15th Triennial World Congress, Barcelona, Spain

CHARACTERIZATION OF INVARIANT CODISTRIBUTIONS FOR DISCRETE-TIME NONLINEAR DYNAMICAL SYSTEMS

E. Aranda-Bricaire [*,1] **C.H. Moog** [**]

[*] *Departamento de Ingeniería Eléctrica, Sección de Mecatrónica*
CINVESTAV, México.
earanda@mail.cinvestav.mx
[**] *IRCCYN, UMR 6597, ECN, France.*
Claude.Moog@irccyn.ec-nantes.fr

Abstract: The goal of this paper is two-fold. First, given an arbitrary n-dimensional discrete-time nonlinear dynamical system, necessary and sufficient conditions for the existence of a one-dimensional invariant codistributions are obtained. Second, it is shown that the previous conditions can be used iteratively to obtain a nested sequence of n invariant codistributions with the properties that each codistribution contains the previous one and the last one coincides with the cotangent bundle of the state manifold. As a byproduct, necessary and sufficient conditions are obtained for a discrete-time nonlinear dynamical system to be equivalent to the so-called feedforward form.

Keywords: Nonlinear systems, Discrete-time systems, Dynamic systems, Invariance, Distributions.

1. INTRODUCTION

Invariant distributions and their dual, invariant codistributions, occupy a prominent place in nonlinear control theory. They have been used to study controllability and observability properties of nonlinear control systems, and to solve various nonlinear synthesis problems. For discrete-time nonlinear control systems, invariant distributions were introduced in (Grizzle, 1985; Monaco and Normand-Cyrot, 1984).

Recently, a generalized notion of invariance has been introduced for both continuous- and discrete-time nonlinear systems. This new notion has been used to solve the dynamic disturbance decoupling problem (DDDP), see (Aranda-Bricaire and Kotta, 2001; Huijberts *et al.*, 1997). Also, in the case of continuous-time systems, invariant distributions provide a geometric characterization of nonlinear control systems which are equivalent to the so-called feedforward form (Astolfi and Mazenc, 2000).

Despite the widespread use of invariant codistributions in control theory, the following question does not seem to have received an answer: given a nonlinear control system, what are all possible invariant codistributions with respect to the system dynamics? Of course answers to particular cases of this questions are well known. For instance, checking whether a given codistribution is invariant or not is a simple exercise. Also, explicit methods are available which allow to construct the smallest invariant codistribution containing a given codistribution.

The main goal of this paper is to give a partial answer to the above general question. More specifically, given a discrete-time nonlinear dynamical

[1] The work of the first author was done while he was visiting IRCCYN, partially supported by the French Ministry of Research and Technology.

system, a characterization of all one-dimensional codistributions which are invariant with respect to the system dynamics will be given. As we shall see, the solution of this apparently simple problem is by no means obvious and, moreover, suggests the solution to various equivalence problems.

The paper is organized as follows. In Section 2, we adapt the linear algebraic formalism introduced in (Aranda-Bricaire *et al.*, 1996; Grizzle, 1993) to the case of uncontrolled systems. In Section 3, the notion of eigenform is presented, as well as its application to the characterization of one dimensional invariant codistributions. In Section 4, the results of the previous Section are used iteratively in order to construct nested sequences of invariant codistributions. In Section 5, it is shown that integrability of these codistributions is a necessary and sufficient condition for equivalence to the so-called feedforward form. Finally, concluding remarks are offered in Section 6.

2. PRELIMINARIES

Throughout the rest of this paper we will make extensive use of the linear algebraic framework introduced in (Aranda-Bricaire *et al.*, 1996; Grizzle, 1993). It will be necessary, however, to adapt this framework to the situation of uncontrolled systems. At some places of the paper, notions from exterior differential systems will be used. For these matters, the reader is referred to (Abraham *et al.*, 1988; Bryant *et al.*, 1991; Choquet-Bruhat *et al.*, 1989).

In this paper we will be dealing with discrete-time nonlinear dynamical systems described by the following difference equation:

$$x(t+1) = f[x(t)], \quad x(0) = x_0, \quad t \geq 0, \quad (1)$$

where the state $x(t) \in \Re^n$, and $f : \Re^n \to \Re^n$ is a real analytic mapping.

Define the operator $\varpi : \Re^n \to \Re^n$ by

$$\nu \mapsto f(\nu),$$

where $f(\cdot)$ is the same mapping as in (1).

Let $\mathcal{K}$ denote the field of meromorphic functions of the scalar components of

$$x(0) = [x_1(0), \ldots, x_n(0)]^T \in \Re^n.$$

The elements of $\mathcal{K}$ can be viewed as functions $\varphi : \Re^n \to \Re$. Using this interpretation, the forward-shift operator $\delta : \mathcal{K} \to \mathcal{K}$ is defined by $\delta\varphi = \varphi \circ \varpi$. Sometimes, the abridged notation $\varphi^+(\cdot) = \delta\varphi(\cdot)$ is used.

Define the vector space $\mathcal{E} = \mathrm{span}_{\mathcal{K}} \{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}$. The elements of $\mathcal{E}$ are called one-forms. The operator $\delta : \mathcal{K} \to \mathcal{K}$ induces the operator $\Delta : \mathcal{E} \to \mathcal{E}$ in the following way. Let $\omega = \sum_i a_i \mathrm{d}\varphi_i \in \mathcal{E}$. Then

$$\omega^+ = \Delta\omega = \Delta(\sum_i a_i \mathrm{d}\varphi_i) = \sum_i a_i^+ \mathrm{d}\varphi_i^+.$$

Throughout the paper it will be assumed that the dynamics of system (1) is reversible. More precisely, we make the following technical assumption

Assumption 2.1. Generically, $\mathrm{rank}\frac{\partial f}{\partial x} = n$.

Assumption 2.1 guarantees that the mapping $\delta : \mathcal{K} \to \mathcal{K}$ is well defined. It is satisfied for discrete-time systems which arise from sampling a continuous time system (Jakubczyk and Sontag, 1990). The following example displays the type of pathologies that can appear for non reversible systems.

Example 2.2. Consider the discrete-time nonlinear system

$$\begin{aligned} x_1^+ &= x_2 \\ x_2^+ &= -x_1 \\ x_3^+ &= x_1 x_2. \end{aligned} \quad (2)$$

Easy computations show that system (2) does not satisfy Assumption 2.1. Define the function $\mu = \frac{1}{x_3 + x_1 x_2} \in \mathcal{K}$. It can be checked that the forward-shift μ^+ is not defined.

Under Assumption 2.1, the mapping $\varpi : \Re^n \to \Re^n$ is well defined and invertible. Therefore, the backward-shift operator $\delta^{-1} : \mathcal{K} \to \mathcal{K}$ exists and is defined by $\delta^{-1}\varphi = \varphi \circ \varpi^{-1}$. Sometimes, the abridged notation $\varphi^-(\cdot) = \delta^{-1}\varphi(\cdot)$ will be used.

The operator $\Delta^{-1} : \mathcal{E} \to \mathcal{E}$ is defined in the following way. Let $\omega = \sum_i a_i \mathrm{d}\varphi_i \in \mathcal{E}$. Then

$$\omega^- = \Delta^{-1}\omega = \Delta^{-1}(\sum_i a_i \mathrm{d}\varphi_i) = \sum_i a_i^- \mathrm{d}\varphi_i^-.$$

Given a codistribution or subspace

$$\Omega = \mathrm{span}_{\mathcal{K}} \{\omega_1, \ldots, \omega_r\} \subset \mathcal{E},$$

define $\Omega^+ = \Delta\Omega = \mathrm{span}_{\mathcal{K}} \{\omega^+ \mid \omega \in \Omega\}$.

Definition 2.3. A codistribution $\Omega \subset \mathcal{E}$ is said to be invariant with respect to the dynamics (1) if $\Omega^+ \subset \Omega$.

3. ONE-DIMENSIONAL INVARIANT CODISTRIBUTIONS

The goal of this Section is to give necessary and sufficient conditions for the existence of a one-dimensional codistribution which is invariant with respect to the dynamics (1). The solution of this apparently simple problem constitutes the fundamental brick upon which solutions to different equivalence problems can be obtained.

To begin with, we need to introduce some notation. Let $f : \Re^n \to \Re^n$ be the mapping which defines the system dynamics (1). Define the mapping $\mathcal{C}(\delta) : \mathcal{K}^n \to \mathcal{K}^n$ by

$$\mathcal{C}(\delta) = \left[\frac{\partial f_j}{\partial x_i}\delta\right] = \left[\frac{\partial f}{\partial x}\delta\right]^T. \tag{3}$$

In the rest of the paper, $[\mathrm{d}x]$ stands for the *column* vector $[\mathrm{d}x_1, \ldots, \mathrm{d}x_n]^T$. With this notation, it is easy to see that $[\mathrm{d}x^+] = [\frac{\partial f}{\partial x}][\mathrm{d}x]$. Since $\{\mathrm{d}x_1, ..., \mathrm{d}x_n\}$ is a basis for $\mathcal{E}$, any one-form $\omega \in \mathcal{E}$ can be written as

$$\omega = \sum_{i=1}^{i=n} a_i \mathrm{d}x_i = [a_1, \ldots, a_n][\mathrm{d}x] = [a][\mathrm{d}x].$$

Note that $\omega^+ = [a^+][\mathrm{d}x^+] = [\mathrm{d}x]^T \mathcal{C}(\delta)[a]^T$. Finally, define the family of operators $\Gamma_\lambda(x,\delta) = [\mathcal{C}(\delta) - \lambda I]$, parameterized by a function $\lambda \in \mathcal{K}$.

Definition 3.1. (Eigenform). A one-form $\omega \in \mathcal{E}$ is said to be an eigenform if there exists a function $\lambda \in \mathcal{K}$ such that $\omega^+ = \lambda\omega$.

Clearly, if ω is an eigenform, then $\Omega = \mathrm{span}_{\mathcal{K}}\{\omega\}$ is a one-dimensional invariant codistribution. Therefore, the characterization of one-dimensional invariant codistributions is equivalent to the characterization of eigenforms.

Theorem 3.2. A one-form $\omega = [a][\mathrm{d}x] \in \mathcal{E}$ is an eigenform if and only if there exists a function $\lambda \in \mathcal{K}$ such that $[a] \in \ker \Gamma_\lambda(x,\delta)$.

PROOF. Let $\omega = [a][\mathrm{d}x]$ and recall that $\omega^+ = [\mathrm{d}x]^T\mathcal{C}(\delta)[a]^T$. Now suppose that ω is an eigenform. Therefore, there exists a function $\lambda \in \mathcal{K}$ such that $\omega^+ = \lambda\omega$. The last two expressions imply that $[\mathrm{d}x]^T[\mathcal{C}(\delta) - \lambda I][a]^T = 0$ or, equivalently, that $[a] \in \ker\Gamma_\lambda(x,\delta)$. The converse is obvious. □

Theorem 3.2 provides a complete characterization of all one-dimensional codistributions which are invariant with respect to the dynamics of the system (1). From a practical point of view, the problem has been reduced to that of finding a function $\lambda \in \mathcal{K}$ such that the operator $\Gamma_\lambda(x,\delta) : \mathcal{K}^n \to \mathcal{K}^n$ becomes singular. This problem can be tackled by usual Gaussian elimination thanks to the following technical Lemma, proven in (Márquez-Martínez *et al.*, 2000).

Lemma 3.3. Let $\mathcal{K}[\delta]$ denote the ring of polynomials in the operator δ whose coefficients belong to the field $\mathcal{K}$. Then, for all $a(\delta), b(\delta) \in \mathcal{K}[\delta]$ there exists polynomials $p(\delta), q(\delta) \in \mathcal{K}[\delta]$ such that $p(\delta)a(\delta) + q(\delta)b(\delta) = 0$.

The following simple example serves to illustrate the typical procedure.

Example 3.4. Consider the following discrete-time (linear) system

$$\begin{aligned} x_1^+ &= x_2 \\ x_2^+ &= -x_1. \end{aligned} \tag{4}$$

For system (4) the family of operators $\Gamma_\lambda(x,\delta)$ is given by

$$\Gamma_\lambda(x,\delta) = \begin{bmatrix} -\lambda & -\delta \\ \delta & -\lambda \end{bmatrix}.$$

The operator $\Gamma_\lambda(x,\delta)$ can be brought to a triangular form by performing elementary row operations. Straightforward computations show that, whenever $\lambda \neq 0$, it holds that

$$\begin{bmatrix} -1 & 0 \\ -\delta & -\lambda^+ \end{bmatrix} \Gamma_\lambda(x,\delta) = \begin{bmatrix} \lambda & \delta \\ 0 & \delta^2 + \lambda\lambda^+ \end{bmatrix}.$$

At this point, the computation of $\ker\Gamma_\lambda(x,\delta)$ amounts to solve the difference equation $a_2^{++} + \lambda\lambda^+ a_2 = 0$ in the unknown a_2, and then solve the equation $\lambda a_1 + a_2^+$ in the unknown a_1. In general, the solutions to these equations are not unique. Table 1 displays various solutions, corresponding to different choices of the parameter λ.

Table 1. Possible choices of coefficients for system (4)

Parameter λ	Coefficient a_1	Coefficient a_2
1	x_1	x_2
1	$-x_2$	x_1
-1	$-x_1$	x_2
-1	x_2	x_1

Each one of the choices displayed in Table 1 defines a eigenform $\omega = a_1\mathrm{d}x_1 + a_2\mathrm{d}x_2$ and, consequently, a codistribution $\Omega = \mathrm{span}_{\mathcal{K}}\{\omega\}$ which is invariant with respect to the dynamics of the system (4).

4. NESTED SEQUENCES OF INVARIANT CODISTRIBUTIONS

In this Section an algorithm will be presented which allows to construct a sequence of invariant codistributions with the property that their dimensions increase by one at each step. Applications of this construction will be presented in the following section.

The tangent linear system associated to the discrete-time nonlinear system (1) is given by $[\mathrm{d}x^+] = [\frac{\partial f}{\partial x}][\mathrm{d}x]$. In order to develop the Algorithm, an alternative representation of the tangent linear system will be presented.

Let $\{\omega_1, \ldots, \omega_n\}$ be an arbitrary basis of the space $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$. Then, necessarily, there exist coefficients $a_{ij} \in \mathcal{K}$, such that $\omega_i^+ = \sum_{j=1}^n a_{ij}\omega_j$, for $i = 1, \ldots, n$. Define $\omega = [\omega_1, \ldots, \omega_n]^T$. Then the above relations can be written in the following matrix form:

$$\omega^+ = \begin{bmatrix} \omega_1^+ \\ \vdots \\ \omega_n^+ \end{bmatrix} = A \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix} = A\,\omega.$$

Invariant Codistribution Algorithm (ICA)

step n

If $\ker \Gamma_\lambda(x, \delta) = 0$, then this step can not be accomplished and the algorithm terminates. Otherwise, pick $[a]^T \in \ker \Gamma_\lambda(x, \delta)$, and define

$$\omega_n = [a][\mathrm{d}x] = \sum_{j=1}^n a_j \mathrm{d}x_j.$$

Choose $n-1$ one-forms $\{\omega_1^n, \ldots, \omega_{n-1}^n\}$ such that

$$\mathrm{span}_{\mathcal{K}}\left\{\omega_1^n, \ldots, \omega_{n-1}^n, \omega_n\right\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}.$$

Let $A_{n-1} \in \mathcal{K}^{(n-1)\times(n-1)}$ be the unique matrix such that

$$\begin{bmatrix} (\omega_1^n)^+ \\ \vdots \\ (\omega_{n-1}^n)^+ \end{bmatrix} \equiv A_{n-1} \begin{bmatrix} \omega_1^n \\ \vdots \\ \omega_{n-1}^n \end{bmatrix} \mod \{\omega_n\}.$$

step i**, for** $i = n-1, \ldots, 2$

Define $\Gamma_\lambda^i(x, \delta) = [A_i^T\delta - \lambda I]$. If $\ker \Gamma_\lambda^i(x, \delta) = 0$, then this step can not be accomplished and the algorithm terminates. Otherwise, pick $[a]^T \in \ker \Gamma_\lambda^i(x, \delta)$, and define

$$\omega_i = [a] \begin{bmatrix} \omega_1^{i+1} \\ \vdots \\ \omega_i^{i+1} \end{bmatrix} = \sum_{j=1}^i a_j \omega_j^{i+1}.$$

Choose $i-1$ one-forms $\{\omega_1^i, \ldots, \omega_{i-1}^i\}$ such that

$$\mathrm{span}_{\mathcal{K}}\left\{\omega_1^i, \ldots, \omega_{i-1}^i, \omega_i\right\} = \mathrm{span}_{\mathcal{K}}\left\{\omega_1^{i+1}, \ldots, \omega_i^{i+1}\right\}.$$

Let $A_{i-1} \in \mathcal{K}^{(i-1)\times(i-1)}$ be the unique matrix such that

$$\begin{bmatrix} (\omega_1^i)^+ \\ \vdots \\ (\omega_{i-1}^i)^+ \end{bmatrix} \equiv A_{i-1} \begin{bmatrix} \omega_1^i \\ \vdots \\ \omega_{i-1}^i \end{bmatrix} \mod \{\omega_i, \ldots, \omega_n\}.$$

step 1

Pick $\omega_1 = \omega_1^2$. It follows that $\{\omega_1, \ldots, \omega_n\}$ is a basis of $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$.

Theorem 4.1. There exist a sequence of invariant codistributions

$$\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n,$$

with $\dim \Omega_i = (n+1) - i$, if and only if all the steps of Algorithm 1 can be accomplished.

PROOF. Sufficiency: Step i of the algorithm defines the codistribution $\Omega_i := \mathrm{span}_{\mathcal{K}}\{\omega_i, \ldots, \omega_n\}$ which is obviously invariant. The converse is obvious. □

The following example illustrates the application of the ICA.

Example 4.2. Consider the following discrete-time nonlinear system

$$\begin{aligned} x_1^+ &= x_2 \\ x_2^+ &= -x_1 \\ x_3^+ &= x_3 + x_1x_2. \end{aligned} \tag{5}$$

For system (5) the family of operators $\Gamma_\lambda(x, \delta)$ is given by

$$\Gamma_\lambda(x, \delta) = \begin{bmatrix} -\lambda & -\delta & x_2\delta \\ \delta & -\lambda & x_1\delta \\ 0 & 0 & \delta - \lambda \end{bmatrix}.$$

Now we proceed to apply ICA to system (5).

step 3 First apply elementary row operations to bring the operator $\Gamma_\lambda(x, \delta)$ into triangular form. Define the unimodular matrix

$$B(\delta) = \begin{bmatrix} -1 & 0 & 0 \\ -\delta & -\lambda^+ & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is easy to verify that

$$B(\delta)\Gamma_\lambda(x, \delta) = \begin{bmatrix} \lambda & \delta & -x_2\delta \\ 0 & \delta^2 + \lambda\lambda^+ & x_1\delta^2 - \lambda^+x_1\delta \\ 0 & 0 & \delta - \lambda \end{bmatrix}.$$

Choosing $\lambda = -1$, it follows that the vector $[a_1, a_2, a_3]^T$, with $a_1 = x_2$, $a_2 = x_1$, and $a_3 = 0$ annihilates the operator $\Gamma_\lambda(x, \delta)$. Therefore, we choose $\omega_3 = x_2\mathrm{d}x_1 + x_1\mathrm{d}x_2$. We complete a basis of $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$ by $\omega_1^3 = x_1\mathrm{d}x_1 + x_2\mathrm{d}x_2$, and $\omega_2^3 = \mathrm{d}x_3$. Straightforward computations show that $(\omega_1^3)^+ = \omega_1^3$, and $(\omega_2^3)^+ = \omega_2^3 + \omega_3$. Therefore,

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

step 2 The family of operators $\Gamma_\lambda^2(x, \delta)$ is given by

$$\Gamma^2_\lambda(x,\delta)=\begin{bmatrix}\delta-\lambda & 0\\ 0 & \delta-\lambda\end{bmatrix}.$$

It is easy to see that the vector $[a_1,a_2]^T$ annihilates the operator $\Gamma^2_\lambda(x,\delta)$, whenever $a_1=a_2=\alpha$ and $\lambda=\frac{\alpha^+}{\alpha}$, $\alpha\in\mathcal{K}$ being a free parameter. Choose for instance $\alpha=1$. Therefore, the form ω_2 is defined by

$$\omega_2=\omega_1^3+\omega_2^3=\mathrm{d}x_3+x_1\mathrm{d}x_1+x_2\mathrm{d}x_2.$$

A basis of $\mathrm{span}_\mathcal{K}\{\omega_1^3,\omega_2^3\}$ can be completed by taking $\omega_1^2=\mathrm{d}x_3$.

step 1. Pick $\omega_1=\omega_1^2=\mathrm{d}x_3$.

Since all the steps of Algorithm 1 can be accomplished, the sequence of invariant codistributions $\Omega_1\supset\Omega_2\supset\Omega_3$ exists, and is defined as follows:

$$\begin{aligned}\Omega_3 &= \mathrm{span}_\mathcal{K}\{\omega_3\}\\ &= \mathrm{span}_\mathcal{K}\{x_2\mathrm{d}x_1+x_1\mathrm{d}x_2\}\\ \Omega_2 &= \mathrm{span}_\mathcal{K}\{\omega_2,\omega_3\}\\ &= \mathrm{span}_\mathcal{K}\{\mathrm{d}x_3+x_1\mathrm{d}x_1+x_2\mathrm{d}x_2,\\ &\quad x_2\mathrm{d}x_1+x_1\mathrm{d}x_2\}\\ \Omega_1 &= \mathrm{span}_\mathcal{K}\{\omega_1,\omega_2,\omega_3\}\\ &= \mathrm{span}_\mathcal{K}\{\mathrm{d}x\}.\end{aligned}\tag{6}$$

5. EQUIVALENCE TO FEEDFORWARD FORM

Definition 5.1. System (1) is equivalent feedforward form if there exists a local change of coordinates $z=\varphi(x_1,...,x_n)$ such that

$$\begin{aligned}z_1(t+1) &= f_1(z_1,...,z_n)\\ z_2(t+1) &= f_2(z_2,...,z_n)\\ &\vdots\\ z_n(t+1) &= f_n(z_n)\end{aligned}$$

In the continuous-time case, a nice geometric characterization of those systems that are equivalent to feedforward form can be found in (Astolfi and Mazenc, 2000), and is recasted below in a dual form.

Theorem 5.2. System (1) can be transformed into feedforward form if and only if there exists a sequence of completely integrable codistributions

$$\Omega_1\supset\Omega_2\supset\cdots\supset\Omega_n$$

such that $\dim\Omega_i=n+1-i$.

Corollary 5.3. System (1) is equivalent to feedforward form if all the steps of Algorithm 1 can be accomplished and the set of forms $\{\omega_1,\ldots,\omega_n\}$ thereby identified satisfy

$$\mathrm{d}\omega_i\wedge\omega_i\equiv 0 \mod \{\omega_{i+1},\ldots,\omega_n\}.$$

It should be strengthened that Corollary 5.3 provides only sufficient conditions for equivalence to feedforward form. The main obstacle to obtain necessary and sufficient conditions is the fact that the sequences of invariant codistributions constructed by an application of the ICA are not unique. The following simple example illustrates this situation.

Example 5.4. Consider again system (4), and define the following codistributions:

$$\begin{aligned}\Omega_2 &= \mathrm{span}_\mathcal{K}\{x_1\mathrm{d}x_1+x_2\mathrm{d}x_2\}\\ \tilde\Omega_2 &= \mathrm{span}_\mathcal{K}\{x_1\mathrm{d}x_2-x_2\mathrm{d}x_1\}\\ \Omega_1 &= \mathrm{span}_\mathcal{K}\{\mathrm{d}x\}.\end{aligned}$$

From Example 3.4, it follows that Ω_1, Ω_2, and $\tilde\Omega_2$ are invariant codistributions. Therefore, different applications of the ICA would lead to the sequences $\Omega_1\supset\tilde\Omega_2$ or $\Omega_1\supset\Omega_2$.

Remark 5.5. The pathology exhibited by Example 5.4 is not a consequence of the application of ICA. It comes from the fact that a given nonlinear system can be equivalent to two different feedforward forms, through the appropriate change of coordinates.

Example 5.6. Consider system (5), and the sequence of invariant codistributions $\Omega_1\supset\Omega_2\supset\Omega_3$ obtained in Example (4.2). It can be easily checked that the codistributions Ω_1, Ω_2 and Ω_3 are completely integrable. Therefore, system (5) is equivalent to feedforward form. The corresponding change of coordinates is obtained by integration of the one-forms ω_1, ω_2, ω_3. This leads to the change of coordinates $z_1=x_3$, $z_2=x_3+x_1x_2$, $z_3=x_1x_2$. In z coordinates, system (5) becomes:

$$\begin{aligned}z_1^+ &= z_2\\ z_2^+ &= z_2-z_3\\ z_3^+ &= -z_3,\end{aligned}$$

which is in feedforward form.

6. PERSPECTIVES AND CONCLUDING REMARKS

In this paper we have introduced the notion of eigenform. This notion allows to give a characterization of one-dimensional codistributions which are invariant with respect a given discrete-time nonlinear system. We have also presented an algorithm that allows to construct sequences of invariant codistributions. As an application of these technical developments, explicit sufficient conditions for equivalence to feedforward form have been obtained. It is interesting to note that this is an improvement with respect to the same problem in the continuous-time case.

A natural continuation of this work would be the characterization of nested sequences of *controlled* *invariant codistributions* for discrete-time nonlinear control systems.

Finally, it is worth mentioning that equivalence to feedforward form can be used for the design of stabilizers for discrete-time nonlinear systems (Mazenc and Nijmeijer, 1998).

7. REFERENCES

Abraham, R., J. E. Marsden and T. Ratiu (1988). *Manifolds, Tensor Analysis and Applications.* number 75 In: *Applied Mathematical Sciences.* 2nd ed.. Springer-Verlag. New York.

Aranda-Bricaire, E. and U. Kotta (2001). Generalized controlled invariance for discrete-time nonlinear systems with an application to the dynamic disturbance decoupling problem. *IEEE Transactions on Automatic Control* **46**(1), 165–171.

Aranda-Bricaire, E., U. Kotta and C.H. Moog (1996). Linearization of discrete-time systems. *SIAM Journal on Control and Optimization* **34**, 1999–2023.

Astolfi, A. and F. Mazenc (2000). A geometric characterization of feedforward forms. In: *Proc. Int. Symposium MTNS-2000.* Perpignan.

Bryant, R.L., S.S. Chern, R.B. Gardner, H.L. Goldschmidt and P. A. Griffiths (1991). *Exterior Differential Systems.* Vol. 18 of *Mathematical Sciences Research Institute Publications.* Springer-Verlag. New York.

Choquet-Bruhat, Y., C. DeWitt-Morette and M. Dillard-Bleick (1989). *Analysis, Manifolds and Physics, Part I: Basics.* North-Holland. Amsterdam.

Grizzle, J. W. (1993). A linear algebraic framework for the analysis of discrete-time nonlinear systems. *SIAM Journal on Control and Optimization* **31**, 1026–1044.

Grizzle, J.W. (1985). Controlled invariance for discrete-time nonlinear systems with an application to the disturbance decoupling problem. *IEEE Transactions on Automatic Control* **30**(9), 868–874.

Huijberts, H.J.C., C.H. Moog and R. Andiarti (1997). Generalized controlled invariance for nonlinear systems. *SIAM Journal on Control and Optimization* **35**(3), 953–979.

Jakubczyk, B. and E. D. Sontag (1990). Controllability of nonlinear discrete-time systems: a lie-algebraic approach. *SIAM Journal on Control and Optimization* **28**, 1–33.

Márquez-Martínez, L.A., C.H. Moog and M. Velasco-Villa (2000). The structure of nonlinear time-delay systems. *Kybernetika* **36**(1), 53–62.

Mazenc, F. and H. Nijmeijer (1998). Forwarding in discrete-time nonlinear systems. *Int. J. Control* **71**(5), 823–835.

Monaco, S. and D. Normand-Cyrot (1984). Invariant distributions for discrete-time nonlinear systems. *Systems and Control Letters* **5**, 191–196.