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STABILITY, CONVERGENCE, AND FEEDBACK EQUIVALENCE OF LTI ITERATIVE LEARNING CONTROL

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Abstract: The goal of Iterative Learning Control (ILC) is to improve the accuracy of a system that repeatedly follows a reference trajectory. This paper proves that for any causal ILC, there is an equivalent feedback that achieves or approaches the ultimate ILC error with no iterations. Remarkably, this equivalent feedback depends only on the ILC operators and hence requires no plant knowledge. This equivalence is obtained whether or not the ILC includes current-cycle feedback. The equivalence is proved for general nonlinear systems, except for the special case of zero ultimate ILC error, which is investigated for LTI systems only. Conditions are obtained for internal stability and convergence of ILC, as these are used to prove equivalence in the zero error case. Since conventional feedback requires no iterations, there is no reason to use causal ILC.

Keywords: Iterative learning control; Feedback control; Stability; Convergence; Linear systems.

1. INTRODUCTION

In many practical control applications, a reference trajectory is repeated several times. For example, a robotic manipulator might perform the same movements thousands of times. This provides an opportunity to increase the accuracy of the system by learning from previous trials. This approach is called Iterative Learning Control (ILC) and was originally proposed in (Arimoto *et al.*, 1984).

The literature on ILC is vast. A survey of the field citing 254 papers is presented in (Moore, 1999). ILC research includes linear time-invariant (LTI) systems (Moore, 1993), discrete-time systems (Phan *et al.*, 2000), and nonlinear systems (Ahn *et al.*, 1993). Analyses consider ILC algorithms that contain general linear operators (Moore, 1993), as well as specific algorithms, such as derivative ILC (Arimoto *et al.*, 1984) and proportional ILC (Saab, 1994). Many analyses include 'current-cycle feedback' in the ILC algorithm (Chen *et al.*, 1996) to stabilize the plant and improve performance. The effect of disturbances and initial conditions is considered in (Heinzinger *et al.*, 1992). Most ILC algorithms are based on causal operators, although non-causal operators have been considered in discrete-time ILC (Phan *et al.*, 2000) and in recent work on continuous LTI ILC (Chen and Moore, 2000).

In this paper, we consider causal LTI ILC with current cycle feedback. The LTI assumption allows the use of frequency-domain techniques. Related work on LTI ILC investigates performance and robustness (Liang and Looze, 1993) and optimization (Amann *et al.*, 1998). A comparison between time and frequency domain stability results appears in (Judd *et al.*, 1993). The design of ILC for non-minimum phase plants is considered in (Amann and Owens, 1994) and (Roh *et al.*, 1996).

This paper proves that for any causal ILC algorithm, there exists a feedback control that matches the ultimate tracking error of the ILC without any iterations, unless the ultimate ILC error is zero. In the latter case, there exists an internally stabilizing controller that approaches zero error at high gain. The purpose of this paper is not to propose a new control method or a new approach to feedback design. Rather, it is to show that the performance of causal ILC is limited to that of conventional feedback control, which is preferred since it does not require iterations. Hence, only non-causal ILC methods should be considered.

This paper is organized as follows. In Section 2, we introduce the general ILC and derive an expression for the ultimate tracking error. In Section 3, we obtain an equivalent feedback control that achieves the ultimate ILC error with no iterations for the case of nonzero ultimate ILC error. Sections 4 and 5 report conditions for ILC internal stability and convergence. These results are used in Section 6 to obtain a stable equivalent feedback system for the zero error case. Section 7 concludes the paper.

2. ITERATIVE LEARNING CONTROL

In this section, we apply ILC to a nonlinear plant. The ILC includes a current-cycle feedback term, which may be set to zero if the plant is stable.

Let $T > 0$, and let L^m denote the space of piecewise continuous, square integrable functions $u : [0,T] \to R^m$, with norm

$$||u|| = \sqrt{\int_0^T u^T(t)u(t)dt} < \infty. \qquad (1)$$

The system to be controlled is modelled as

$$y_i = Pu_i, \qquad (2)$$

where $u_i \in L^m$ is a control signal, $P : L^m \to L^n$ is a causal, nonlinear time-varying operator representing the plant, $y_i \in L^n$ is the plant output, and $i \in \{0, 1, 2, ...\}$ is the trial number. It is desired that y_i follow a reference trajectory $y_d \in L^n$ and that the tracking accuracy improve as the number of trials increases. Let us define the error in trial i as $e_i = y_d - y_i$. Then, (2) may be written as

$$e_i = y_d - Pu_i. \qquad (3)$$

The general ILC algorithm considered in this paper is given by

$$u_i = Fu_{i-1} + Ce_i + De_{i-1}, \qquad (4)$$

where F, C, and D are bounded causal operators. These operators may be nonlinear, but are assumed to map null signals into null signals (i.e. $F(0) = 0$, etc.) As noted in the survey article (Moore, 1999), this general ILC law has appeared in several papers, usually with linear operators assumed. The term Ce_i is the feedback term and is sometimes called 'current-cycle feedback' to distinguish it from De_{i-1}.

Let $u_\infty \in L_m$ and $e_\infty \in L_n$ denote fixed points of the ILC system (3) and (4). These signals satisfy $u_i = u_\infty$ and $e_i = e_\infty$ if $u_{i-1} = u_\infty$ and $e_{i-1} = e_\infty$ (Moore, 1993). Setting $u_i = u_{i-1} = u_\infty$ and $e_i = e_{i-1} = e_\infty$ in (3) and (4) gives

$$e_\infty = y_d - Pu_\infty \qquad (5)$$

$$(I - F)u_\infty = (C + D)e_\infty, \qquad (6)$$

where the (nonlinear) operator $C+D$ is defined as $(C+D)x = Cx+Dx$, and $I-F$ is defined similarly. It is assumed that the fixed point system (5) and (6) has a unique solution for u_∞ and e_∞.

If $(I - F)^{-1}$ is defined, then (5) and (6) may be written as

$$e_\infty = y_d - Pu_\infty \qquad (7)$$

$$u_\infty = Ke_\infty, \qquad (8)$$

where

$$K = (I - F)^{-1}(C + D). \qquad (9)$$

Note that K is a causal operator since F, C, and D are. Since the solution to (7) and (8) is assumed to be unique, it may be written as

$$e_\infty = (I + PK)^{-1}y_d \qquad (10)$$

$$u_\infty = K(I + PK)^{-1}y_d. \qquad (11)$$

Also of interest is the case $F = I$, which makes $(I - F)^{-1}$ undefined. This case gives $e_\infty = 0$ as a solution to (6), but is somewhat pathological due to the strict conditions it imposes on P for ILC convergence. These conditions are obtained for LTI plants in Section 5.

3. EQUIVALENT FEEDBACK CONTROL

In this section, we show that when $(I - F)^{-1}$ is defined, e_∞ of the ILC can achieved in a single trial using feedback control. As we are considering only one trial of feedback control, we drop the subscript i on the signals in (3), and write the open-loop system as

$$e = y_d - Pu \qquad (12)$$

Conventional feedback control has the form

$$u = Ke, \quad (13)$$

where K is necessarily a causal operator.

Theorem 1. Suppose $(I - F)^{-1}$ is defined on L^m, and choose K as in (9). Then the feedback control (13) applied to (12) gives $u = u_\infty$ and $e = e_\infty$.

PROOF. The closed-loop system (12), (13) is identical to the fixed point system (7), (8) and therefore has the identical solution $u = u_\infty$ and $e = e_\infty$ given by (10) and (11).

Remark 1. Since K in the equivalent feedback control depends only on the ILC operators C, D, and F, no additional plant knowledge is required to construct the equivalent feedback control.

Remark 2. Since Theorem 1 includes the case $C = 0$, the equivalent feedback K exists whether or not the ILC includes current cycle feedback C.

Remark 3. The equivalent feedback achieves the fixed point of the ILC whether or not the ILC converges to the fixed point.

4. INTERNAL STABILITY OF ILC

Theorem 1 does not apply if $(I - F)^{-1}$ is undefined. This case is of some interest because $F = I$ gives $e_\infty = 0$ in (10). In Section 6, we extend our equivalence result to the case $e_\infty = 0$ (for LTI systems). It is shown that if the ILC system is internally stable and converges to zero error, then there is an internally stable high-gain feedback that achieves arbitrarily small error (without iterations). In this section, we obtain necessary and sufficient conditions for internal stability of ILC.

All signals are now assumed to be on the infinite interval, and all operators are assumed to be linear and time-invariant (LTI). The space L^m is redefined accordingly with $T = \infty$ in (1). Since we will be considering systems that may be unstable, we also define the extended space

$$L_e^m = \{u | u_\tau \in L^m, \forall \tau \geq 0\}, \quad (14)$$

where

$$u_\tau(t) = \begin{cases} u(t), & 0 \leq t \leq \tau \\ 0, & t > \tau \end{cases} \quad (15)$$

The ILC system is internally stable if e_i and u_i remain bounded in each trial in the presence of bounded disturbances and noise. Let $d_i \in L^m$ be an input disturbance on trial i, and let $n_i \in L^n$ be a measurement noise. We may rewrite the ILC system (3) and (4) as

$$e_i = y_d - P(u_i + d_i) \quad (16)$$
$$u_i = Fu_{i-1} + C(e_i + n_i) + D(e_{i-1} + n_{i-1}) \quad (17)$$

Since the current cycle feedback C may stabilize the system, P, F, C, and D need not be stable to ensure bounded e_i and u_i. It is assumed that the disturbance sensitivity function

$$Q = (I + CP)^{-1} \quad (18)$$

is defined.

Substitution of (16) into (17) gives

$$u_i = QFu_{i-1} + QCy_d + (Q - I)d_i + QCn_i - QDe_{i-1} + QDn_{i-1}. \quad (19)$$

Here, we have use the fact that $QCP = I - Q$. Substitution of (19) into (16) gives

$$e_i = (I - PQC)y_d - PQFu_{i-1} - PQd_i - PQCn_i + PQDe_{i-1} - PQDn_{i-1}. \quad (20)$$

Note that $I - PQC = (I + PC)^{-1}$.

Definition 1. The closed-loop ILC system (19) and (20) is internally stable if for all $u_{i-1} \in L^m$, $d_i \in L^m$, $e_{i-1} \in L^n$, $n_i \in L^n$, and $n_{i-1} \in L^n$, the outputs u_i and e_i are in L^m and L^n, respectively (i.e. bounded for all time).

Lemma 1. The ILC system (19) and (20) is internally stable if and only if the operators Q, QF, QC, QD, PQF, PQ, PQC, and PQD are stable.

PROOF. If each of the eight operators listed in the lemma is stable, then each term in the right hand side of (19) and (20) is bounded for all bounded inputs, and hence their sums u_i and e_i are bounded. This proves sufficiency. To prove necessity, we will show that if any of the eight operators are unstable, then u_i and e_i are unbounded for certain bounded inputs $u_{i-1}, d_i, e_{i-1}, n_i, n_{i-1}$. Suppose that QC is unstable. Then QCy_d is either bounded or unbounded, depending on y_d. If QCy_d is unbounded, and the other five inputs are zero, then u_i is unbounded . If QCy_d is bounded, then choose $n_i \in L^n$ such that QCn_i is unbounded and choose the remaining four inputs equal to zero, again making u_i unbounded. Hence, QC must be stable for internal stability. By similar reasoning, the other seven operators must be stable as well. □

Lemma 1 in (Goldsmith, 2002) incorrectly lists stability of $PQDP$ as a requirement for internal stability. Note that since

$$e_{i-1} = y_d - P(u_{i-1} + d_{i-1}), \quad (21)$$

the signals e_{i-1}, u_{i-1}, and d_{i-1} are dependent. This is why d_{i-1} does not appear in (19) and (20). For example, $e_{i-1} = u_{i-1} = 0$ implies $d_{i-1} = u_d$.

5. CONVERGENCE OF ILC

To investigate the convergence of ILC, the d_i and n_i in (19) are set to zero, which gives

$$u_i = Q(F - DP)u_{i-1} + (I - Q + QDP)u_d \quad (22)$$

The fixed point of (22) is given by

$$u_\infty = Q(F - DP)u_\infty + (I - Q + QDP)u_d. \quad (23)$$

We may shift the fixed point u_∞ to the origin by defining

$$v_i = u_i - u_\infty. \quad (24)$$

Substituting $u_i = v_i + u_\infty$, $u_{i-1} = v_{i-1} + u_\infty$, and (23) into (22) gives

$$v_i = Q(F - DP)v_{i-1}, \quad (25)$$

which implies that

$$v_i = H^i v_0, \quad (26)$$

where

$$H = Q(F - DP). \quad (27)$$

Note that H is proper since F, D, and P are all assumed proper.

We similarly define a shifted error

$$z_i = e_i - e_\infty \quad (28)$$

Substitution of (3), (10), and (26) into (28) gives

$$z_i = -Pv_i \quad (29)$$

Substitution of (26) into (29) gives

$$z_i = -PH^i v_0 \quad (30)$$

Definition 2. The ILC system (30) *converges* if for each $v_0 \in L^m$, $lim_{i\to\infty} z_i = 0$

Since $||H^i|| \leq ||H||^i$, a sufficient condition for convergence is PH stable and $||H|| < 1$. We may write H as

$$H = I - M, \quad (31)$$

where

$$M = I - Q(F - DP). \quad (32)$$

For the remainder of this paper, it is assumed that the system is SISO ($n = m = 1$) and that P and u_d are not identically zero. We will use a common symbol to represent a signal and its Laplace transform, and similarly for operators; the meaning should be clear from the context. The norm $||u||$ may be viewed either as the 2-norm of the signal $u \in L$ or the 2-norm of its Laplace transform, as these are equal. We denote the closed right half plane as $\Omega = \{s|\text{Re } s \geq 0\}$.

Lemma 2. Suppose that $n = m = 1$ and the ILC system (19) and (20) is internally stable. Then (30) converges only if $M(s) \in \Omega$ for all $s \in \Omega$.

PROOF. Suppose that there is an $s_0 \in \Omega$ such that $M(s_0)$ is not in Ω. Then the real part of $-M(s)$ is strictly greater than zero, and so

$$|H(s_0)|^2 = |1 - M(s_0)|^2 \quad (33)$$
$$> 1 + |M(s_0)|^2. \quad (34)$$

If H is a constant, then $|H|^i$ grows unbounded as i grows large, and the system does not converge. Therefore, suppose H is not constant. Since H is also stable, the Maximum Modulus Theorem (Doyle *et al.*, 1992) states that $|H(s)|$ does not achieve its maximum at an interior point of Ω. Hence, there is an $\omega_0 \in R$ such that

$$|H(j\omega_0)|^2 > 1 + |M(s_0)|^2. \quad (35)$$

Since $|H(j\omega)|$ is continuous ($H(s)$ is stable and finite-dimensional), there exists $b \in (1, 1 + |M(j\omega_0)|^2)$ and $\delta > 0$ such that

$$|H(j\omega)|^2 > b \quad (36)$$

for all $\omega \in [\omega_0 - \delta, \omega_0 + \delta]$. Since PH^i is stable for each $i \geq 1$, the size of z_i is given by

$$||z_i||^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |P(j\omega)|^2 |H(j\omega)|^{2i} |v_0(j\omega)|^2 d\omega \quad (37)$$

$$\geq \frac{b^i}{2\pi} \int_{\omega_0-\delta}^{\omega_0+\delta} |P(j\omega)|^2 |v_0(j\omega)|^2 d\omega. \quad (38)$$

Since P is not identically zero and $P(s)$ is finite-dimensional, $|P(j\omega)|$ can be zero at only a finite number of points in $[\omega_0 - \delta, \omega_0 + \delta]$. Choose the input $v_0 \in L$ such that $v_0(j\omega)$ is nonzero over a finite sub-interval of $[\omega_0 - \delta, \omega_0 + \delta]$. Then the integral in (38) is nonzero. Taking the limit of both sides of (38) gives

$$\lim_{i\to\infty} ||z_i|| = \infty. \quad (39)$$

Hence, the system does not converge. □

With $n = m = 1$, the fixed point of the ILC error is obtained from (10) and $y_d = Pu_d$ as

$$e_\infty = [1 - P(1 - F + CP + DP)^{-1}(C + D)]Pu_d. \quad (40)$$

Lemma 3. If $e_\infty = 0$ in (40), then $F = 1$.

PROOF. Setting $e_\infty = 0$ in (40), applying P^{-1} to both sides, and rearranging gives

$$u_d = (1 - F + CP + DP)^{-1}(C + D)Pu_d. \quad (41)$$

Applying $(1 - F + CP + DP)$ to both sides of (41) gives

$$(1 - F)u_d + (C + D)Pu_d = (C + D)Pu_d, \quad (42)$$

which gives

$$(1 - F)u_d = 0. \quad (43)$$

Clearly, $F = 1$ is a solution to (43). To see that this solution is unique, suppose there is another solution, $F \neq 1$. Then $(1 - F)^{-1}$ exists, and multiplying (43) through by $(1 - F)^{-1}$ gives $u_d = 0$, which contradicts the assumption that $u_d \neq 0$. □

If $e_\infty = 0$ ($F = 1$), then $z_i = e_i$ and (30) becomes

$$e_i = -PH^i v_0 \quad (44)$$

Setting $F = 1$ in (32) gives

$$M = 1 - Q + QDP \quad (45)$$
$$= QCP + QDP. \quad (46)$$

For an internally stable ILC, Lemma 2 and (46) can be used to prove that e_i in (44) converges to zero for all $v_0 \in L$ only if $P(s)$ is minimum phase and the relative degree of P is less than or equal to one. Non-minimum phase plants are considered in (Amann and Owens, 1994), while the relative degree requirement is mentioned in (Ahn *et al.*, 1993). In the next section, we will use Lemma 2 to extend our feedback equivalence result to the zero error case.

6. FEEDBACK EQUIVALENCE WHEN ULTIMATE ILC ERROR IS ZERO

In Section 4, conditions were obtained for ILC to be internally stable (bounded u_i and e_i in the presence of bounded noise and disturbance inputs). Section 5 reported conditions for ILC to converge in the absence of noise and disturbance inputs. In this section, we present the following result: if the ILC is internally stable and converges to *zero error*, then there exists an internally stabilizing high-gain feedback that converges (as the gain grows large) to zero error without iterations.

SISO systems are assumed ($n = m = 1$). The equivalent feedback is required to converge to zero error in the absence of initial conditions, noise, and disturbances. Substitution of the feedback control (13) into (12), along with the achievability condition $y_d = Pu_d$, gives

$$e = (1 + PK)^{-1}Pu_d \quad (47)$$
$$u_d - u = (1 + PK)^{-1}u_d. \quad (48)$$

Hence, the feedback K must give arbitrarily small e and $u_d - u$ in (47) and (48).

The second requirement on K is that it internally stabilizes P (i.e. in the presence of initial conditions, noise, and disturbances). This requirement is equivalent to the stability of $(1 + PK)^{-1}$, $(1 + PK)^{-1}K$, and $(1 + PK)^{-1}P$ (Doyle *et al.*, 1992). Lemma 1 and Lemma 2 may be used to prove the following theorem (Goldsmith, 2002).

Theorem 2. Suppose that the ILC system (44) is internally stable and that for each $v_0 \in L$, $\lim_{i\to\infty} e_i = 0$. Then, the feedback

$$K = C + k(C + D) \quad (49)$$

internally stabilizes P for all $k \geq 0$ and gives $lim_{k\to\infty} e = 0$ and $lim_{k\to\infty} u = u_d$ in (47) and (48).

Remark 4. As in Theorem 1, K depends only on the ILC operators C and D, not on the plant P (although the size of the gain factor k required for a given $||e||$ may depend on P).

Remark 5. Since Theorem 2 includes the case $C = 0$, the equivalent feedback K exists whether or not the ILC includes current cycle feedback C.

Remark 6. The achievability assumption, $y_d = Pu_d$ for bounded u_d, permits tracking of *unbounded* y_d when P is unstable (for example, when P is an integrator).

Theorem 2 implies that even as the gain approaches infinity, the control signal u remains bounded, since it converges to a bounded u_d. This is because the product ke remains bounded as k approaches infinity and e approaches zero. Also, for a given k, internal stability implies that u remains bounded in the presence of initial conditions, disturbances, and noise. However, for given initial conditions or noise, $||u||$ grows with k, so a saturation function and/or noise filter may be required (if the filtering provided by $C + D$ is inadequate).

7. CONCLUSION

Conventional feedback control is preferrable to causal ILC because it can achieve in one trial the same error that ILC achieves in an arbitrarily large number of trials. An equivalent feedback can always be constructed from the ILC parameters with no additional plant knowledge, whether or not the ILC itself includes feedback. If the ILC converges to *zero* error, the equivalent feedback is a high gain controller whose gain k plays the role of the iteration number i in the ILC. The internal stability and convergence (as $i \to \infty$) of the ILC guarantees the internal stability and convergence (as $k \to \infty$) of the equivalent high-gain feedback. This result is extended to discrete-time systems in (Goldsmith, 2001).

The feedback equivalence result holds only when F, C, and D are causal (so that K is also causal and hence implementable as a feedback operator). However, non-causal F and D can be used in ILC since the full signals u_{i-1} and e_{i-1} are available in trial i. This approach is considered in (Phan *et al.*, 2000) and (Chen and Moore, 2000). The potential benefit of non-causal ILC over conventional feedback control requires investigation. Non-causal algorithms need not be restricted to the *fixed-point algorithms* currently identified with ILC, which may give limited performance even with non-causal operators.

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