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ON SUPERVISORY CONTROL OF LINEAR SYSTEMS WITH INPUT CONSTRAINTS

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Abstract: A supervisory control system which *globally* regulates to zero the state of a very poorly modeled linear process in the presence of input constraints is proposed in this paper. The model of the process is an unknown member of a family of systems which are open-loop unstable but not exponentially unstable. For the analysis, two switching logics are considered: The hysteresis switching logic and a new type of dwell time switching logic. In both cases, the supervisory control system is shown to regulate to zero all the continuous states of the system.

Keywords: Supervisory Control, Constraints, Uncertain Dynamic Systems, Adaptive Control, Regulation.

1. INTRODUCTION

We propose a supervisory control system to globally regulate to zero the state of a very poorly modeled linear process $\mathbb{P}$ in the presence of input constraints. The model of $\mathbb{P}$ is an unknown member of a family of dynamical systems of the form $\mathcal{F} = \cup_{p\in\mathcal{P}}\mathcal{F}_p$ where each $\mathcal{F}_p$ is a subfamily consisting of a given *nominal process model* $\mathbb{N}_p$ together with a collection of "perturbed versions" of $\mathbb{N}_p$, and $\mathcal{P}$ is a finite set of indices. The approach of supervisory control (Morse, 1996; Morse, 1995) is that of employing a family of candidate controllers $\mathcal{C} := \{\mathbb{C}_p : p \in \mathcal{P}\}$, chosen in such a way that for each $p \in \mathcal{P}$, $\mathbb{C}_p$ would "solve" the regulation problem were $\mathbb{P}$ to be any element of $\mathcal{F}_p$. The idea then is to generate a *switching signal* σ taking values in $\mathcal{P}$, which causes the state $x_{\mathbb{P}}$ of the process $\mathbb{P}$ in closed-loop with switched controller $\mathbb{C}_\sigma$ to be regulated to zero. The algorithm which generates σ is called a "supervisor". An estimator-based supervisor consists of three subsystems, a multi-estimator $\mathbb{E}$, a bank of monitoring signal generators $\mathbb{M}_p$, $p \in \mathcal{P}$, and a switching logic $\mathbb{S}$. $\mathbb{E}$ is a finite-dimensional, input-to-state stable (Sontag and Wang, 1996) dynamical system with state x whose input is the pair $\{v,\ y\}$ (the pair of input and, respectively, output vectors of $\mathbb{P}$), and whose p-th output is a signal y_p which would be an asymptotically correct estimate of y, if $\mathbb{N}_p$ were the actual process model and there were no measurement noise or disturbances. For $\mathbb{E}$ to have this property, its p-th candidate model would have to exhibit (under appropriate feedback interconnection and initialization (Hespanha and Morse, 1999b)) the same input-output behavior between v and y_p as $\mathbb{N}_p$ does between its input and output. A *monitoring signal generator* $\mathbb{M}_p$ is a dynamical system whose input is the p-th output estimation error $e_p := y_p - y$ and whose output μ_p is a suitably defined signal which measures the size of the e_p. The third subsystem of an estimator-based supervisor is a switching logic $\mathbb{S}$ whose role is to generate σ.

In this paper, we consider the case in which $\mathcal{F}_p = \{\mathbb{N}_p\}$ and design each controller $\mathbb{C}_p$ so as to make the closed-loop system $\mathbb{N}_p - \mathbb{C}_p$ integral input-to-state stable (Sontag, 1998) with respect to the output estimation error e_p, while fulfilling the input constraint. When no noise, disturbances or

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unmodelled dynamics are present, the hysteresis switching logic generates a signal σ which stops switching in finite time (Hespanha, 1998; Hespanha and Morse, 1999a) and the analysis of the overall system is straightforward. In particular (Hespanha *et al.*, 2001), this family of integral input-to-state stabilizing controllers along with a supervisor employing an hysteresis switching logic globally regulates to zero the state of $\mathbb{P}$. In less idealistic situations, however, the switching to stop in finite time is an event which is unlikely to happen. In (Morse, 1996), the analysis of robustness for linear supervisory control systems is carried out by adopting the so-called "dwell time" switching logic, which constrains the switching signal to "dwell" at a value $p \in \mathcal{P}$ for at least τ_{D} units of time before switching again, with τ_{D} a positive *constant* – the "dwell time". We use here a similar switching logic, introduced in (De Persis *et al.*, 2002a) (see also De Persis *et al.*, 2002b), with the major difference being that, since the system is non-linear (because of the nonlinearities that model the constraint on the input), the dwell time is allowed to change with the time according to a suitable law. It is proven that the supervisory control with time-varying dwell time switching logic regulates the state to zero even in the case the switching does not stop in finite time. Most of the results presented in this paper are given without a proof, for which we refer the reader to (De Persis *et al.*, 2002a). The approach can be extended to deal with the case in which the set $\mathcal{P}$ is a continuum of points.
After formulating the problem in Section 2, we design each component of the supervisory control system, namely the multi-estimator, the monitoring signal generators and the multi-controller in Sections 3, 4 and 5, respectively. In Section 6, we analyze the supervisory control system considering both the hysteresis switching logic and the dwell time switching logic. In both cases, we show global regulation to zero of all the continuous states of the system. Conclusions are drawn in Section 7.

2. PROBLEM FORMULATION

The process $\mathbb{P}$ is presumed to admit the model

$$\begin{aligned} \dot{x}_{\mathbb{P}} &= A_{\mathbb{P}} x_{\mathbb{P}} + B_{\mathbb{P}} \mathrm{sat}(v) \\ y &= C_{\mathbb{P}} x_{\mathbb{P}} , \end{aligned} \tag{1}$$

where $x_{\mathbb{P}} \in \mathbb{R}^n$, $v \in \mathbb{R}^s$, $y \in \mathbb{R}^p$, and $\mathrm{sat}(\cdot)$, which models constraints on the control magnitude, is an $\mathbb{R}^s$-valued saturation function, i.e. a function with the following properties (see e.g. Isidori, 1999).

Definition. A locally Lipschitz function $\mathrm{sat}(\cdot) : \mathbb{R} \to \mathbb{R}$ is said to be a saturation function if

(i) $\mathrm{sat}(0) = 0$ and $r\mathrm{sat}(r) > 0$ for all $r \neq 0$,

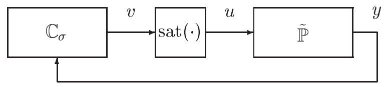

Fig. 1. Feedback interconnection.

(ii) there exist $\underline{k}, \bar{k} > 0$ such that $|\mathrm{sat}(r)| \leq \bar{k}$ for all r and $\liminf_{|r|\to\infty} |\mathrm{sat}(r)| \geq \underline{k}$,
(iii) $\mathrm{sat}(\cdot)$ is differentiable in a neighborhood of the origin and $\mathrm{sat}'(0) = 1$.

A function $\mathrm{sat}(\cdot) : \mathbb{R}^s \to \mathbb{R}^s$ is an $\mathbb{R}^s$-valued saturation function if $\mathrm{sat}(v_1, \ldots, v_s) = (\mathrm{sat}_1(v_1), \ldots, \mathrm{sat}_s(v_s))$ and $\mathrm{sat}_i(\cdot)$ is a saturation function for all $i = 1, \ldots, s$. $\triangleleft$

The input to this function is the signal v generated by the switched controller $\mathbb{C}_\sigma$ and its output is the bounded control signal $u = \mathrm{sat}(v)$ acting on the underlying linear process $\tilde{\mathbb{P}}$. The feedback loop we are considering is thus that depicted in Figure 1.

The actual state-space representation (1) of $\mathbb{P}$ is unknown but it is assumed to belong to a known family of nominal model plants $\mathbb{N}_p$, each one admitting the following state-space representation

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}_p \bar{x} + \bar{B}_p \mathrm{sat}(v) \\ y &= \bar{C}_p \bar{x} , \end{aligned} \tag{2}$$

with $\mathcal{P} = \{p_1, \ldots, p_m\}$ a finite set. These systems are assumed to satisfy the following

Assumption 1. The models $\mathbb{N}_p$ for each $p \in \mathcal{P}$ are stabilizable and detectable and all the eigenvalues of $\bar{A}_p$ are in the closed left-half plane of the complex plane.

We consider the so-called "exact matching" case, that is the case in which $\mathbb{P} \in \mathcal{F} = \cup_{p\in\mathcal{P}} \mathcal{F}_p$, with $\mathcal{F}_p = \{\mathbb{N}_p\}$. We note that, as a consequence of the exact matching case assumption, there exists a value $p^* \in \mathcal{P}$ such that $A_{\mathbb{P}} = \bar{A}_{p^*}$, $B_{\mathbb{P}} = \bar{B}_{p^*}$, $C_{\mathbb{P}} = \bar{C}_{p^*}$.

The problem of interest is (cf. Morse, 1996): For the plant (1) design the family of output-feedback controllers $\mathcal{C} = \{\mathbb{C}_p \ : \ p \in \mathcal{P}\}$ and a supervisor (multi-estimator $\mathbb{E}$, monitoring signal generators $\mathbb{M}_p$, switching logic $\mathbb{S}$) which generates a switching signal σ so as to achieve asymptotic regulation to zero of the state of the process $\mathbb{P}$ and boundedness of all the system signals.

3. IDENTIFIER-BASED MULTI-ESTIMATOR AND MONITORING SIGNAL GENERATOR

The most convenient and simple way to design a multi-estimator is that of designing single estimators for the nominal model plants $\mathbb{N}_p$ and then stack them all together. A multi-estimator of this kind will typically have a structure like

$$\begin{aligned} \dot{x}_p &= (\bar{A}_p - \bar{K}_p\bar{C}_p)x_p + \bar{B}_p\mathrm{sat}(v) + \bar{K}_p y \\ y_p &= \bar{C}_p x_p \;, \quad p \in \mathcal{P} \;, \end{aligned} \tag{3}$$

with $\bar{K}_p$ designed in such a way that $(\bar{A}_p - \bar{K}_p\bar{C}_p)$ is a Hurwitz matrix. It is immediate to verify that the multi-estimator (3) is input-to-state stable with respect to the input pair $\{v, y\}$ and that its p-th output y_p would be an asymptotically correct estimate of the process output y were the actual process model equal to $\mathbb{N}_p$. Multi-estimator (3) can also be rewritten in the more compact form

$$\begin{aligned} \dot{x} &= Ax + B\mathrm{sat}(v) + Ky \\ y_p &= C_p x \;, \quad p \in \mathcal{P} \;. \end{aligned} \tag{4}$$

The outputs y_p, $p \in \mathcal{P}$, generated by the multi-estimator (4) are used to obtain the *output estimation errors* $e_p = y_p - y$ which feed the monitoring signal generators $\mathbb{M}_p$

$$\dot{\mu}_p = -\lambda\mu_p + |e_p|^\ell \;,\; \mu_p(0) > 0 \;,\; p \in \mathcal{P} \;, \tag{5}$$

in which ℓ is a fixed integer in the set $\{1, 2\}$. The monitoring signal generators are input-to-state stable, provided that $\lambda > 0$. Also note that the exact matching condition and the equations of the output estimation errors show that e_{p^*} decays exponentially to zero, i.e. $|e_{p^*}(t)| \le \bar{C}\exp(-\bar{\lambda}t)$, for some positive numbers $\bar{C}, \bar{\lambda}$.

4. MULTI-CONTROLLER

Following (Hespanha and Morse, 1999b; Hespanha *et al.*, 2001; De Persis *et al.*, 2002a), the controller is designed for a system obtained from the multi-estimator in such a way that it is input-output equivalent to the p-th model $\mathbb{N}_p$ (cf. Hespanha and Morse, 1999b, Section 6). Namely, consider the multi-estimator (4) under the feedback interconnection $y = y_p - e_p$

$$\begin{aligned} \dot{x} &= (A + KC_p)x + B\mathrm{sat}(v) - Ke_p \\ y_p &= C_p x \;. \end{aligned} \tag{6}$$

A system like (6) can be made integral input-to-state stable (iISS) with a suitable feedback. We recall that (Sontag, 1998)

Definition. A system $\dot{\xi} = f(\xi, u)$ is iISS if there exist functions[2] $\alpha(\cdot), \tilde{\theta}_1(\cdot), \tilde{\theta}_2(\cdot) \in \mathcal{K}_\infty$, $\gamma(\cdot) \in \mathcal{K}$, such that for all ξ_0, all u, and for all $t \ge 0$,

$$\alpha(|\xi(t, \xi_0, u)|) \le \tilde{\theta}_1(\tilde{\theta}_2(|\xi_0|)e^{-t}) + \int_0^t \gamma(|u(s)|)ds \,. \tag{7}$$

Hereafter $\gamma(\cdot)$ is referred to as the gain function.

The feedback we use to make the system iISS is the same which has been already proposed for the robust stabilization of linear systems subject to input saturation (Teel, 1996).

Lemma 1. Consider the system

$$\dot{\xi} = \tilde{A}\xi + \tilde{B}\mathrm{sat}(v) + \tilde{K}e \,. \tag{8}$$

If the pair $(\tilde{A}, \tilde{B})$ is stabilizable and all the eigenvalues of $\tilde{A}$ are in the closed left-half plane of the complex plane, then there exist a positive integer ν, matrices L_i, $i = 1, \ldots, \nu$, constants c_j, $j = 2, \ldots, \nu$ and a feedback of the form[3]

$$v = L_1\xi + c_2\mathrm{sat}(L_2\xi + c_3\mathrm{sat}(\ldots + c_\nu\mathrm{sat}(L_\nu\xi))) \tag{9}$$

such that the closed-loop system (8), (9) is iISS with respect to the input e with gain function $\gamma(r) = r^2$, and is locally exponentially stable when $e = 0$.

Remark. The same result holds for the more general class of nonlinear feedforward systems (Teel, 1996; De Persis *et al.*, 2002b; De Persis *et al.*, 2002a). It is also possible to prove the integral input-to-state stability of the closed-loop system (8), (9) with different gain functions $\gamma(\cdot)$. However, we are interested in a *quadratic* function because it allows to extend the analysis to the case in which $\mathcal{P}$ is a continuum of points (Morse, 1996). When $\mathcal{P}$ is a finite set, as in this paper, other gain functions are acceptable. In Section 5.3, we give a sketch of the proof of the integral input-to-state stability of the closed-loop system (8), (9) with a *linear* gain function $\gamma(r) = \bar{\gamma}r$ in the case in which the models $\mathbb{N}_p$ are critically stable.

Application of the previous result to system (6) is straightforward since, for each $p \in \mathcal{P}$, system (6) is stabilizable and has no eigenvalue in the open right-half plane of the complex plane. The controller $\mathbb{C}_p$ has the equation

$$v = \chi_p(x) \;, \quad p \in \mathcal{P} \;, \tag{10}$$

with $\chi_p(x)$ the function on the right-hand side of (9).

Corollary 1. For each $p \in \mathcal{P}$, system (6) in closed-loop with $v = \chi_p(x)$, namely

$$\dot{x} = (A + KC_p)x + B\mathrm{sat}(\chi_p(x)) - Ke_p \;, \tag{11}$$

is iISS with respect to e_p with quadratic gain function. In particular, there exist class-$\mathcal{K}_\infty$ functions $\alpha(\cdot), \tilde{\theta}_1(\cdot), \tilde{\theta}_2(\cdot)$, and a constant $\bar{\gamma} > 0$ such that the solution $x(t)$ of (11) from the initial condition

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[2] $\mathcal{K}$ is the class of functions $[0, \infty) \to [0, \infty)$ which are zero at zero, strictly increasing and continuous. $\mathcal{K}_\infty$ is the subclass of functions $\mathcal{K}$ which are unbounded.

[3] The number ν depends on the eigenstructure of $\tilde{A}$. If $\tilde{A}$ is critically stable, i.e. there exists a $\tilde{P} > 0$ such that $\tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} \le 0$, then $\nu = 1$ and the feedback (9) is linear.

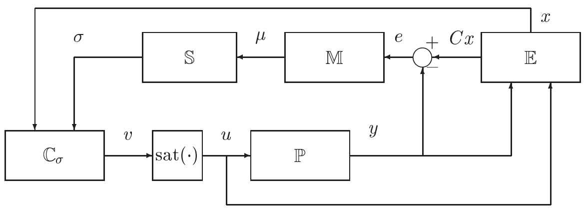

Fig. 2. Supervisory control system in the presence of input saturation.

$x(t_0) = x_0$ under the input e_p satisfies, for all $t \geq t_0 \geq 0$, all x_0 and all e_p,

$$\alpha(|x(t)|) \leq \tilde{\theta}_1(\tilde{\theta}_2(|x_0|)e^{-(t-t_0)}) + \int_{t_0}^{t} \bar{\gamma}|e_p(\tau)|^\ell d\tau \ ,$$

with $\ell = 2$. Also, there exist positive real numbers $a_1, a_2, a_3, \bar{s}$, and smooth functions $W_p : \mathbb{R}^n \to \mathbb{R}$ such that, for all $|x| \in [0, \bar{s}]$, $a_1|x|^2 \leq W_p(x) \leq a_2|x|^2$ and

$$\frac{\partial W_p}{\partial x}((A+KC_p)x+B\text{sat}(\chi_p(x)))\leq -a_3|x|^2 \ .$$

5. ANALYSIS

To complete the supervisory control architecture we need to define the switching logic $\mathbb{S}$.

5.1 *Hysteresis switching logic*

We consider the scale-independent hysteresis switching logic proposed in (Hespanha, 1998; Hespanha and Morse, 1999a).

Hysteresis Switching Logic $\mathbb{S}_{\text{H}}$ (Hespanha, 1998; Hespanha and Morse, 1999a). Fix a constant $h > 0$ (the so-called hysteresis constant) and consider the monitoring signals μ_p generated according to the equations (5). Set $\sigma(0) = \text{argmin}_{p\in\mathcal{P}}\{\mu_p(0)\}$. Suppose that at a certain time t_i the value of σ has just switched to some $q \in \mathcal{P}$. σ is then kept fixed until a time $t_{i+1} > t_i$ such that $(1 + h)\min_{p\in\mathcal{P}}\{\mu_p(t_{i+1})\} \leq \mu_q(t_{i+1})$, at which point $\sigma(t_{i+1}) = \text{argmin}_{p\in\mathcal{P}}\{\mu_p(t_{i+1})\}$. Repeating this procedure a piecewise constant switching signal σ is generated.

We recall that the system (see Figure 2) is composed by an unknown process $\mathbb{P}$ of the form (1), a multi-estimator $\mathbb{E}$ described by the equations (4), a controller $\mathbb{C}_\sigma$

$$v = \chi_\sigma(x) \qquad (12)$$

and monitoring signal generators $\mathbb{M}_p$ described by the set of equations (5). Equations (1), (4), (12), (5) define a hybrid dynamical system of the form

$$\dot{z} = f_\sigma(z) \ , \quad \text{with } z = [x_{\mathbb{P}}^T \ x^T \ \mu_1 \ \ldots \ \mu_m]^T \ , \quad (13)$$

whose solution, along with the signal σ generated by the hysteresis switching logic $\mathbb{S}_{\text{H}}$, exists and is unique on $[0,\infty)$ for each initial condition $\{z(0), \sigma(0)\}$ (see Hespanha and Morse, 1999b). The switching logic $\mathbb{S}_{\text{H}}$ in the present context has the desirable property to stop switching in finite time without any extra assumption. Indeed, in view of the construction of the multi-estimator, and of the exact matching assumption, the hypothesis of the so-called Scale-Independent Hysteresis Switching Theorem (Hespanha and Morse, 1999a) are fulfilled, provided that the parameter λ in (5) is taken sufficiently small. In particular, we have (Hespanha and Morse, 1999a)

Lemma 2. If $\lambda \in (0, \ell\bar{\lambda})$, then for each initial state $z(0)$ with $\mu_p(0) > 0$ for $p \in \mathcal{P}$, for each $\sigma(0) \in \mathcal{P}$, the solution z of (13) and the output σ of $\mathbb{S}_{\text{H}}$ exist and are unique for all $t \in [0,\infty)$. Moreover, there exists a time $T^* < \infty$ such that $\sigma(t) = q^*$ for all $t \geq T^*$. Finally, $e_{q^*} \in \mathcal{L}_\ell[0,\infty)$.

Let's consider the multi-estimator (4) under the feedback interconnection $y = C_p x - e_p$, $p \in \mathcal{P}$, (cf. (6)) in closed-loop with the controller (12), and the monitoring signal generators (5) with $\ell = 2$. In view of Lemma 2, we know that from T^* on $\sigma = q^*$. Hence, from T^* on this feedback interconnection has equations (11) written for $p = q^*$. From Corollary 1, we know that the system is iISS with a quadratic gain function with respect to the input e_{q^*}. As $e_{q^*} \in \mathcal{L}_2[0,\infty)$, by Lemma 2, then by Proposition 6 in (Sontag, 1998) x must converge to zero as t tends to infinity. The analysis now follows that in (Hespanha *et al.*, 2001) and the following holds.

Theorem 1. Let $\mathbb{P}$ be the process (1), unknown member of the family of nominal plant models (2). Suppose that Assumption 1 holds. Consider the supervisory control system described by the equations (4), (12), and (5), with $\ell = 2$, along with the hysteresis switching logic $\mathbb{S}_{\text{H}}$. Then, for each set of initial conditions $x_{\mathbb{P}}(0)$, $x(0)$, $\mu_p(0) > 0$, $p \in \mathcal{P}$, $\sigma(0)$, the response of the supervisory control system exists for all $t \geq 0$ and all the continuous states converge to zero as t goes to infinity.

5.2 *Dwell time switching logic*

Let $\alpha(\cdot)$, $\tilde{\theta}_1(\cdot)$, $\tilde{\theta}_2(\cdot) \in \mathcal{K}_\infty$ and $a_1, a_2, a_3, \bar{s} > 0$ be as in Corollary 1. Define the functions

$$\theta_1(r) = \tilde{\theta}_1^{-1}\left(1/2\alpha\left(1/3r\right)\right) \ , \quad \theta_2(r) = \tilde{\theta}_2(r) \ , \quad (14)$$

and set [4]

[4] Note that $\theta_2(r)/\theta_1(r) > 1$ for all $r > 0$, and (15) is well-posed (cf. De Persis *et al.*, 2002b).

$$\tau_\Delta(r) := \ln(\theta_2(r)/\theta_1(r)) \;, \quad r > 0 \;. \qquad (15)$$

Let $\bar{r} := \theta_2^{-1}(\theta_1(3\bar{s}))$, and fix a "dwell-time" function $\tau_{\rm D} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ satisfying

$$\tau_{\rm D}(r) \geq \begin{cases} \tau_\Delta(r) \;, & r \geq \bar{r} \\ \max\{\tau_\Delta(\bar{r}), \dfrac{3a_2}{a_3} \ln \dfrac{a_2}{a_1}\} \;, & r < \bar{r} \;. \end{cases} \qquad (16)$$

With the notations introduced above, we consider the following switching logic.

State Dependent Switching Logic $\mathbb{S}_{\rm SD}$ (Morse, 1996; De Persis *et al.*, 2002b). Set $\sigma(0) = \mathrm{argmin}_{p\in\mathcal{P}}\{\mu_p(0)\}$. Suppose that at some time t_0, σ has just changed its value to p. At this time, a *timing signal* τ is reset to 0 and a variable X is set equal to $|x(t_0)|$, that is in X is "stored" the magnitude of the state of the plant at that switching time. Compute now the *dwell-time* $\tau_{\rm D}(X)$. At the end of the switching period, when $\tau = \tau_{\rm D}(X)$, if there exists the minimal value $q \in \mathcal{P}$ such that μ_q is smaller than μ_σ, then σ is set equal to q, τ is reset to zero and the entire process is repeated. Otherwise, a new search for the minimal value $q \in \mathcal{P}$ such that μ_q is smaller than μ_σ is carried out.

Along the same lines of the proof of Lemma 1 in (Morse, 1996), it is possible to prove the following property for the switching signal σ generated by $\mathbb{S}_{\rm SD}$ driven by the μ_p's (5).

Lemma 3. Let $\mathcal{T} := \{0 =: t_0, t_1, \ldots, t_j, \ldots\}$ be the sequence of switching times of σ. Then there exists a finite subset $\mathcal{P}^* \subset \mathcal{P}$ containing p^* with the following properties.

(i) There exists a finite switching time $t^* \in \mathcal{T}$ such that $\sigma(t) \in \mathcal{P}^*$ for all $t \geq t^*$;
(ii) For each $p \in \mathcal{P}^*$, $e_p \in \mathcal{L}_\ell[0,\infty)$.

The lemma is instrumental in proving the following theorem.

Theorem 2. Let $\mathbb{P}$ be the process (1), unknown member of the family of nominal plant models (2). Suppose that Assumption 1 holds and that the function sat$(\cdot)$ is continuously differentiable in a neighborhood of the origin. Consider the supervisory control system described by the equations (4), (12), and (5), with $\ell = 2$, along with the state dependent dwell time switching logic $\mathbb{S}_{\rm SD}$, with $\tau_{\rm D}(\cdot)$ satisfying (16). Then, for each set of initial conditions $x_{\mathbb{P}}(0)$, $x(0)$, $\mu_p(0) > 0, p \in \mathcal{P}$, $\sigma(0)$, the response of the supervisory control system exists for all $t \geq 0$ and all the continuous states converge to zero as t goes to infinity.

Remark. It is worth noting that the controller (12) actually depends on x_σ only. Moreover, if, for each $p \in \mathcal{P}$, the controller $\mathbb{C}_p$ (10) is such that system (11) is iISS with respect to e_p with *linear* gain function (cf. Corollary 1), then, for the supervisory control system described by the equations (4), (12), and (5), with $\ell = 1$, along with the state dependent dwell time switching logic $\mathbb{S}_{\rm SD}$, the conclusion of Theorem 2 still holds.

5.3 *The dwell time function*

In this section, we explicitly compute the function τ_Δ in (15), which allows to define the dwell-time function $\tau_{\rm D}$ given in (16). As is seen from (14), the calculation of τ_Δ amounts to determine the functions $\alpha(\cdot)$, $\tilde{\theta}_1(\cdot)$, $\tilde{\theta}_2(\cdot)$ which appear in Corollary 1. We carry out the computation for systems (2) which are critically stable. Hence, for each $p \in \mathcal{P}$, there exists a symmetric matrix $P_p > 0$ satisfying $(A + KC_p)^T P_p + P_p(A + KC_p) \leq 0$. In this case the controller $\mathbb{C}_p$ (10) is simply

$$v = F_p x \;, \quad p \in \mathcal{P} \;, \qquad (17)$$

with $F_p = -B^T P_p$. In fact, (see (Liu *et al.*, 1996) and also (Isidori, 1999, proof of Proposition 14.1.5)) for each $p \in \mathcal{P}$ there exists a symmetric matrix $Q_p > 0$ and a positive real number $\nu_p > 0$ such that the function

$$W_p(x) = 1/2 x^T Q_p x + \nu_p/3 (x^T P_p x)^{\frac{3}{2}} \;,$$

satisfies, for all $p \in \mathcal{P}$,

$$\frac{\partial W_p}{\partial x}((A + KC_p)x + B\mathrm{sat}(F_p x)) \leq -|x|^2 \;,$$

and, for some $a, b, c > 0$,

$$a|x|^2 \leq W_p(x) \leq b|x|^2 + c|x|^3 \;. \qquad (18)$$

Consider the function $V_p(x) = \ln(1 + W_p(x))$. This function is definite positive and proper and satisfies

$$\dot{V}_p := \frac{\partial V_p}{\partial x}((A+KC_p)x + B\mathrm{sat}(F_p x) - Ke_p) \leq -\frac{|x|^2}{1 + W_p(x)} + \bar{\gamma}|e_p| \leq -\rho(|x|) + \bar{\gamma}|e_p| \;,$$

with $\rho(r) = r^2/(1 + br^2 + cr^3)$ a continuous and positive definite function, and $\bar{\gamma}$ a positive real number. That is to say (cf. Angeli *et al.*, 2000), the closed-loop system (6), (17) is iISS with linear gain function.

The previous inequality and simple manipulations also imply that there exists a $\kappa > 0$ such that

$$\dot{V}_p \leq -\kappa(\exp(V_p/3) - 1)^2 \exp(-V_p) + \bar{\gamma}|e_p| \;.$$

Following the arguments in (Angeli *et al.*, 2000, Lemma IV.2 and Corollary IV.3), the Lemma below can be proven.

Lemma 4. Let $\theta_1(\cdot), \theta_2(\cdot) \in \mathcal{K}_\infty$ be such that, if $r(t)$ satisfies the scalar differential inequality

$$\dot{r}(t) \leq -\tilde{\rho}(r(t)) \ , \quad r(0) \geq 0 \ , \tag{19}$$

with $\tilde{\rho}(r) := \kappa(\exp(r/3) - 1)^2 \exp(-2r)$, then $r(t) \leq \theta_1(\theta_2(r(0))e^{-t})$, for all $t \geq 0$. Then the function $V_p(\cdot)$ computed along the solution of the system (6), (17) satisfies, for all $t \geq 0$,

$$V_p(x(t)) \leq \theta_1(\theta_2(V_p(x(0))e^{-t}) + \int_0^t 2\bar{\gamma}|e_{\sigma(s)}(s)|ds \ .$$

To compute $\theta_1(\cdot)$ and $\theta_2(\cdot)$, we consider the $\mathcal{K}_\infty$ function (Praly and Wang, 1996)

$$\varrho(0) = 0 \ , \ \varrho(s) = \exp(\int_1^s \frac{dw}{\tilde{\rho}(w)}) \text{ for } s > 0 \ ,$$

which is continuously differentiable and satisfies $\varrho'(r)\tilde{\rho}(r) = \varrho(r)$. Setting $s = \varrho(r)$, from (19), we have

$$\dot{s}(t) = \varrho'(r(t))\dot{r}(t) \leq -\varrho(r(t)) = -s \ ,$$

which implies $s(t) = \varrho(r(t)) \leq \exp(-t)\varrho(r(0))$, and

$$r(t) \leq \varrho^{-1}(\varrho(r(0))e^{-t}) =: \theta_1(\theta_2(r(0))e^{-t}) \ .$$

Using Lemma 4 and (18), it is easy then to conclude that the inequality in Corollary 1 holds with $\ell = 1$ and

$$\begin{aligned} \alpha(r) = \ln(1 + ar^2) \ , \quad \tilde{\theta}_1(r) = \varrho^{-1}(r) \ , \\ \tilde{\theta}_2(r) = \varrho(\ln(1 + br^2 + cr^3)) \ . \end{aligned} \tag{20}$$

A straightforward calculation yields the expression of $\varrho(\cdot)$, which, along with (14) and (20), gives, for $s > 0$,

$$\tau_\Delta(s) = \tilde{b}^2 \left[\frac{1}{4}\zeta^4 + \frac{5}{3}\zeta^3 + 5\zeta^2 + 10\zeta + 5\ln\zeta - \frac{1}{\zeta} \right]_{\zeta=\zeta_1}^{\zeta=\zeta_2} ,$$

where $\tilde{b}$ is such that $1 + bs^2 + cs^3 \leq (1 + \tilde{b}s)^3$ for all $s \geq 0$, $\zeta_1 = (1 + as^2/9)^{1/6} - 1$ and $\zeta_2 = (1 + bs^2 + cs^3)^{1/3} - 1$. The expression shows that $\tau_\Delta(s)$ grows *polynomially* with s.

6. CONCLUSIONS

In this paper we have presented a solution to the problem of supervisory control of largely uncertain systems under input constraint. It has been shown that if the unknown process belongs to a finite class of linear open-loop unstable but not exponentially unstable systems which are stabilizable and detectable, then it is possible to design a supervisory control architecture so as to achieve global regulation of the state to zero.

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