraint is always satisfied in the classical arithmetic on real numbers. Because ignoring equality constraints will lead to results that are less precise than necessary, it is essential to include the constraints, when applicable, into the general definition of basic arithmetic operations on fuzzy numbers. In general, each constraint *R* on $A*B$ is a relation (crisp or fuzzy) on $A \times B$. For the extension principle of the fuzzy set theory, the constrained arithmetic operations $(A*B)_R$ are defined by the following equation

$$ m_{(A*B)_R}(z) = \sup_{z=x*y} \min(m_A(x), m_B(y), m_R(x,y)) \quad (3)$$

For the cut representation of the fuzzy intervals,

$$ {}^{a}(A*B)_R = \{x*y \mid < x, y > \in ({}^{a}A \times {}^{a}B) \cap {}^{a}R\} \quad (4)$$

Any operations $A*B$ or $B*A$ are unconstrained, even though *A* = *B*, while operations $A*B$ and $B*A$ are subject to the equality constraint. These constrained operations, for example on *A*, may conveniently be expressed as follows, where *E* denotes the relation R representing the equality constraint.

$$ {}^{a}(A+A)_E = \{x+x \mid x \in {}^{a}A\} = {}^{a}[2\underline{a}, 2\overline{a}] \quad (5)$$

$$ {}^{a}(A-A)_E = \{x-x \mid x \in {}^{a}A\} = 0 \quad (6)$$

$$ {}^{a}(A \cdot A)_E = \{x \cdot x \mid x \in {}^{a}A\} \quad (7)$$

$$ {}^{a}(A/A)_E = \{x/x \mid x \in {}^{a}A, 0 \notin {}^{a}A\} = 1 \quad (8)$$

Under the equality constraint for *X*, where *A*, *B*, $X \in \Re$, we can obtain

$$ A + X = B \Leftrightarrow X = B - A \quad (9)$$

$$ A \cdot X = B \Leftrightarrow X = B/A \quad (0 \notin {}^{a}A). \quad (10)$$

But these are not, in general, solutions in the standard fuzzy arithmetic.

3. STANDARD-FUZZY-ARITHMETIC-BASED LYAPUNOV FUNCTION

The inverted pendulum is frequently used as a benchmark dynamic nonlinear plant for evaluating a control algorithm or a combination of them. It has been extensively studied by numerous researchers (Li and Shieh, 2000, Wang, 1996, Zak, 1999). Its state variables are $x_1 = q$ (the pendulum's angle), and $x_2 = \dot{q}$ (the pendulum's angular velocity). The system's dynamic equations are given as follows (Slotine and Li, 1991)

$$ \begin{cases} \dot{x}_1 = x_2 = F_1(x) \\ \dot{x}_2 = f(x_1, x_2) + g(x_1, x_2)u = F_2(x) \end{cases} \quad (11)$$

where,

$$ f(x_1, x_2) = \frac{9.8 \sin x_1 - \frac{mlx_2^2 \cos x_1 \sin x_1}{m_c + m}}{l(\frac{4}{3} - \frac{m \cos^2 x_1}{m_c + m})}$$

$$ g(x_1, x_2) = \frac{\frac{\cos x_1}{m_c + m}}{l(\frac{4}{3} - \frac{m \cos^2 x_1}{m_c + m})}$$

Where m_c is the mass of the cart, m is the mass of the pole, $2l$ is the pole's length, and u is the applied force (control).

3.1 Lyapunov Stability Analysis of Inverted Pendulum Systems

In this subsection, the use of Lyapunov's indirect method for stability analysis of an inverted pendulum is illustrated (Jenkins and Passino, 1999, Passino and Yurkovich, 1999). The stability conditions derived below will be used to verify the perception-based information for the balancing of a pendulum. From Eq. (11),

$$\overline{A} = \left.\begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix}\right|_{x=0} =$$

$$\left.\begin{bmatrix} 0 & 1 \\ \frac{9.8(m_c+m)}{\frac{1}{3}l(4m_c+m))} + \frac{1}{\frac{1}{3}(4m_c+m))}\frac{\partial u}{\partial x_1} & \frac{1}{\frac{1}{3}(4m_c+m))}\frac{\partial u}{\partial x_2} \end{bmatrix}\right|_{x=0} \quad (12)$$

The eigenvalues of $\overline{A}$ are given by the determinant of $lI - \overline{A}$.

$$l\overline{A} - I =$$

$$\left.\begin{bmatrix} l & -1 \\ -(\frac{9.8(m_c+m)}{\frac{1}{3}l(4m_c+m))} + \frac{1}{\frac{1}{3}(4m_c+m))}\frac{\partial u}{\partial x_1}) & l - \frac{1}{\frac{1}{3}(4m_c+m))}\frac{\partial u}{\partial x_2} \end{bmatrix}\right|_{x=0}$$

To ensure that the origin $x_e = 0$ is asymptotically stable, the eigenvalues l_i of $\overline{A}$ must be in the left half of the complex plane. It is sufficient that

$$l^2 - (\frac{1}{\frac{1}{3}(4m_c+m)}\frac{\partial u}{\partial x_2})\, l - (\frac{9.8(m_c+m)}{\frac{1}{3}(4m_c+m)} + \frac{1}{\frac{1}{3}(4m_c+m)}\frac{\partial u}{\partial x_1}) = 0 \quad (13)$$

has its roots in the left half-plane. Eq. (13) will have its roots in the left half-plane if each of its coefficients is positive. Hence, to ensure the asymptotic stability, the following conditions must be satisfied

$$\frac{\partial u}{\partial x_1} < -9.8(m_c+m)\,, \quad \frac{\partial u}{\partial x_2} < 0 \quad (14)$$

From Eq. (14), we can easily conclude that the force u is inversely proportional to the pendulum's angular velocity x_2, and is also inversely proportional to the pendulum's angle. This is exactly reflected by perceptions on balance of a inverted pendulum. For example, as the pole is falling over to the right hand side, one must move his/her finger to the right hand side at once.

Table 1. Perceptions for balancing a pole

	Perceptions	Remarks
S1	$\dot{x}_1 = x_2$	From the state description.
S2	$\dot{x}_2 = \ddot{q}$ is proportional to the control u	The angular acceleration is proportional to the force applied to the cart.
S3	u is inversely proportional to $x_1 = q$	From the knowledge of balancing a pole. It is also verified by Eq. (14).
S4	u is inversely proportional to $x_2 = \dot{q}$	From the knowledge of balancing a pole. It is also verified by Eq. (14).

3.2 Design of a Stable Fuzzy Controller using Perception-based Information

Assume that the model (11) is not known. However, based on the physical intuition and the experience on balancing a pole, the perception-based information can be got as shown in Table 1. In the following, it will demonstrate that the fuzzy control rules can be derived from the perceptions by means of fuzzy-arithmetic-based Lyapunov function, and the stability of the fuzzy controller can be guaranteed.

Consider the Lyapunov function candidate $V(x_1, x_2)$ = $\frac{1}{2}(x_1^2 + x_2^2)$ which can be used to represent a measure of the distance of the pendulum's actual state (x_1, x_2) and the desired state $(x_1, x_2) = (0,0)$. Differentiating V yields:

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 \quad (15)$$

Using S2 in Table 1, Eq. (15) can be rewritten as

$$\dot{V} \approx x_1\dot{x}_1 + x_2 u = x_1 x_2 + x_2 u = x_2(x_1 + u) \quad (16)$$

Its linguistic description is given below

$$LV(\dot{V}(x)) = LVx_2(LVx_1 + LVu) \quad (17)$$

Where $LV(\dot{V}(x))$, LVx_1, LVx_2, and LVu are linguistic values of $\dot{V}(x)$, x_1, x_2, and u respectively.

Theorem 1. If $V(x)$ is a Lyapunov function and the linguistic value $LV(\dot{V}(x)) = Negative$ and $Supp(Negative)$ $\subset (-\infty, 0]$, then the fuzzy controller designed by fuzzy Lyapunov synthesis is locally stable. Furthermore, if $Supp(Negative) \subset (-\infty, 0)$, then the stability is asymptotic.

The proof of Theorem 1 is given in Zhou and Ruan (2001). It provides a guidance to design a *stable* fuzzy controller only using the perception-based information. For example, if $x_2 = PM$, and choose $x_1 + u = NM$, then a set of fuzzy control rules as shown in Table 2 can be derived by using standard fuzzy arithmetic operations defined in Eq. (1) and (2). From Eq. (17), we have $LV(\dot{V}(x))$ $= PM \cdot NM = Negative$. This is illustrated in Fig. 2. It can be seen that $Supp(Negative)$ $\subset [-9, -1] \subset (-\infty, 0]$. From Theorem 1, It can be seen that the fuzzy control rules in Table 2 are stable.

Table 2. Fuzzy control rules (x2 = PM)

x_1	$x_1 + u$ = NM	u	Remarks
NM	NM + u = NM	ZE	NM + ZE = NM
NS	NS + u = NM	NS	NS + NS = NM
ZE	ZE + u = NM	NM	ZE + NM = NM
PS	PS + u = NM	NB	PS + NB = NM
PM	PM + u = NM	NL	PM + NL = NM

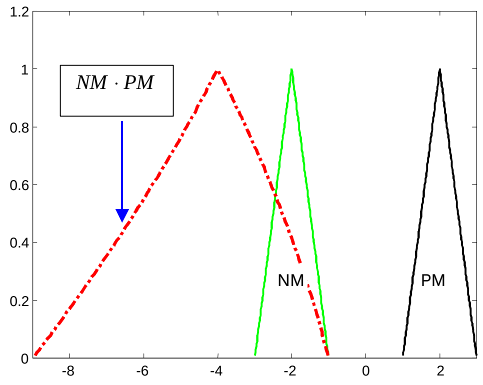

Fig. 2. Illustration of $LV(\dot{V}(x)) = PM \cdot NM$

Table 3. Fuzzy control rules derived from the perception-based information

u		x_1				
		NM	NS	ZE	PS	PM
x_2	NM	PL	PB	PM	PS	ZE
	NS	PB	PM	PS	ZE	PS
	ZE	PM	PS	ZE	PS	PM
	PS	PS	ZE	PS	PM	PB
	PM	ZE	PS	PM	PB	PL

Repeating the similar procedure, a set of fuzzy control rules as shown in Table 3 can be derived from the perception-based information (Table 1) using standard-fuzzy-arithmetic-based Lyapunov function.

4. CONSTRAINED-FUZZY-ARITHMETIC-BASED LYAPUNOV FUNCTION

Consider the following fuzzy rule derived from fuzzy Lyapunov synthesis approach using standard fuzzy arithmetic as shown in Table 3.

If x_1 is NS and x_2 is PS Then u is ZE (18)

From Eq. (17), $LV(\dot{V}(x)) = PS \cdot (NS + ZE)$. This is illustrated in Fig. 3.

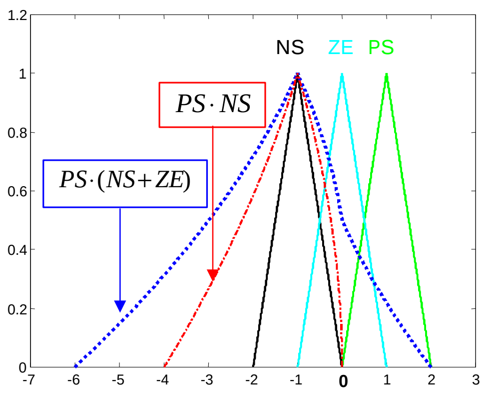

Fig. 3. Illustration of $LV(\dot{V}(x)) = PS \cdot (NS + ZE)$

From Fig. 3, it can be seen that $Supp(PS \cdot (NS + ZE)) = [-6,2] \not\subset (-\infty,0]$. The stability condition given in Theorem 1 is not satisfied. This is caused by the deficiency of the standard fuzzy arithmetic. The standard fuzzy arithmetic does not utilise some of the information available. Therefore, the obtained results may be more imprecise than necessary or, in some cases, even incorrect. To overcome this deficiency, a constrained fuzzy arithmetic (Klir, 1997) is needed to take all available information into account in terms of relevant requisite constrains.

In the following, we will demonstrate how to use the constrained-fuzzy-arithmetic-based Lyapunov function to derive fuzzy control rules from the perception-based information given in Table 1. The same condition as Table 2 is considered here, that is, $x_2 = PM$. By choosing $x_1 + u = NM$, under the equality constraint for u, from (9), we have

$$u = NM - x_1 \quad (19)$$

If $x_1 = NM = <-3, -2, -1>$, under the equality constraint, ${}^a(LVu) = {}^a(NS - NM)_E$. Considering ${}^a(NS) = [-2+a, -a]$ and ${}^a(NM) = [-3+a, -1-a]$, then ${}^a(LVu) = [(-2+a)-(-3+a), (-a)-(-1-a)] = [1,1]$. This leads to $u = 1$. Hence, the following fuzzy control rule can be derived

If x_1 is NM and x_2 is PS Then $u = 1$

It is a fuzzy control rule with singleton consequent (Sugeno, 1999). The rest of fuzzy rules for the condition $x_2 = PM$ are illustrated in Table 4.

Table 4. Fuzzy control rules (x2 = PM)

x_1	${}^a(LVu) = {}^a(NM - x_1)_E$	u
NM	$[(-3+a)-(-3+a), (-1-a)-(-1-a)]$ = [0, 0]	0
NS	$[(-3+a)-(-2+a), (-1-a)-(-a)]$ = [-1, -1]	-1
ZE	$[(-3+a)-(-1+a), (-1-a)-(1-a)]$ = [-2, -2]	-2
PS	$[(-3+a)-a, (-1-a)-(2-a)]$ = [-3, -3]	-3
PM	$[(-3+a)-(1+a), (-1-a)-(3-a)]$ = [-4, -4]	-4

Table 5. Singleton fuzzy rules derived by constrained-fuzzy-arithmetic-based Lyapunov Function

u		x_1				
		NM	NS	ZE	PS	PM
x_2	NM	4	3	2	1	0
	NS	3	2	1	0	-1
	ZE	2	1	0	-1	-2
	PS	1	0	-1	-2	-3
	PM	0	-1	-2	-3	-4

Repeating the same procedure, a set of fuzzy control rules with singleton consequent shown in Table 5 can be produced by using the constrained-fuzzy arithmetic-based Lyapunov function.

To investigate the stability of the above fuzzy control rules with singleton consequent, let's consider the same condition as that of the fuzzy control rule (18). The corresponding rule in Table 5 is

If x_1 is NS and x_2 is PS Then u is 0 (20)

Under the equality constraint, $(LVx_1 + LVu)_E = NS$. From the Eq. (17), we have $LV(\dot{V}(x)) = LVx_2(LVx_1 + LVu) = PS \cdot NS$. From Fig. 3, we can see that $Supp(PS \cdot NS) =$ $[-4,0] \subset (-\infty,0]$. From Theorem 1, the fuzzy control rule (20) is stable. Compare with fuzzy control rule (18), it can be seen that the deficiency of the fuzzy Lyapunov synthesis with the standard fuzzy arithmetic can be overcome by the constrained fuzzy arithmetic.

5. EXPERIMENT

To demonstrate the effectiveness of the proposed fuzzy controller design method, a real-time experiment of the fuzzy control of an autonomous pole-balancing mobile robot with an onboard TMS 320C32 DSP processor was conducted (see Fig. 5). This project aims to design and fabricate an autonomous mobile robot to participate in the Singapore Robotic Games (SRG). The mobile robot is able to balance a free-falling pole by means of horizontal movement. While balancing the pole, it would also travel with a pre-designed slope profile. The mobile robot with the highest number of successful cycles in a single untouched attempt within a predefined time slot will be considered the winning entry.

The parameters of the physical robot are given as follows: the pole's length is $2l = 1m$, the mass of the pole is $m = 0.1kg$, and the mass of the cart is $m_c = 2.5kg$. Fig. 5 shows the trajectory of the pole angle and the velocity tracking results using the fuzzy control rules (Table 3) derived from perception-based information using the standard-fuzzy-arithmetic-based Lyapunov function. It can be seen that the pole never falls down as the mobile robot can always track the desired trajectory though the pole swings very much. This may be due to the limited perception-based information. A similar experiment is also conducted using the singleton fuzzy control rules (Table 5) derived by the constrained-fuzzy-arithmetic-based Lyapunov function. The results are similar to that of Fig. 5. From Fig. 6, it can be found that the pole angle is sometimes bigger than 0.2 *rad*. However, for the fuzzy control rules in Table 3, the pole angle is always less than 0.2 *rad*. This may mean that the traditional fuzzy control rules can achieve better tracking and balancing results than the singleton fuzzy controller.

Fig. 4. An autonomous pole-balancing mobile robot.

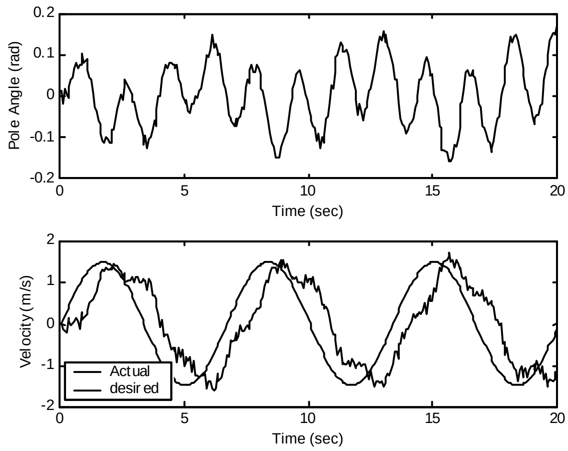

Fig. 5. Balancing and tracking results using the fuzzy control rules derived from the perception-based information by means of the standard-fuzzy-arithmetic-based Lyapunov function.

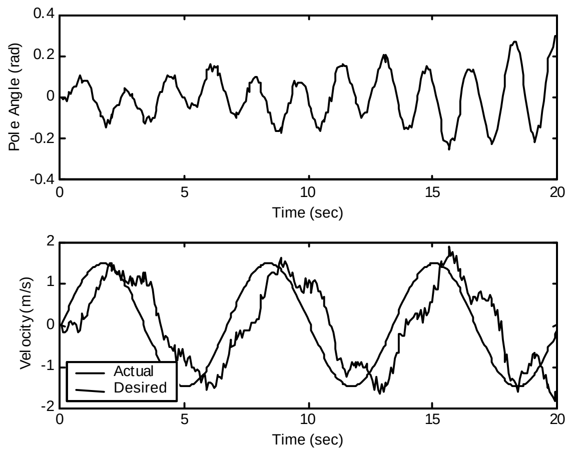

Fig. 6. Balancing and tracking results using the fuzzy control rules derived from the perception-based information by means of the standard-fuzzy-arithmetic-based Lyapunov function.

To improve the pole-balancing performance, further learning is necessary, for example, the fuzzy reinforcement learning methods (Zhou, Yang and Jia, 2001).

6. CONCLUDING REMARKS

A novel approach to design fuzzy controllers using fuzzy-arithmetic-based Lyapunov function that gives a linguistic description on the plant and the control objective is presented in this paper. It is found that by using the standard-fuzzy-arithmetic-based Lyapunov function, the conventional fuzzy control rules can be produced. While by using the constrained-fuzzy-arithmetic-based Lyapunov function, the fuzzy control rules with singleton consequent can be derived. We also demonstrate that the constrained fuzzy arithmetic can be utilised to overcome some deficiencies in the standard fuzzy arithmetic for fuzzy controller design. On the other hand, based on Lyapunov's indirect method, conditions to ensure the stability of a pendulum-cart system are given, and these conditions are then used to verify the perception-based information for balancing a pendulum. In the real-time experiment of the fuzzy control of the autonomous pole-balancing mobile robot, we found that the pole doesn't fall down as the robot tracks the desired trajectory even without further tuning the fuzzy controller proposed in this paper, though it swings very much some times.

The perception-based information is very limited to design a controller. How to integrate both measurement-based information and perception-based information to design an intelligent controller CW will be a new challenge. We will also try to incorporate some other techniques in the fuzzy controller design approach presented in this paper.

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