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STABILITY OF FAULT TOLERANT CONTROL SYSTEMS DRIVEN BY ACTUATORS WITH SATURATION

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Abstract: A Fault Tolerant Control System (FTCS) model subject to random faults and saturation in actuators is developed. Actuators are assumed to have non-linear saturation characteristics, they operate linearly within their limits and saturate to fixed levels if any limit is exceeded. This paper derives sufficient conditions for the exponential stability in the mean square for a FTCS with saturated actuators using Lyapunov's second method. Sufficient conditions involve the solution of Riccati-like matrix equations. An algorithm to investigate the stability of the FTCS is constructed. Stability of other control models driven by actuators with saturation can be established as interesting special cases of this work. Theoretical results are illustrated by a numerical example.

Keywords: Fault Tolerant Control Systems, Actuator Saturation, Exponential Stability.

1. INTRODUCTION

Modern technological systems, such as aircraft, space stations and nuclear power plants, rely on sophisticated control functions to meet increased performance requirements. For these safety-critical systems Fault Tolerant Control Systems (FTCS) have been developed to improve reliability, maintainability and survivability. The objective of FTCS is to achieve acceptable performance not only during normal system operation but also when there are malfunctions in sensors, actuators, or plants. In general, the task of FTCS can be decomposed into two functions: the first is to detect the existence of faults and to identify the fault-induced changes using a Fault Detection and Identification (FDI) algorithm and the second is to reconfigure the control law by certain reconfiguration mechanism. Since faults are random in nature and FDI decision is based on statistical tests, FTCS can be represented by stochastic differential equations. The stochastic description of FTCS can be viewed as a general hybrid system. Fundamentally, FTCS should be concerned with practical control systems in real environment. Therefore, several substantial results on the stability of FTCS have been developed in an attempt to deal with issues arising in practical applications. The stochastic stability of FTCS was studied by Srichander and Walker (1993), the effect of detection-delays and false alarms was considered by Mariton (1989). More recently, stochastic stability with multiple faults was treated in Mahmoud *et al* (2001a), stability of FTCS subject to environment noises was developed in Mahmoud *et al* (2001b). Closely related to FTCS, another major class of hybrid systems is Jump Linear Systems (JLS). Research in JLS is classified into two categories: The first concerns with deriving necessary and/or sufficient conditions for the existence of an optimal quadratic regulator (Boukas, 1993; Wonham, 1971). The second category deals with the properties of JLS such as stability, controllability and observability (Feng *et al*, 1992; Ji and Chezick, 1990).

A vital problem which usually arises in a practical control system is actuator saturation. Valves are examples of actuators with saturation used in process control. A valve has a range of operation limited by being fully opened and fully closed. Unfortunately, the physical limitation of actuator saturation is usually unavoidable. If such behavior is not taken into account, an integral wind-up may be induced which leads to a large overshoot in system response, a limit cycle or an unstable closed-loop system (Glattfelder 1983). The consequences of actuator saturation are more serious when the system encounters sudden changes such as faults.

Special interest was devoted to the problem of feedback control with nonlinear saturating actuator for deterministic systems. The stability analysis of a continuous system with saturating actuators was studied for SISO systems employing the Popov's criteria (Glattfelder, 1983; Krikelis, 1980). The tracking problem was considered by Krikelis and Barakas (1984). The problem was treated in time domain by Chen and Wang (1986, 1988). The combined problem of actuator saturation with state-delays was reported by Chen *et al* (1988), and with parameter uncertainties by Niculescu (1996).

Despite the urgent need to consider this crucial issue, to our knowledge, the problem of nonlinear saturation for FTCS has not been addressed. Very recently, constrained quadratic state feedback control of discrete-time JLS was discussed by Costa *et al* (1999). In this work, actuators are forced to operate only in the linear region to avoid saturation. However, the affect of actuator saturation on system stability was not considered. Therefore, the current work mainly elaborates how actuator saturation affect stability of FTCS.

In this paper, a FTCS model subject to random faults and saturation in actuators is developed. Actuators have non-linear characteristics, they operate linearly within their upper and lower limits and may saturate to fixed levels if any limit is exceeded. In particular, this work defines and derives sufficient conditions for the exponential stability in the mean square of FTCS with saturated actuators. The derivations will be completed employing Lyapunov's second method. Sufficient conditions involve the solution of Riccati-like matrix equations. Moreover, stochastic stability of JLS driven by saturated actuators can be established as an interesting special case of this work. Stability of FTCS without saturating actuators is compared with the stability of FTCS with saturation.

2. MATHEMATICAL FORMULATION

2.1 *Dynamical Model*

A general Fault Tolerant Control System with Saturated Actuators (FTCSSA) is shown in Figure 1. The FTCSSA subject to random faults in actuators is described by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(\eta(t))Sat[u(x(t), \Psi(t), t)] \\ Sat[u(x(t), \Psi(t), t)] &= Sat[-K(\Psi(t))x(t)] \end{aligned} \quad (1)$$

where $x(t) \in \Re^n$ is the system state, $u(x(t), \Psi(t), t) \in \Re^m$ is the input, $\eta(t)$ is the failure process, and $\Psi(t)$ is the FDI process. $\eta(t)$ and $\Psi(t)$ are homogeneous Markov processes with finite state spaces $S = \{1, 2, ..., s\}$ and $R = \{1, 2, ..., r\}$, respectively. The transition probability for the actuator failure

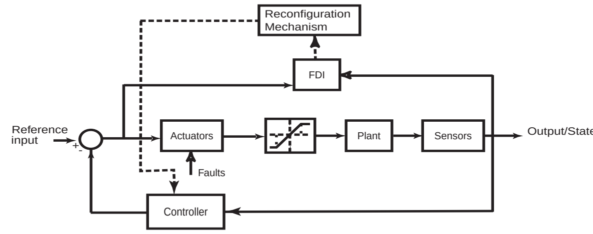

Fig. 1: General schematic diagram for FTCSSA.

process, $\eta(t)$, is defined as

$$p_{kj}(\Delta t) = \alpha_{kj}\Delta t + o(\Delta t) \;\; (k \neq j) \quad (2)$$

where α_{kj} represents the actuator failure rate. Given that $\eta(t) = k \in S$, the conditional transition probability of the FDI process, $\Psi(t)$, is defined as

$$p^k_{ij}(\Delta t) = q^k_{ij}\Delta t + o(\Delta t) \;\; (i \neq j) \quad (3)$$

where q^k_{ij} is the rate that the FDI process will decide that the next state is j leaving the state i, given that the actuator failure process is in the state k (Mahmoud *et al* 2001,b; Srichander and Walker, 1993).

In the sequel, we will use the following notations: $B(\eta(t)) = B_k$ when $\eta(t) = k \in S$ and $Sat[u(x(t), \Psi(t), t)] = Sat(u_i)$ when $\Psi(t) = i \in R$. Also denote $x(t) = x,\ \eta(t) = \eta,\ \Psi(t) = \Psi$ and the initial conditions $x(t_o) = x_o,\ \eta(t_o) = \eta_o,\ \Psi(t_o) = \Psi_o$.

2.2 *Actuator With Saturation*

Figure 2 shows the characteristic diagram of actuators with saturation. The actuator, described by a static non-linear function which saturates at u_H and u_L, is defined as

$$Sat[u(x(t), \Psi(t), t)] = \begin{cases} u_H & u(x(t), \Psi(t), t) > u_H \\ -K(\Psi(t))x(t), & u(x(t), \Psi(t), t) \in [u_L, u_H] \\ u_L & u(x(t), \Psi(t), t) < u_L \end{cases} \quad (4)$$

In which the operation of the $Sat[u(x(t), \Psi(t), t)]$ is linear for all $u(x(t), \Psi(t), t) \in [u_L, u_H]$. u_L and u_H are the upper and the lower limits of the actuator, respectively. In view of Figure 2 and the axioms of the norm function, we have

$$\| \; Sat[u(x(t), \Psi(t), t)] - \frac{1}{2}u(x(t), \Psi(t), t) \; \| \leq \frac{1}{2} \; \| \; u(x(t), \Psi(t), t) \; \| \quad (5)$$

The system (1) in the linear region, $u(x(t), \Psi(t), t) \in [u_L, u_H]$,

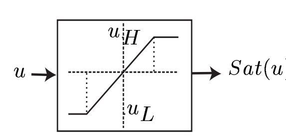

Fig. 2: Characteristics of actuator with saturation.

is assumed to satisfy both the growth and the uniform Lipschitz conditions. Under these conditions, the joint process $\{x(t), \eta(t), \Psi(t)\}$ is a Markov process. It is assumed that perfect state information is available for feedback.

3. EXPONENTIAL STABILITY OF FTCSSA

Without loss of generality, we assume that the equilibrium solution, $x = 0$, is the solution whose stability properties are being tested. At this equilibrium solution, the exponential stability in the mean square of FTCSSA (1) is defined as

Definition 1. The equilibrium solution, $x = 0$, of the FTCSSA (1) is said to be exponentially stable in the mean square, if for any $\eta_o \in S$ and $\Psi_o \in R$, there exist $\delta(\eta_o, \Psi_o) > 0$ and some positive constants $a > 0$ and $b > 0$, such that when $\|x_o = x(\eta_o, \Psi_o, t_o)\| \leq \delta$, the following inequality holds $\forall t \geq t_o$,

$$\mathcal{E}\{||x(t; x_o, t_o)||^2\} \leq a||x_o||^2 \exp\{-b(t - t_o)\}$$

Sufficient conditions for the exponential stability in the mean square are stated in the following Theorem

Theorem 1. The equilibrium solution, $x = 0$, of the FTCSSA (1) is exponentially stable in the mean square $\forall\, t \geq 0$, if there exists a stochastic Lyapunov function $V(x, \eta, \Psi)$ for some constants $k_1 > 0, k_2 > 0$, and $k_3 > 0$, such that

(a) $k_1 \parallel x \parallel^2 \leq V(x, \eta, \Psi) \leq k_2 \parallel x \parallel^2$

(b) $\ell V(x, \eta, \Psi) \leq -k_3 \parallel x \parallel^2$

The proof of this theorem employs the supermartingale properties of a stochastic Lyapunov function $V(x, \eta, \Psi)$. Similar theorem has been derived in Mahmoud *et al* (2001a), therefore, proof will not be detailed here to avoid repetition. In Theorem 1, $\ell V(x, \eta, \Psi)$ is the weak infinitesimal operator of the FTCSSA (1). For a quadratic stochastic Lyapunov function

$$V(x, \eta, \Psi) = x^T P(\eta, \Psi) x \tag{6}$$

where $P(\eta, \Psi)$ is positive symmetric matrices $\forall\ \eta = k \in S$ and $\Psi = i \in R$, the weak infinitesimal operator is

$$\begin{aligned}\ell V(x, \eta, \Psi) = {} & x^T A^T P_{ki} x + x^T P_{ki} A x + x^T \{\sum_{\substack{j \in S \\ j \neq k}} \alpha_{kj} \left[P_{ji} - P_{ki}\right]\} x \\ & + x^T \{\sum_{\substack{j \in R \\ j \neq i}} q_{ij}^k \left[P_{kj} - P_{ki}\right]\} x + x^T P_{ki} B_k [Sat(u_i) - \frac{1}{2} u_i + \frac{1}{2} u_i] \\ & + [Sat(u_i) - \frac{1}{2} u_i + \frac{1}{2} u_i]^T B_k^T P_{ki} x\end{aligned} \tag{7}$$

Define

$$\tilde{A}_{ki} = A - \frac{1}{2} B_k K_i - \frac{1}{2} I \sum_{\substack{j \in S \\ j \neq k}} \alpha_{kj} - \frac{1}{2} I \sum_{\substack{j \in R \\ j \neq i}} q_{ij}^k \tag{8}$$

Under the state feedback $u_i = -K_i x$, the weak infinitesimal operator is rewritten as

$$\begin{aligned}\ell V(x, \eta, \Psi) = {} & x^T \{\tilde{A}_{ki}^T P_{ki} + P_{ki} \tilde{A}_{ki} + \sum_{\substack{j \in S \\ j \neq k}} \alpha_{kj} P_{ji} + \sum_{\substack{j \in R \\ j \neq i}} q_{ij}^k P_{kj}\} x \\ & + x^T P_{ki} B_k [Sat(u_i) - \frac{1}{2} u_i] + [Sat(u_i) - \frac{1}{2} u_i]^T B_k^T P_{ki} x\end{aligned} \tag{9}$$

For the selected Lyapunov function (6), we have

$$\lambda_{\min}(P(\eta, \Psi)) \parallel x \parallel \leq V(x, \eta, \Psi) \leq \lambda_{\max}(P(\eta, \Psi)) \parallel x \parallel \tag{10}$$

Note that (10) satisfies the boundeness condition in Theorem 2 in Mahmoud *et al* (2001b). As per Theorem 1, The FTCSSA is exponentially stable in the mean square if the weak infinitesimal operator (9) satisfies the smoothness condition in Theorem 2. Therefore, the exponential stability of FTCSSA can be defined as follows

Definition 2. The FTCSSA (1) is said to be exponentially stable in the mean square under the linear feedback control law $u_i(t) = -K_i x(t)$, if there exist positive-definite symmetric matrices P_{ki} such that the following matrix inequality holds

$$\begin{aligned}& x^T \{\tilde{A}_{ki}^T P_{ki} + P_{ki} \tilde{A}_{ki} + \sum_{\substack{j \in S \\ j \neq k}} \alpha_{kj} P_{ji} + \sum_{\substack{j \in R \\ j \neq i}} q_{ij}^k P_{kj}\} x + \\ & x^T P_{ki} B_k [Sat(u_i) - \frac{1}{2} u_i] + [Sat(u_i) - \frac{1}{2} u_i]^T B_k^T P_{ki} x \leq -\alpha \parallel x \parallel^2\end{aligned}$$

$\tilde{A}_{ki}$ is defined in (8).

4. A TESTABLE SUFFICIENT CONDITION FOR EXPONENTIAL STABILITY OF FTCSSA

In this section, an easy-to-test sufficient condition will be derived. This condition involves the solution of Riccati-like matrix equation. In the literature, several algorithms are available to solve Riccati matrix equation. These algorithms are easily extended to solve the matrix equation which results from this work.

The following theorem states sufficient condition for the exponential stability of the FTCSSA in terms of Riccati-like matrix equations.

Theorem 2. If the Riccati-like matrix equations

$$\hat{A}_{ki}^T P_{ki} + P_{ki} \hat{A}_{ki} + \sum_{\substack{j \in S \\ j \neq k}} \alpha_{kj} P_{ji} + \sum_{\substack{j \in R \\ j \neq i}} q_{ij}^k P_{kj} - P_{ki} B_k R_{ki}^{-1} B_k^T P_{ki} + Q_{ki} = 0 \tag{11}$$

have positive-definite solutions, P_{ki}, for given positive-definite weighting matrices Q_{ki} and R_{ki}. Then, the FTCSSA (1) is exponentially stable in the mean square under the control law

$$u_{ki}(t) = -R_{ki}^{-1} B_k^T P_{ki} x(t) \tag{12}$$

if

$$\lambda_{\min}(Q_{ki}) - \frac{\left[\lambda_{\max}(P_{ki})\right]^2}{\lambda_{\min}(R_{ki})} \parallel B_k \parallel^2 > 0 \tag{13}$$

where

$$\hat{A}_{ki} = A - \frac{1}{2} I \sum_{\substack{j \in S \\ j \neq k}} \alpha_{kj} - \frac{1}{2} I \sum_{\substack{j \in R \\ j \neq i}} q_{ij}^k \tag{14}$$

Proof: The weak infinitesimal operator of the FTCSSA (1) is given in (7). If we define $\hat{A}_{ki}$ as in (14), under the control law (12), the weak infinitesimal operator becomes:

$$\begin{aligned}\ell V(x,\eta,\Psi) &= x^T \hat{A}_{ki}^T P_{ki} x + x^T P_{ki} \hat{A}_{ki} x \\ -x^T P_{ki} B_k R_{ki}^{-1} B_k^T P_{ki} x &+ x^T \{\sum_{\substack{j \in S \\ j \neq k}} \alpha_{kj} P_{ji}\} x + x^T \{\sum_{\substack{j \in R \\ j \neq i}} q_{ij}^k P_{kj}\} x \\ +x^T P_{ki} B_k [Sat(u_i) - \frac{1}{2} u_i] &+ [Sat(u_i) - \frac{1}{2} u_i]^T B_k^T P_{ki} x \end{aligned} \tag{15}$$

Since

$$x^T P_{ki} B_k [Sat(u_i) - \frac{1}{2} u_i] + [Sat(u_i) - \frac{1}{2} u_i]^T B_k^T P_{ki} x \in R^1 \tag{16}$$

Then

$$\begin{aligned} &x^T P_{ki} B_k [Sat(u_i) - \frac{1}{2} u_i] + [Sat(u_i) - \frac{1}{2} u_i]^T B_k^T P_{ki} x = \\ &2x^T P_{ki} B_k [Sat(u_i) - \frac{1}{2} u_i] \leq \parallel 2x^T P_{ki} B_k [Sat(u_i) - \frac{1}{2} u_i] \parallel \end{aligned} \tag{17}$$

The axioms of norm and (5) yield

$$\parallel 2x^T P_{ki} B_k [Sat(u_i) - \frac{1}{2} u_i] \parallel \leq \parallel x^T P_{ki} B_k \parallel \parallel R_{ki}^{-1} B_k^T P_{ki} x \parallel \tag{18}$$

Then

$$\begin{aligned} \ell V(x,\eta,\Psi) &\leq x^T \hat{A}_{ki}^T P_{ki} x + x^T P_{ki} \hat{A}_{ki} x \\ &+ x^T \{\sum_{\substack{j \in S \\ j \neq k}} \alpha_{kj} P_{ji}\} x + x^T \{\sum_{\substack{j \in R \\ j \neq i}} q_{ij}^k P_{kj}\} x - x^T P_{ki} B_k R_{ki}^{-1} B_k^T P_{ki} \\ &+ x \parallel x^T P_{ki} B_k \parallel \parallel R_{ki}^{-1} B_k^T P_{ki} x \parallel \end{aligned} \tag{19}$$

From (11), we have

$$\begin{aligned} \ell V(x,\eta,\Psi) &\leq -x^T Q_{ki} x + \parallel x^T P_{ki} B_k \parallel \parallel R_{ki}^{-1} B_k^T P_{ki} x \parallel \\ &\leq -x^T Q_{ki} x + \parallel P_{ki} \parallel^2 \parallel R_{ki}^{-1} \parallel \parallel B_k \parallel \parallel B_k^T \parallel \parallel x \parallel^2 \end{aligned}$$

The definition of the induced Euclidean norm gives

$$\ell V(x,\eta,\Psi) \leq -\left[\lambda_{\min}(Q_{ki}) - \frac{\left[\lambda_{\max}(P_{ki})\right]^2}{\lambda_{\min}(R_{ki})} \parallel B_k \parallel^2 \right] \parallel x \parallel^2 \tag{20}$$

Define real constants γ_{ki}

$$\gamma_{ki} = \lambda_{\min}(Q_{ki}) - \frac{\left[\lambda_{\max}(P_{ki})\right]^2}{\lambda_{\min}(R_{ki})} \parallel B_k \parallel^2 > 0 \tag{21}$$

Hence, there exist some $\gamma_{ki} > 0$, such that

$$\ell V(x,\eta,\Psi) \leq -\gamma_{ki} \parallel x \parallel^2 < 0 \tag{22}$$

The conditions of Theorem 1 are satisfied, therefore, the FTCSSA is exponentially stable in the mean square. The proof is completed.

Test algorithm

The exponential stability in the mean square of the FTCSSA (1) can be tested as follows:

1) Select positive-definite matrices Q_{ki}^o and R_{ki}^o.
2) Solve the matrix Riccati-like equations in Theorem 2. If positive-definite solutions exist and satisfy condition (13), then FTCSSA is exponentially stable in the mean square. Otherwise, go to Step 3.
3) Increase Q_{ki} by some factor. Say $Q_{ki}^1 = 2Q_{ki}^o$. Goto Step 2 and iterate.
4) If the algorithm did not succeed to give positive-definite solutions, then stop. Declare that exponential stability of FTCSSA cannot be judged.

5. REMARKS AND SPECIAL CASES

Remark 1: Under the assumption of perfect FDI performance, i.e. instantaneous fault detection and perfect fault isolation, both the failure process and the FDI process will have identical state spaces. That is, the two random processes $\eta(t)$ and $\Psi(t)$ can be replaced by a single process denoted as $r(t)$. Similar to $\eta(t)$ and $\Psi(t)$, the process $r(t)$ represents a continuous time discrete state Markov process with values in a finite set $\Upsilon = \{1, 2, ..., N\}$ with transition probability rate matrix $\Xi = [\Phi_{ij}]_{i,j=1,N}$.

In this case, the transition probability for the jump process, $r(t)$, can be defined as

$$P_{kj}(\Delta t) = \Phi_{kj} \Delta t + o(\Delta t) \;\; (k \neq j) \tag{23}$$

with $\sum_{\substack{j=1 \\ j \neq i}}^{N} \Phi_{ij} = -\Phi_{ii} = \Phi_i$.

With this assumption, the system when is driven by actuators with saturation, can be modeled as

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(r(t)) u_s(x(t), r(t), t) \\ u_s(x(t), r(t), t) &= \begin{cases} u_H & u(x(t), r(t), t) > u_H \\ -K(r(t))x(t) & u_L < u(x(t), r(t), t) < u_H \\ u_L & u(x(t), r(t), t) < u_L \end{cases} \end{aligned} \tag{24}$$

Following similar arguments, Corollary 1 states sufficient condition for the exponentially stability of the system (24).

Corollary 1. The system with saturating actuators (24) is exponentially stable in the mean square if there exist positive-definite symmetric solutions, P_i, to the following Riccati-like matrix equation

$$\breve{A}_i^T P_i + P_i \breve{A}_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \Phi_{ij} P_j - P_i B_i R_i^{-1} B_i^T P_i + Q_i = 0 \quad (25)$$

and

$$\lambda_{\min}(Q_i) - \frac{\left[\lambda_{\max}(P_i)\right]^2}{\lambda_{\min}(R_i)} \parallel B_i \parallel^2 > 0 \quad (26)$$

$\forall\ Q_i > 0,\ \ R_i > 0$. The linear control law is given as

$$u_i = -R_i^{-1} B_i^T P_i x(t) \quad (27)$$

where

$$\breve{A}_i = A - \frac{1}{2} \Phi_i I \quad (28)$$

The model of the system (24) is similar to the model of JLS. Therefore, Corollary 1 can be used to examine the exponential stability of JLS driven by actuators with saturation.

Remark 2: If only part of the nonlinear saturation characteristic is to be considered during the actual system operation, then less conservative results can be obtained. In this case, the saturation will be restricted to the sector $[a, 1]$, with $0 \leq a \leq 1$ instead of the original sector $[0, 1]$ as shown in Figure 3. The axioms of norm function gives the following inequality

$$\parallel Sat(u_i) - \frac{1}{2}(1+a)u_i \parallel \leq \frac{1}{2}(1-a) \parallel u_i \parallel \quad (29)$$

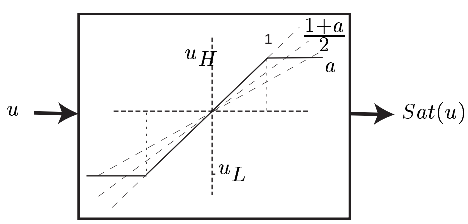

Fig. 3: Characteristics of actuator with saturation in a sector [a,1].

Following similar arguments used to derive sufficient conditions for the stability of FTCSSA (1) with saturation in the sector $[0, 1]$, we obtain the following lemma

Lemma 1. The FTCSSA (1) with saturation non-linearities in the sector $[a, 1]$ is exponentially stable in the mean square under the control law

$$u_{ki}(t) = -R_{ki}^{-1} B_k^T P_{ki} x(t) \quad (30)$$

where P_{ki} are bounded positive-definite symmetric solutions to the following Riccati-like matrix equation

$$\hat{A}_{ki}^T P_{ki} + P_{ki} \hat{A}_{ki} + \sum_{\substack{j \in S \\ j \neq k}} \alpha_{kj} P_{ji} + \sum_{\substack{j \in R \\ j \neq i}} q_{ij}^k P_{kj} - (1+a) P_{ki} B_k R_{ki}^{-1} B_k^T P_{ki} + Q_{ki} = 0 \quad (31)$$

$\forall\ Q_{ki} > 0,\ \ R_{ki} > 0$, and $\hat{A}_{ki}$ is given in (14).

Proof: can be adopted similar to Theorem 2.

This result can be considered as a general form to examine exponential stability of FTCS.

- If $a = 0$, all actuator saturation is considered during system operation, we obtain the results of Theorem 2.
- If $a = 1$, actuators are assumed to be linear without saturation during system operation, we obtain the results of Theorem 5 in Srichander and Walker 1993.

6. A NUMERICAL EXAMPLE

Consider a system with one possible actuator fault. The system and other design parameters are given as follows:

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

Actuator failure rates are assumed to be

$$[\alpha_{jk}] = \begin{bmatrix} -.005 & .005 \\ .001 & -.001 \end{bmatrix}$$

The FDI conditional transition rates are

$$[q_{ij}^1] = \begin{bmatrix} -0.575 & 0.575 \\ 2.90 & -2.90 \end{bmatrix},\ [q_{ij}^2] = \begin{bmatrix} -2.10 & 2.10 \\ 1.06 & -1.06 \end{bmatrix}$$

The initial weighting matrices used are

$$Q_{11} = Q_{12} = 5I,\ Q_{21} = Q_{22} = 10I,\ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$R_{11} = 0.75\ \ R_{12} = 1.25\ \ R_{21} = 0.75\ \ R_{22} = 1.25$$

The stochastic stability of the FTCSSA with possible actuator fault is to be investigated. To illustrate the theoretical results developed in this work, the example is solved for different scenarios.

CASE 1: Actuator saturation and stability of FTCS

If the conditions of Theorem 2 are satisfied then the FTCSSA is exponentially stable in the mean square, otherwise, stability cannot be judged. Three scenarios are considered: FTCSSA with all non-linearities due to saturation; FTCSSA with only part of non-linearities, that is, actuators operate within the sector $[a, 1]$; and FTCS without actuator saturation. positive-definite solutions of Riccati-like matrix equations and associated constants γ_{ki} for the three scenarios are summarized in Table 1 and 2, respectively.

Sufficient conditions for the stability of FTCSSA necessitate that the Riccati-like equations have positive-definite solutions and the constants γ_{ki} are positive for all $k \in S, i \in R$. Even though Table 1 shows that symmetric positive-solutions exist for the three scenarios, we still have to check the positiveness of γ_{ki}. Table 2 lists these constants. As can be seen, when all saturation non-linearities are considered the FTCSSA is not stable. On the other hand, both FTCSSA with the actuator saturation in the sector $[0.75, 1]$ and without any saturation are

Table 1. Positive-definite solutions of P_{ki}.

a	P_{ki}			
0.0	$P_{11} =$	$\begin{bmatrix} 4.028 & 1.304 \\ 1.304 & 1.353 \end{bmatrix}$	$P_{12} =$	$\begin{bmatrix} 5.242 & 1.657 \\ 1.657 & 1.457 \end{bmatrix}$
	$P_{21} =$	$\begin{bmatrix} 15.28 & 4.559 \\ 4.559 & 3.242 \end{bmatrix}$	$P_{22} =$	$\begin{bmatrix} 20.042 & 5.884 \\ 5.884 & 3.612 \end{bmatrix}$
.75	$P_{11} =$	$\begin{bmatrix} 2.636 & 0.908 \\ 0.908 & 1.236 \end{bmatrix}$	$P_{12} =$	$\begin{bmatrix} 3.433 & 1.150 \\ 1.150 & 1.310 \end{bmatrix}$
	$P_{21} =$	$\begin{bmatrix} 9.571 & 3.008 \\ 3.008 & 2.814 \end{bmatrix}$	$P_{22} =$	$\begin{bmatrix} 12.547 & 3.862 \\ 3.862 & 3.060 \end{bmatrix}$
1.0	$P_{11} =$	$\begin{bmatrix} 2.391 & 0.836 \\ 0.836 & 1.214 \end{bmatrix}$	$P_{12} =$	$\begin{bmatrix} 3.115 & 1.058 \\ 1.058 & 1.283 \end{bmatrix}$
	$P_{21} =$	$\begin{bmatrix} 8.593 & 2.738 \\ 2.738 & 2.737 \end{bmatrix}$	$P_{22} =$	$\begin{bmatrix} 11.264 & 3.511 \\ 3.511 & 2.963 \end{bmatrix}$

Table 2. Constants γ_{ki}, FTCSSA in the sector [a,1].

Sector [a,1]	γ_{11}	γ_{12}	γ_{21}	γ_{22}
$a = 0.0$	-22.7025	-22.705	-84.2859	-86.1988
$a = .75$	.8329	1.9009	.4309	.3114
$a = 1.0$	5.0	5.0	10.0	10.0

stable. The state feedback control law gains which stabilize the FTCSSA in the sector [0.75,1] are shown in Table 3.

Table 3. Controller gains for the FTCSSA.

K_{11}^T	K_{12}^T	K_{21}^T	K_{22}^T
$\begin{bmatrix} 3.5146 \\ 1.2107 \end{bmatrix}$	$\begin{bmatrix} 2.7464 \\ .9199 \end{bmatrix}$	$\begin{bmatrix} 6.3805 \\ 2.0054 \end{bmatrix}$	$\begin{bmatrix} 5.0188 \\ 1.5447 \end{bmatrix}$

7. CONCLUSION

The effect of actuator saturation due to physical limitations on the stability of FTCS has been addressed. Sufficient conditions for exponential stability in the mean square of FTCSSA have been derived. Two conditions have to be satisfied, namely, the existence of positive-definite solutions, P_{ki}, for Riccati-like matrix equations and the existence of positive constants γ_{ki}. A test algorithm has been constructed. The results revealed that a state feedback controller can be designed to stabilize FTCS driven by actuators with saturation. Other control models including JLS were developed as special cases. The potential of the developed theory was verified by a numerical example.

8. REFERENCES

Boukas, E. K. (1993). Control of Systems with Controlled Jump Markov Disturbances, *Control Theory and Advanced Technology*, **9**, 577-595.

Chen, B. S. and S. S. Wang (1986). The Design of Feedback Controller with Nonlinear Saturating Actuator: Time Domian Approach, *Proc. of Conference on Decision and Control*, Athens, Greece, 2048-2053.

Chen, B. S. and S. S. Wang (1988). The Stability of Feedback Control with Nonlinear Saturating Actuator: Time domain Approach, *IEEE Trans. on Auto. Control*, **33**, no. 5, 483-487.

Chen, B. S., S. S. Wang and H. C. Lu (1988). Stabilization of Time-Delay Systems Containing Saturating Actuators, *Int. Journal of Control*, **47**, no. 3, 867-881.

Costa, O. V., E. O. Filho, E. K. Boukas and R. P. Marques (1999). Constrianed Quadratic State Feedback Control of Discrete- Time Markovian Jump Linear Systems, *Automatica*, **35**, 617-626.

Feng, X., K. A. Loparo, Y. Ji, and H. J. Chezick (1992). Stochastic Stability Properties of Jump Linear Systems, *IEEE Trans. on Auto. Control*, **37**, 38-53.

Glattfelder A. H. and W. Schaufelberger (1983). Stability Analysis of Single Loop Control Systems with Saturation and Antireset-Windup Circuits, *IEEE Trans. on Auto. Control*, **28**, 1074-1080.

Ji, Y. and H. J. Chezick, Controllability (1990). Stabilizability, and Continuous-Time Markovian Jump Linear Quadratic Control, *IEEE Trans. on Auto. Control*, **35**, 777-788.

Krikelis, N. J.(1980). State Feedback Integral Control with Intelligent Integrator, *Int. Journal of Control*, **32**, no. 3, 465-473.

Krikelis N. J. and S. K. Barakas (1984). Design of Tracking System Subject to Actuator Saturation and Integrator Wind-up, *Int. Journal of Control*, **39**, no. 4, 667-682.

Mahmoud, M. M., J. Jiang and Y. M. Zhang (2001). Stochastic Stability Analysis of Fault Tolerant Control Systems with Multiple Failure Processes, *To appear in International Journal of Systems Science*.

Mahmoud, M. M., J. Jiang and Y. M. Zhang (2001). Stochastic Stability Analysis of Fault Tolerant Control Systems in The Presence of Noise, *IEEE Trans. on Auto. Control*, **46**, no. 11, 1810-1815.

Mariton, M. (1989). Detection Delays, False Alarm Rates and Reconfiguration of Control Systems, *Int. Journal of Control*, **49**, 981-992.

Niculescu, S. I., J. M. Dion and L. Dugard (1996). Robust Stabilization for Uncertain Time-Delay Systems Containing Saturating Actuators, *IEEE Trans. on Auto. Control*, **41**, no. 5, 742-747.

Srichander R. and B. K. Walker (1993). Stochastic Stability Analysis for Continuous-Time Fault Tolerant Control Systems, *Int. Journal of Control*, **57**, no. 2, 433-452.

Wonham, W. M. (1971). Random Differential Equations in Control Theory. In: *Probabilistic Methods in Applied Mathematics*, **2**, pp. 131-212, Academic Press, New York.