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SOME RESULTS ON STABILITY OF REDUCED-ORDER FILTER USING LEADING SINGULAR MODES

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Abstract: Stability of the filter whose gain is constructed on the basis of several leading singular modes of the system dynamics, is studied. Simple numerical examples and simulation experiment are presented to illustrate the theory and to compare it with an eigenvalue decomposition filter.

Keywords: Eigenvalue decomposition, Singular value decomposition, Time-varying system, Exponential stability.

1. INTRODUCTION

In the work Hoang *et al.* (2001), stability of the reduced-order adaptive filter (ROAF) is studied for time-invariant system based on eigenvalue decomposition approach (EVD) from which one sees that detectability is necessary and sufficient condition for ensuring a stability of the filter whose gain is constructed on the basis of all unstable and neutral modes of the system.

This EVD approach, however, is difficult to apply to time-varying systems. At the same time, Cohn and Todling (1996) has proposed the Partial Singular Value Decomposition Filter (PSF) in which the system dynamics is approximated by its leading SVD (Singular Value Decomposition) part. The main conclusion drawn from the twin-experiment on the data assimilation for the two-dimensional, linear shallow-water model in Cohn and Todling (1996) is that the PSF must account for all modes with singular values larger or equal to 1 for otherwise the PSF diverges. The theoretical question arising here is whether this conclusion has a global character, i.e. is it valid for all dynamical systems ? In the present paper it will be shown that involving all unstable and neutral singular modes (supposing they are all observable) in the construction of the projection subspace is *sufficient* (not necessary) for the convergence of the filter.

2. PARTIAL SINGULAR VALUE DECOMPOSITION FILTER

Suppose that the system is described by

$$x(t+1) = \Phi x(t) + w(t), \qquad (1)$$

and we are given the observations

$$z(t+1) = Hx(t+1) + v(t+1). \qquad (2)$$

In (1)(2), Φ, H are known (nxn) and (pxn) matrices, $w(t), v(t)$ represent the model and observation errors which are assumed to be white with zero mean and covariance $Q(t), R(t)$ respectively.

When the dimension of the system state $x(t)$ is very high (order of $10^6 - 10^7$ for typical meteorological and oceanographic numerical models), there is no possibility to apply directly a standard Kalman filter (KF) to generate the estimate for $x(t)$. The main idea underlying the approach in Cohn and Todling (1996) is

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to utilize a partial SVD of the tangent linear dynamics (i.e., Φ for linear system (1)) between consecutive observation times : the tangent linear propagator is approximated by a leading part of its SVD. Introduce the following class of filters

$$\hat{x}(t+1) = \Phi\hat{x}(t) + K\zeta(t+1), \qquad (3)$$

where ζ is the innovation vector, K is the gain matrix. Let $\Phi = UDV^T$ be the SVD of Φ, where $U = [U_1, U_2], V = [V_1, V_2]$, $D = \text{diag}[D_1, D_2]$ U_1, V_1 are of dimensions $(n\text{x}\nu)$, U_2, V_2 are $[n\text{x}(n-\nu)]$ matrices, and D_1, D_2 are $(\nu\text{x}\nu)$ and $[(n-\nu)\text{x}(n-\nu)]$ matrices respectively; $D_1 = \text{diag}\ [\sigma_1, ..., \sigma_\nu]$ is composed from the first ν leading singular values of Φ (and the columns of U_1 and V_1 are the corresponding right and left singular vectors of Φ). Write

$$\Phi = \Phi_1 + \Phi_2, \qquad (4)$$

where $\Phi_1 := U_1 D_1 V_1^T$, $\Phi_2 := U_2 D_2 V_2^T$. The PSF in Cohn and Todling (1996) is in fact a Kalman-like filter in which the leading part Φ_1 is proposed to be used instead of Φ in the Algebraic Riccati equation (ARE).

3. REDUCED-ORDER ADAPTIVE FILTER

Consider the filter (3) and let the gain K be parametrized by some vector θ. In Hoang *et al.* (1997) an optimal AF is determined which is aimed at minimizing the mean prediction error

$$J_\theta(\zeta) = E[\Psi(\zeta)] \to \min_\theta [\Psi(\zeta) = \zeta^T\zeta] \qquad (5)$$

The optimal parameters θ are updated on-line by applying stochastic optimization algorithms.

To reduce the number of adjusted parameters in the gain, let K be assumed to be of the form (cf., Hoang *et al.*, 1997)

$$K = P_r K_e \qquad (6)$$

where P_r is an *a priori* known $(n\text{x}n_e)$ matrix and K_e is an $(n_e\text{x}p)$ matrix whose parameters are estimated by minimization of the objective function (5). Usually $n_e << n$.

3.1. PSF as a reduced-order filter

If we replace Φ by Φ_1 and assume that the initial error covariance matrix (ECM) for the system state $P(0)$ as well as the ECM of the model error $w(t)$ belong to $R[U_1]$, then the forecast ECM derived from the ARE belongs to $R[U_1]$ too. This fact allows us to represent the gain of the PSF in the form (7) with $P_r = U_1$. The method developed in Hoang *et al.* (2001) is then very helpful for studyng the question on stability of the PSF and its adaptive version in next sections.

4. STABILITY OF THE PSF

4.1 A simplified case

Let $\sigma_i(\Phi)$ be the i^{th} largest singular value of Φ. The vector and matrix norms are defined as $||x|| = [\sum_{i=1}^n x_i^2]^{1/2}$, $||\Phi|| = [\sigma_1(AA^T)]^{1/2}$

Consider filter (3) and the SVD (4). Let ν be a nonnegative integer number such that for some fixed $\epsilon \in (0,1)$,

$$\sigma_{\nu+1}(t) \leq \frac{\epsilon}{||B||}, B = K_e H \qquad (7)$$

Introduce

$$\text{rank}\ [H_e] = \nu, H_e = HU_1. \qquad (8)$$

Theorem 4.1. Consider system (1)(2) and let ν be a nonnegative integer satisfying constraint (7). Consider filter (3) and let $P_r = U_1$. Then under condition (8) there exists a reduced gain K_e ensuring the exponential stability in the filter.

$Proof.$

Introduce the gain K_e in the reduced space $K_e = H_e^+$ where H_e^+ is the Moore-Penrose pseudoinverse of H_e (Albert, 1972).

Due to condition (8),

$$K_e H_e = H_e^+ H_e = I_\nu, \nu \neq 0 \qquad (9)$$

We show now that $K = P_r K_e$ subject to $P_r = U_1$ will ensure the exponential stability of the filter. Let us look in detail the transition matrix of the filter $L = A\Phi = A\Phi_1 + A\Phi L_1 + L_2$ where $A := I - KH$, $L_1 := A\Phi_1 = (I - U_1 K_e H)\Phi_1$, $L_2 := A\Phi_2 = (I - U_1 K_e H)\Phi_2$ and Φ_1, Φ_2 are defined in (4).

(i) For K defined above, $L_1 = 0 : L_1 = [U_1 - U_1 K_e H U_1] D_1 V_1^T = 0$ since $K_e H U_1 = I_\nu$.

(ii) Consider $U^T L_2(t) V = \tilde{D}(t) = \begin{vmatrix} U_1^T L_2 V_1 & U_1^T L_2 V_2 \\ U_2^T L_2 V_1 & U_2^T L_2 V_2 \end{vmatrix}$ where $\tilde{D}_2 = \begin{vmatrix} 0 & 0 \\ 0 & D_2 \end{vmatrix}$. We have $||L_2|| \leq ||\tilde{D}_1||||\tilde{D}_2||$. Let us look at $||\tilde{D}_1|| = ||\tilde{D}^{(2)}||$. Introduce the matrix $\tilde{D}^{(1)} := K_e HU = [K_e HU_1, K_e HU_2] = [I, K_e HU_2]$. It can be shown that $||[I, K_e HU_2]|| = ||\tilde{D}^{(2)}|| = ||K_e HU|| = ||K_e H||$ since U is orthonormal. Hence $||L_2|| \leq ||\tilde{D}_1||||\tilde{D}_2|| = ||B||\sigma_{\nu+1}$ where B is defined in (7). The requirement $||B||\sigma_{\nu+1} \leq \epsilon$ for some fixed $\epsilon \in (0,1)$ then is automatically satisfied if we impose constraint (7). (End of proof)

Comment 4.1. Let ω denote the observational index of (H, Φ), i.e. ω is the minimal integer such that rank $[\mathcal{O}_\omega]$ = rank $[\mathcal{O}_\beta]$= $\tilde{n}$ for all integer $\beta, \beta \geq \omega$, $\mathcal{O}_\omega := [H^T, (H\Phi)^T, ..., (H\Phi^\omega)^T]^T$. Let $\tilde{n}$ be the dimension of the space range $[\mathcal{O}_\omega]$. Let $\tilde{H}$ be the $(\tilde{n}\text{x}n)$-matrix whose rows are the $\tilde{n}$ linearly independent rows of $\mathcal{O}_\omega$. Then for the existence of a stable filter, condition (8) may be replaced by

$$\text{rank } [\tilde{H}_e] = \nu, \tilde{H}_e := \tilde{H} U_1 \qquad (10)$$

4.2. General case.

4.2.1. Condition (7) imposes the constraint on the maximum singular value among those which do not participate in construction of the projection subspace. Introduce now

$$U_1 \text{ or } V_1 \subset R[H^T] \qquad (11)$$

Lemma 4.1. Consider system (1)(2) and let ν be a nonnegative integer satisfying constraint (8). Consider filter (3) and let $P_r = H^T, K_e = [HH^T]^+$ or $K = H^+$. Then under condition (11) the filter is exponentially stable.

Proof.

(i) $U_1 \subset R[H^T]$: $||I - KH|| = 1$ and $||L_2|| = ||[I - KH]U_2 D_2 V^T(2)|| \leq ||D_2|| < 1$.

(ii) $V_1 \subset R[H^T]$: $V_1^T[I - KH] = 0$ and for $\Phi = \Phi_1 + \Phi_2$ one has $\Phi_1[I - KH] = 0$ hence $\Phi[I - KH] = \Phi_2[I - KH]$ and $||\Phi_2[I - KH]|| < 1$. (End of proof)

Remark. Requirement (11) is relatively strong for H. Consider $\mathcal{O}_\omega$ defined in Comment 4.1. Suppose that there exists a positive number ω' such that

$$U_1 \text{ or } V_1 \subset R[\mathcal{O}_{\omega'}^T] \qquad (12)$$

Let ω_0 be a minimum integer number for which the above relationship holds. For convenience introduce the notation $\tilde{H} := \mathcal{O}_{\omega_0}$. Consider the filter

$$\hat{x}(t+1) = \Phi^\omega \tilde{x}(t+1-\omega), \qquad (13)$$

where

$$\begin{gathered} \tilde{x}(t+1-\omega) = \Phi\tilde{x}(t-\omega) + \tilde{K}\,\tilde{\zeta}(t+1-\omega), \\ \tilde{\zeta}(t+1-\omega) := \tilde{z}(t+1-\omega) - \tilde{H}\Phi\tilde{x}(t-\omega), \\ \tilde{z}(t+1-\omega) := \\ [z^T(t+1-\omega), z^T(t+2-\omega), ..., z^T(t+1)]^T \\ = \tilde{H}x(t+1-\omega) + \tilde{v}(t+1) \end{gathered}$$

Then the observation system is written as $\tilde{z} = \tilde{H}x$

Corollary 4.1. Consider system (1)(2) and let condition (12) hold. Consider filter (34) subject to the gain $K = \tilde{H}^+$. Then the filter is exponentially stable.

4.2.2. Next we will present the general result stating that there exists a stable PSF if all the left or the right unstable and neutral singular modes of the system are observable.

Definition 4.1 (Strejic, 1981). System state $x(t)$ is observable if it can be determined by future values of the system output $z(T), T > t$ and if $T - t$ is finite.

In what follows let in (1)(2) $w(t) = 0, v(t) = 0$.

Assumption 4.1. Any system state $x \in R[U_1]$ or $x \in R[V_1]$ is observable.

Lemma 4.2. Assumption 4.1 is equivalent to (12).

Proof. The proof is given for $x \in R[U_1]$ only.

(a) (12) holds: Consider the system of equations $\tilde{z} = \tilde{H}x'$. This system has solution since $w(t) = 0, v(t) = 0$. Let $x = U_1 y$ be a system state for some $y \neq 0$. From (12) it follows $x \in R[\tilde{H}^T]$. Theorem 3.1 (b) of Albert (1972) says that in this case the system $\tilde{z} = \tilde{H}x'$ has a unique solution and this solution is defined by $\hat{x} = \tilde{H}^+\tilde{z}$. Since x is a solution of $\tilde{z} = \tilde{H}x'$, i.e. $\tilde{z} \equiv \tilde{H}x$, one concludes that x can be defined exactly as $x = \hat{x}$ or x is observable.
(b) Let $x \in R[U_1]$ be observable. Again consider the system $\tilde{z} = \tilde{H}x'$. As this system has solution, all its solutions can be written as $\hat{x}_y = \tilde{H}^+\tilde{z} + \eta(y), \eta(y) := [I - \tilde{H}^+\tilde{H}]y, \forall y \in R^n$. Suppose for instant that (12) does not hold. It means that there exists $y' \neq 0$ such that $x = \tilde{H}^+\tilde{z} + \eta(y')$. The relation $\tilde{H}\eta(y') \equiv 0$ implies that one cannot obtain any information on the component $\eta(y')$ from $\tilde{z}$ hence it is impossible to define x from the system $\tilde{z} = \tilde{H}x'$. The last fact is contrary to observability of x. Thus observability of x implies $x = \tilde{H}^+\tilde{z}$ or $x \in R[\tilde{H}^T]$. (End of proof) .

Results of lemmas 4.1-4.2 prove

Theorem 4.2. Consider system (1)(2) and suppose that all the unstable and neutral left or right singular modes of the system dynamics are observable. Then there exists an exponentially stable filter.

Comment 4.2. From Theorem 4.2 it is reasonable to introduce the definition of *s-detectability* for the system as observability of all the left or the right unstable and neutral singular modes of the system dynamics.

Comment 4.3. As seen from proofs of Theorems 4.1-4.2, stability of a filter based on SVD approach does not depend on time-invariant character of the system dynamics. This constitutes one of major advantages of the SVD approach in comparison with the EVD approach in Hoang *et al.* (2001).

5. NUMERICAL EXAMPLES

5.1 Example 1

Let Φ be 2x2 matrix, all elements of Φ are zero except $\phi_{2,1} = a > 1$ (a may be time-varying). Evidently the SVD of Φ has $U = I_2, V^T = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, D =$ diag $[a, 0]$.

Let $H = (1,1)^T$. Let us take $P_r = u_1, K_e = H_e^+ = 1$. It is easy to check then that conditions (7)(8) are satisfied (rank $[H_e] = 1, \sigma_2(t) = 0 \leq \frac{\epsilon}{\sqrt{2}}$, $\epsilon \in (0,1)$) hence the filter in Theorem 4.1 is exponentially stable. Mention that for constant a, Φ has two stable eigenvalues (they are 0) and even the open-loop filter is stable. It means that in general it is not necessary to construct the projection subspace from all the unstable and neutral singular modes to ensure the stability of the PSF.

5.2 *Example 2*

Let Φ be 2x2 matrix, $\Phi = UDV^T$ with $U = I$, $D =$ diag $[\sigma_1, \sigma_2]$, $V^T = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$.

(a) Let $H = (1,1)$. Suppose that $\sigma_1 > 1, \sigma_2 < 1$, i.e. only the first singular mode is unstable.

$(a.1)$ First we show that involving all the left unstable and neutral modes in construction of the projection subspace is unsufficient for ensuring the stability of the PSF. Let $P_r = u_1 = (1,0)^T$. One can check that u_1 is observable. The gain in the reduced space is scalar and is denoted by $K_e = k_e$. Since

$$\Phi = \frac{1}{\sqrt{2}}\begin{vmatrix} \sigma_1 & \sigma_1 \\ -\sigma_2 & \sigma_2 \end{vmatrix}$$
$$I - KH = I - P_r K_e H = \begin{vmatrix} 1-k_e & -k_e \\ 0 & 1 \end{vmatrix}$$
$$L = [I - KH]\Phi =$$
$$\frac{1}{\sqrt{2}}\begin{vmatrix} (1-k_e)\sigma_1 + k_e\sigma_2 & (1-k_e)\sigma_1 - k_e\sigma_2 \\ -\sigma_2 & \sigma_2 \end{vmatrix}$$

the eigenvalues of $\tilde{L} := \frac{1}{\sqrt{2}}L$ satisfy the equation $a\tilde{\lambda}^2 + b\tilde{\lambda} + c = 0$ where $a = 1, b = -[(1-k_e)\sigma_1 + (1 + k_e\sigma_2)], c = 2\sigma_1\sigma_2(1-k_e)$ and they are given by $\tilde{\lambda_1} = \frac{[-b+\sqrt{\Delta}]}{2}$, $\tilde{\lambda_2} = \frac{[-b-\sqrt{\Delta}]}{2}$, $\Delta = b^2 - 4ac$. If two eigenvalues λ_1, λ_2 of L are stable, then it is necessary $|\lambda_1\lambda_2| < 1$. We have $\lambda_1\lambda_2 = \frac{\tilde{\lambda_1}\tilde{\lambda_2}}{2} = \frac{ac}{2} = \sigma_1\sigma_2(1-k_e)$. The requirement $|\lambda_1\lambda_2| < 1$ leads to $1 - \frac{1}{\sigma_1\sigma_2} < k_e < 1 + \frac{1}{\sigma_1\sigma_2}$. For (very) large σ_1 this constraint is equivalent to $k_e = 1$. Let us look at L for $k_e = 1$. Then $L = \frac{\sigma_2}{\sqrt{2}}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$ and L has two eigenvalues $\lambda_1 = \sqrt{2}\sigma_2$ and 0. Thus two eigenvalues of L are stable only if $\sigma_2 < \frac{1}{\sqrt{2}}$. For constant Φ, H, K the filter is time-invariant system and it is stable if and only if all eigenvalues of L are stable. We conclude then that the filter with the gain $K = u_1 k_e$ is stable iff $\sigma_2 < \frac{1}{\sqrt{2}}$ (at least after a finite time instant t). Evidently if $\frac{1}{\sqrt{2}} \leq \sigma_2 < 1$ there does not exist k_e ensuring the stability of the PSF.

$(a.2)$ Looking at V one sees that the right unstable v_1 belongs to $R[H^T]$ hence one can design a stable filter by projecting the correction onto subspace $R[P_r], P_r = v_1$ with $K = H^+$. No constraint like $\sigma_2 < \frac{1}{\sqrt{2}}$ is needed for ensuring a stability of the filter.

(b) Another way to remove the constraint $\sigma_2 < \frac{1}{\sqrt{2}}$ using the left singular vector u_1 is to follow Corollary 4.1. Since u_1 is observable ($\tilde{H}$ is nonsingular), one can design a stable filter (13) on the basis of the observation system $\tilde{z} = \tilde{H}x$. Since $\tilde{H} = [H^T, (H\Phi)^T]^T$ is nonsingular, the gain $\tilde{K} = \tilde{H}^+ = \tilde{H}^{-1} = \frac{1}{\delta}\begin{vmatrix} \frac{\sigma_1+\sigma_2}{\sqrt{2}} & -1 \\ -\frac{\sigma_1-\sigma_2}{\sqrt{2}} & 1 \end{vmatrix}$, $\tilde{L} = 0$ and

$$\tilde{x}(t) = \tilde{L}\tilde{x}(t-1) + \tilde{K}\tilde{z}(t),$$
$$\tilde{K}\tilde{z}(t) = \frac{1}{\delta}\begin{vmatrix} \frac{\sigma_1+\sigma_2}{\sqrt{2}}z(t) - z(t+1) \\ -\frac{\sigma_1-\sigma_2}{\sqrt{2}}z(t) + z(t+1) \end{vmatrix}$$

from which one obtains

$\hat{x}(t+1) = \Phi\tilde{x}(t) = [\frac{\sigma_1}{\sqrt{2}}z(t), z(t+1) - \frac{\sigma_1}{\sqrt{2}}z(t)]^T$.

For noisy free system, $x(t+1) = \Phi x(t)$,

$$x(t+1) = \begin{vmatrix} \frac{\sigma_1}{\sqrt{2}}[x_1(t) + x_2(t)] \\ \frac{\sigma_2}{\sqrt{2}}[x_1(t) - x_2(t)] \end{vmatrix}$$

.

For $H = (1,1)$ one finds $z(t) = x_1(t) + x_2(t)$ hence $\frac{\sigma_1}{\sqrt{2}}z(t) = \frac{\sigma_1}{\sqrt{2}}[x_1(t) + x_2(t)] = x_1(t+1)$ and $z(t+1) - \frac{\sigma_1}{\sqrt{2}}z(t) = x_2(t+1)$. Thus the filter yields $\hat{x}(t+1) = x(t+1)$. The reason by which filter in Corollary 4.1 produces exactly the system state for noisy-free system is that both u_1, u_2 are observable and $\tilde{H}$ with minimal rank satisfying the condition $u_1 \in R[\tilde{H}^T]$ is nonsingular. The system $\tilde{z}(t) = \tilde{H}x(t)$ hence allows to define exactly $x(t)$.

5.3. Example 3

$\Phi(t)$ has $\phi_{11} = \phi_{22} = 0$ for all t but $\phi_{12} = a > 1$, $\phi_{21} = 0$ for odd t and $\phi_{21} = a$, $\phi_{12} = 0$ for even t. It is clear that the eigenvalues of $\Phi(t)$ are stable. The open loop filter has the fundamental matrix $L(t+1,\tau) := L(t)...L(\tau), t \geq \tau$, $L(t,t) = I$ which is equal to : (1) if $\mu := t + 1 - \tau$ is odd then all the elements of $L(t+1,\tau)$ are equal to 0 except its (1,2)-element which is equal to a^μ; (2) for even μ, all the elements of $L(t+1,\tau)$ are zero except its (1,1)-element equal to a^μ. Thus the open-loop estimator is unstable. The filter in Hoang *et al.* (2001) is thus applicable only for time-invariant systems.
On the other hand, the system dynamics has only one unstable singular mode $u_1 = u_1(t)$ ($u_1(t) = (1,0)^T$ for odd t and $u_1(t) = (0,1)^T$ for even t). Let $H = (1,1)$. Evidently the unstable singular mode is observable. The closed-loop estimator in Theorem 4.1 subject to $\nu = 1$ has the transition matrix $L(1) = 0$ (from $\sigma_2 = 0$ it follows $\tilde{D} = 0$ and hence $L_2 = 0$. Hence the filter is stable.

6. TWIN EXPERIMENT ON ESTIMATION OF PERIODIC DEPENDENCIES

6.1 Numerical model

6.1.1. Continuous model. In this section the experiment on estimation of periodic dependencies which arises in all domaines of scientific research Bellman (1965) is presented. Let the function $f(t)$ have the form

$$f(t) = \sum_{i=1}^{m} \alpha_i cos(\omega_i t) \tag{14}$$

The parameters m, α_i and ω_i are usually unknown and the problem is to estimate these parameters using the set of obsevations $f(t_j), j = 1, ..., N$. Following the approach in Bellman (1965), the observations $z(t) = f(t)$ are presented in the form

$$z(t) = f(t) = \sum_{i=1}^{m} u_i(t) \tag{15}$$

where $u_i(t)$ - solution of the equations

$$u_i''(t) + \omega_i^2 u_i(t) = 0, u_i(0) = \alpha_i, u_i'(0) = 0 \tag{16}$$

In the present experiment we assume that m is known and $m = 3$.

6.1.2. Discreteized model. Applying the scheme $u''(t) \approx \frac{u(t)-2u(t-\delta t)+u(t-2\delta t)}{\delta^2(t)}$, $u'(t) \approx \frac{u(t)-u(t-\delta t)}{\delta t}$ and representing the discretized version of system (16) in the state space form $x(t) = (x_1, ..., x_6)^T = [u_1(t), u_1(t-\delta t), u_2(t), u_2(t-\delta t), u_3(t), u_3(t-\delta t)]^T$ yields the following discrete-time system

$$x(t+1) = Ax(t) \tag{17}$$

where we use the formalism $[t+1] - [t] = \delta t$. The matrix A = block diag $[A_1, A_2, A_3]$ with A_i is (2x2) matrix. The elements of A_i are $a_{11} = 2. - (\delta t \omega_i)^2$, $a_{12} = -1, a_{21} = 1$ and $a_{22} = 0$. Thus in the state space form the system state $x(t) \in R^6$. The true values of the parameters are (Bellman, 1965): $\alpha_1 = 1$, $\alpha_2 = 0.5, \alpha_3 = 0.1$; $\omega_1 = 1.11$, $\omega_2 = 2.03, \omega_3 = 3.42$. The parameter $\delta t = 0.01$. First (16) is integrated subject to true values (α_i, ω_i). The zero mean uncorrelated Gaussian noise is added at each model time step to the components x_1, x_3, x_5 with corresponding variances 0.0025, 0.0001, 0.0001. The obtained thus values of $x(t)$ are used as the "true" state. The noisy observations z_{t_o} are assumed to be available at the moments t_o, $z_{t_o} = f(t_o) = x_1(t_o) + x_3(t_o) + x_5(t_o) + v(t_o)$ where t_o is defined as $[t_o + 1] - [t_o] = 10\delta t$ and $v(t_o)$ is zero mean uncorrelated Gaussian sequence with the variance $R = 0.0001$. The observation matrix H thus is $H = (1, 0, 1, 0, 1, 0)$.

6.2 Experiments

Assume that we are given a-priori $\alpha_1^0 = 0.9$, $\alpha_2^0 = 0.6, \alpha_3^0 = 0.2$. The parameters ω_i are supposed to be known precisely. Thus the transition matrix A between two model time steps as well as $\Phi = A^{10}$ between two assimilation (observation) instants can be computed exactly and one can apply the "true" KF (between two assimilation instants we have $x(t_o + 1) = \Phi x(t_o), [t_o + 1] - [t_o] = 10\delta t)$. On the other hand, the initial gain in the AF will be computed on the basis of Φ_1 which is a leading part in the SVD of Φ, i.e. $\Phi = \Phi_1 + \Phi_2$, $\Phi_1 = U_1 D_1 V_1^T$, U_1, V_1 consist of the left and the right unstable singular modes of Φ respectively.

Three filters will be applied in the experiment:

(i) "True" Kalman filter (TKF): That is a standard KF with all parameters and statistics well specified except for the model error covariance Q. As this matrix is unknown, the following procedure is applied to obtain its best estimate: First we run the KF subject to different $Q = qI$. The total variances of the filtered error, obtained at $t_o = 1000$, are compared for different q varying from 0 to 100. It is found that filtered error decreases as q increases and it is stabilized from about $q \geq 0.6$. For simplicity in the KF we assign $q = 1$.

(ii) Non-adaptive PSF (NAF): this filter has the time-invariant gain $K^0 = \tilde{M}H^T[H\tilde{M}H^T]^+$, $\tilde{M} = \Phi_1\Phi_1^T$. Evidently the correction in the filter is an element of the subspace $R[U_1]$.

(iii) Adaptive filter (AF): In this filter the gain has the form $K = \Lambda K^0$, Λ = diag $[\lambda_1, ..., \lambda_6]$ which are adjusted to minimize the prediction error. The initial values of $\lambda_i, i = 1, ..., 6$ are equal to 1 which correspond to NAF.

6.3 Numerical results

The computation reveals that Φ has 3 unstable singular modes. The matrix Φ_1 has therefore the rank 3.

Fig. 1 shows variances of forecast errors produced by three filters. As expected, the TKF produces the best estimate. In comparison with the NAF, the AF allows to reduce significantly the estimation error and at the end of the assimilation period its performance is comparable with that of the TKF. As seen from Table 1, at $t_o = 200$, the AF is capable of reducing the estimation error in the NAF from 0.752 to 0.366 or equivalently the reduction of 51 % of this error. The reduction attained 97% at the end of the assimilation period $t_o = 1000$. Regarding the AF at two different time instants $t_o = 200$ and $t_o = 1000$ on notes also the reduction of 93 % of the estimation error resulting initially at $t_o = 200$. It is worthy of mentioning that if in the KF one put $Q = 0$ as done in the AF, then at $t_o = 1000$ the variance of the filtered error is 9.3. Moreover the KF is explosed after some iterations if

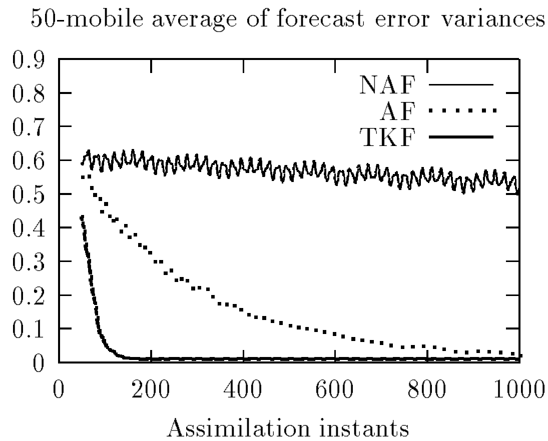

Fig. 1. Performances of three filters NAF, AF and TKF in terms of forecast error variances (mobile averages over 50 assimilation instants) for the system state.

its gain is updated using the approximation $\Phi \approx \Phi_1$ in the RDE as it does in the AF. Thus the KF is really the best filter only under ideal conditions, i.e. when there are given exactly all system parameters and error statistics. Otherwise it may produce a poor estimate or even diverges.

Table 1

Total error variances of filtered estimates

t_o	NAF	AF	TKF
200	0.752	0.366	0.004
1000	0.694	0.023	0.013

7. CONCLUSION

We presented in this work some theoretical results on the (exponential) stability of the PSF. The modifications of the structure of the gain for adaptation purpose can be made following the EVD approach in Hoang *et al.* (2001). In contrast to the EVD approach, where using all the unstable and neutral eigenmodes of the transition matrix is necessary and sufficient for the existence of a stable filter (of course, for time-invariant system), the design of a stable PSF requires much more careful examination of observability of either the left or the right unstable (and neutral) singular modes in the SVD structure.

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