15th Triennial World Congress, Barcelona, Spain

CONSTANT OUTPUT TRACKING AND DISTURBANCE REJECTION FOR SYSTEMS WITH LIPSCHITZ NONLINEARITIES

A. Behçet Açıkmeşe, Martin Corless *

* *School of Aeronautics and Astronautics, Purdue University, West Lafayette, Indiana 47907, USA,*
{*acikmese, corless*}*@ecn.purdue.edu*

Abstract: In this paper, we consider nonlinear systems subject to unknown constant disturbance inputs. We wish to design state feedback and observer-based output feedback controllers which ensure that the system output asymptotically tracks a specified constant reference signal and all states are bounded. Our results establish design procedures of state feedback and observer-based PI controllers for a class of nonlinear systems with globally Lipschitz nonlinearities.

1. INTRODUCTION

Synthesizing feedback controllers to achieve asymptotic tracking of prescribed reference outputs while rejecting disturbances is a fundamental problem in control. The theory of output regulation for multivariable, linear, time-invariant (LTI) systems was studied extensively by Davison (Davison 1976).

Desoer and Lin (Desoer and Lin 1985) showed that, if a nonlinear plant is exponentially stable about a unique equilibrium point for each constant control input and has a strictly increasing steady state input-output map, then a PI controller with sufficiently small gain matrices achieves asymptotic tracking of reference signals tending to constant vectors while rejecting disturbances tending to constant vectors.

The review article written by Byrnes and Isidori (Byrnes and Isidori 1998) outlines some relevant results obtained in output tracking. They considered the problem setup where the disturbance inputs and desired outputs to be tracked range over all possible trajectories of a given LTI system. Necessary and sufficient conditions were given for LTI systems and an extension of the result to nonlinear systems was also described.

Schmitendorf and Barmish (Schmitendorf and Barmish 1986) designed controllers for uncertain LTI systems with constant disturbance inputs. They assumed that the system satisfies "matching conditions" and the uncertain parameters were in known compact sets. Their controller involves linear state feedback and the integral of the tracking error. This controller achieves asymptotic stability of the closed loop system and asymptotic tracking of any desired constant output. Their technique can be viewed as a quadratic stabilization approach for uncertain LTI systems satisfying "matching conditions".

Benson and Schmitendorf (Benson and Schmitendorf 1997) considered a robust tracking problem for a more general class of uncertain LTI systems. They designed state feedback controllers with an augmented state approach, as in (Schmitendorf and Barmish 1986), by solving a robust stabilization problem and achieved asymptotic tracking with zero steady state error. Then they also designed observer based output feedback controllers by solving a robust H_∞ problem. In this case the steady state tracking error is shown to be bounded by a prescribed value for step reference inputs.

In this paper, a constant output tracking problem is considered in the presence of unknown constant disturbances for nonlinear systems with globally Lipschitz nonlinearities. We first consider state feedback PI controllers which has a piece depending linearly on the state and another piece depending on the integral of the tracking error. Then we consider observer based PI controllers where the estimate of state and the integral of the estimate of the tracking error is used. Therefore a nonlinear observer with a model of the nonlinearity in the system is used to estimate the states of the system and the exogenous input, which consists of the disturbance and reference signals. Arcak and Kokotovic (Arcak and Kokotovic 2001) designed nonlinear observers to stabilize systems with slope-restricted nonlinearities. Their design achieved a controller-observer separation, where each component can be designed independently.

Our design procedures achieve that the tracking error exponentially decays to zero and the states of the system and the observer remain bounded for the class of systems considered. In the case of observer based design, a controller-observer separation is achieved and each part of the design involves solution of an LMI (linear matrix inequality) (Boyd *et al.* 1994), (Balakrishnan and Kashyap 1999).

2. A CLASS OF NONLINEAR SYSTEMS

In this paper, we consider nonlinear systems of the following form:

$$\begin{aligned} \dot{x} &= Ax + B_p\Delta(q) + B_u u + B_w w \\ q &= C_q x + D_{qu} u + D_{qw} w \\ y &= C_y x + D_{yu} u + D_{yw} w \\ z &= C_z x + D_{zu} u + D_{zw} w \end{aligned} \tag{1}$$

where Δ is a known globally Lipschitz function (a slope-restricted nonlinearity in the scalar case) satisfying

$$||\Delta(q_1) - \Delta(q_2)|| \leq \sigma ||q_1 - q_2|| \quad \text{for all } q_1, q_2 \tag{2}$$

and for some scalar $\sigma > 0$. The vector $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{m_u}$ is the control input, $y(t) \in \mathbb{R}^{k_y}$ is the measured output, $z(t) \in \mathbb{R}^{k_z}$ is the performance output, $w \in \mathbb{R}^{m_w}$ is the exogenous input containing constant disturbance and reference signals and $q(t) \in \mathbb{R}^{k_q}$. Note that the condition (2) is automatically satisfied when Δ is a differentiable function with a bound on the norm of the derivative.

Our objective is to design a controller which achieves asymptotic tracking of constant reference signals by a performance output, in the presence of constant disturbances while keeping the state and control input bounded. By defining w as a vector composed of disturbance and reference signals and by appropriate definition of the performance output z, the design objective of asymptotic tracking can always be expressed as

$$\lim_{t\to\infty} z(t) = 0 \,. \tag{3}$$

3. STATE FEEDBACK PI CONTROLLERS

In this section we will consider the case in which the state x and the performance output z can be measured, that is, $y = (x, z)$. The analysis and design carried out for this case will be a step to the more general case where the measured output does not contain this information.

We introduce first the following additional dynamics to the system dynamics (1)

$$\dot{x}_I = z \,, \tag{4}$$

where x_I is called the integrator state. Then a proposed controller is a PI (proportional integral) controller given by

$$u = K_P x + K_I x_I \,. \tag{5}$$

where the gain matrices K_P and K_I are to be determined. The following observation will be used as a basis for further analysis and design.

Lemma 1. Suppose that for each constant exogenous input w, the closed loop system described by (1), (4) and (5) has a GAS (globally asymptotically stable) equilibrium state. Then for each constant exogenous input w, the closed loop system has the following properties: The state $x(\cdot)$ and control input $u(\cdot)$ are bounded and $\lim_{t\to\infty} z(t) = 0$.

Proof Let $\xi = (x, x_I)$ be the state for the closed loop system and $\xi^* = (x^*, x_I^*)$ be the equilibrium point corresponding to w. Then the closed loop dynamics can be described by $\dot{\xi} = G(\xi, w)$, for some continuous function G, and $G(\xi^*, w) = 0$. Since the equilibrium point is GAS, it follows that $\xi(\cdot)$, and hence, $x(\cdot)$ and $u(\cdot)$ are bounded.

Since $\lim_{t\to\infty} \xi(t) = \xi^*$, it follows that

$$\lim_{t\to\infty} \dot{\xi}(t) = \lim_{t\to\infty} G(\xi(t), w) = G(\xi^*, w) = 0 \,.$$

Since $\dot{x}_I = z$, it now follows that $\lim_{t\to\infty} z(t) = 0$. □

At this point, we can give the first main result of this paper which establishes a procedure to design controller gain matrices K_P and K_I. This procedure involves solution of a LMI.

Theorem 2. Consider the dynamical system described by (1). Suppose there exist matrices $Q = Q^T > 0$ and S such that the following LMI is satisfied

$$\begin{bmatrix} \hat{A}Q + Q\hat{A}^T + \hat{B}S + S^T\hat{B}^T + \hat{B}_p\hat{B}_p^T & Q\hat{C}_q^T + S^T D_{qu}^T \\ \hat{C}_q Q + D_{qu} S & -\frac{1}{\sigma^2} I \end{bmatrix} < 0 \tag{6}$$

where

$$\hat{A} = \begin{bmatrix} A & 0 \\ C_z & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_u \\ D_{zu} \end{bmatrix}, \quad \hat{B}_p = \begin{bmatrix} B_p \\ 0 \end{bmatrix}$$

and $\hat{C}_q = [C_q \ 0]$. Letting

$$\begin{bmatrix} K_P & K_I \end{bmatrix} = SQ^{-1}, \tag{7}$$

the closed loop system (1), (4), (5), (7) has the following properties for each constant exogenous input (disturbance and reference signals) w:

- **(a)** There exists a GES (globally exponentially stable) equilibrium state
- **(b)** Consequently $x(\cdot)$ and $u(\cdot)$ are bounded and

$$\lim_{t\to\infty} z(t) = 0 .$$

4. OBSERVER BASED PI CONTROLLERS

In this section we consider the case where the only available piece of information for feedback is the measured output y. The measured output does not necessarily contain the full state information or the performance output as required by the PI controller (5). Therefore, an observer is augmented to the system in order to to estimate the state and the exogenous input which are used in obtaining the control input. The integrator and observer dynamics can be described by

$$\dot{x_I} = C_z\hat{x} + D_{zu}u + D_{zw}\hat{w} \tag{8}$$

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B_p\Delta(\hat{q}) + B_u u + B_w\hat{w} + L_1(\hat{y} - y) \\ \dot{\hat{w}} &= L_2(\hat{y} - y) \end{aligned} \tag{9}$$

where

$$\begin{aligned} \hat{q} &= C_q\hat{x} + D_{qu}u + D_{qw}\hat{w} \\ \hat{y} &= C_y\hat{x} + D_{yu}u + D_{yw}\hat{w} \end{aligned}$$

The matrices L_1 and L_2 are called the observer gain matrices. Then the control input is

$$u = K_P\hat{x} + K_I x_I \tag{10}$$

where K_P and K_I are controller gain matrices. The following observation is a straight forward extension of Lemma 1.

Lemma 3. Suppose for each constant exogenous input w, the closed loop system described by (1), (8), (9) and (10) has a GAS equilibrium state. Then for each constant exogenous input w, the closed loop system has the following properties: The state $x(\cdot)$ and control input $u(\cdot)$ are bounded and $\lim_{t\to\infty} z(t) = 0$.

Our other objective is to establish a procedure to design the controller gain matrices K_P, K_I independently from the observer gain matrices L_1, L_2, thus achieving "controller-observer separation" in design. The following theorem which is the second main result of this paper establishes a design procedure achieving all our objectives for the class of nonlinear systems described in (1).

Theorem 4. Consider the dynamical system described by (1). Suppose there exist matrices $Q = Q^T > 0$ and S such that the LMI (6) is satisfied. Also suppose that there exist matrices $P = P^T > 0$ and Y such that the following LMI is satisfied

$$\begin{bmatrix} P\tilde{A} + \tilde{A}^T P + Y\tilde{C} + \tilde{C}^T Y^T + \tilde{C}_e\tilde{C}_e^T & P\tilde{B}_p \\ \tilde{B}_p^T P & -\frac{1}{\sigma^2}I \end{bmatrix} < 0 \tag{11}$$

where

$$\tilde{A} = \begin{bmatrix} A & B_w \\ 0 & 0 \end{bmatrix}, \ \tilde{B}_p = \begin{bmatrix} B_p \\ 0 \end{bmatrix},$$

$\tilde{C}_e = [C_q \ D_{qw}]$ and $\tilde{C} = [C_y \ D_{yw}]$.

Letting

$$\begin{aligned} \begin{bmatrix} K_P & K_I \end{bmatrix} &= SQ^{-1} \\ \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} &= P^{-1}Y, \end{aligned} \tag{12}$$

the closed loop system (1), (8), (9), (10), (12) has the following properties for each constant exogenous input w:

- **(a)** There exists a GES equilibrium state
- **(b)** Consequently $x(\cdot)$ and $u(\cdot)$ are bounded and

$$\lim_{t\to\infty} z(t) = 0 .$$

It is clear to see that the LMI's (6) and (11) can independently be solved. Therefore the design for K_P, K_I and L_1, L_2 can be done independently, which establishes a "controller-observer separation" in synthesis.

5. PROOF OF THE MAIN RESULTS

Before giving the proofs of Theorems 2 and 4, we establish a lemma which will be used in these proofs (see (Açıkmeşe 2001) for a proof of this lemma). For that reason we introduce the following nonlinear autonomous system

$$\dot{x} = f(x), \tag{13}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function.

Definition 5. The dynamical system (13) is GUAS (globally uniformly asymptotically stable) about a solution, $\overline{x}(\cdot)$ if

- For each $\varepsilon > 0$ and $r > 0$ there exists some $T(\varepsilon, r) > 0$ such that, if $||x(t_0) - \overline{x}(t_0)|| < r$ for any solution of (13) then, $||x(t) - \overline{x}(t)|| < \varepsilon$ for all $t \geq t_0 + T(\varepsilon, r)$

- For each $\varepsilon > 0$, there exists some $\delta(\varepsilon)$ such that, if $||x(t_0) - \bar{x}(t_0)|| < \delta(\varepsilon)$ for any solution $x(\cdot)$ of (13) then, $||x(t) - \bar{x}(t)|| < \varepsilon$ for all $t \geq t_0$.

Lemma 6. Suppose a dynamical system described by (13) is GUAS about a bounded solution $\bar{x}(\cdot)$. Then there exists an equilibrium point x^* of (13) such that every solution $x(\cdot)$ of (13) satisfies

$$\lim_{t\to\infty} x(t) = x^* .$$

Now we can present a proof of Theorem 2.

Proof(Theorem 2) Let $\xi = (x, x_I)$ and $K = [K_P \; K_I]$. Then we can describe the closed loop system (1), (4), (5) by

$$\begin{aligned} \dot{\xi} &= \hat{A}_{cl}\xi + \hat{B}_p\Delta(q) + \hat{B}_w w \\ q &= \hat{C}_{q_{cl}}\xi + D_{qw}w , \end{aligned} \tag{14}$$

where

$$\hat{A}_{cl} = \hat{A} + \hat{B}K , \; \hat{C}_{q_{cl}} = \hat{C}_q + D_{qu}K , \; \hat{B}_w = \begin{bmatrix} B_w \\ D_{zw} \end{bmatrix} \tag{15}$$

and $\hat{A}$, $\hat{B}$, $\hat{B}_p$, and $\hat{C}_q$ are as defined in the statement of the theorem.

Using the candidate quadratic Lyapunov function $V(\xi) = \xi^T Q^{-1}\xi$, we first show that all trajectories of the closed loop system (14) are bounded for every constant exogenous input. To achieve this, consider any constant exogenous input w, let $X \equiv Q^{-1}$ and post and pre-multiply inequality (6) by the matrix

$$\begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} .$$

Then using a Schur complement result (Boyd *et al.* 1994), the following matrix inequality is equivalent to the inequality (6):

$$\begin{bmatrix} X\hat{A}_{cl} + \hat{A}_{cl}^T X + \sigma^2 \hat{C}_{q_{cl}}^T \hat{C}_{q_{cl}} & X\hat{B}_p \\ \hat{B}_p^T X & -I \end{bmatrix} < 0 . \tag{16}$$

Let $p = \Delta(q) = \Delta(\hat{C}_{q_{cl}}\xi + D_{qw}w)$ and pre and post-multiply inequality (16) by the vectors $(\xi, p)^T$ and (ξ, p) to obtain

$$\dot{V} - 2\xi^T X\hat{B}_w w + \left[\sigma^2 ||\hat{C}_{q_{cl}}\xi||^2 - ||p||^2\right] \leq -\varepsilon||\xi||^2$$

for some scalar $\varepsilon > 0$. It now follows from the Lipschitz property of Δ that

$$||\Delta(q)|| - ||\Delta(0)|| \leq ||\Delta(q) - \Delta(0)|| \leq \sigma||q|| .$$

Recalling the definition of q we obtain that,

$$||p|| = ||\Delta(q)|| \leq \sigma||\hat{C}_{q_{cl}}\xi|| + c_1$$

where $c_1 = \sigma||D_{qw}w|| + ||\Delta(0)||$. By squaring both sides of the inequality above, we obtain the following,

$$\sigma^2||\hat{C}_{q_{cl}}\xi||^2 - ||p||^2 \geq -2c_1\sigma||\hat{C}_{q_{cl}}|| \, ||\xi|| - c_1^2 .$$

Consequently, for each constant exogenous input w we have

$$\dot{V} \leq -\varepsilon||\xi||^2 + \kappa_1||\xi|| + \kappa_2$$

for some positive scalars κ_1 and κ_2. It is clear from the right-hand side of the inequality above that there is a large enough $r > 0$ such that $||\xi|| > r$ implies that $\dot{V} < 0$. Therefore, for each constant w, every solution of the closed loop system (14) is bounded.

Consider now two trajectories $\xi_1(\cdot)$ and $\xi_2(\cdot)$ of the closed loop system (14) corresponding to a fixed constant exogenous input w. Then, letting $\delta\xi \equiv \xi_1 - \xi_2$ and $\delta p = \Delta(q_1) - \Delta(q_2)$, where q_1 and q_2 are values of q on trajectories $\xi_1(\cdot)$ and $\xi_2(\cdot)$, respectively, the dynamics of $\delta\xi$ can be described by

$$\begin{aligned} \delta\dot{\xi} &= \hat{A}_{cl}\delta\xi + \hat{B}_p\delta p \\ \delta q &= \hat{C}_{q_{cl}}\delta\xi \\ ||\delta p|| &\leq \sigma||\delta q|| . \end{aligned} \tag{17}$$

Choosing $V(\delta\xi) = \delta\xi^T Q^{-1}\delta\xi$ as a candidate quadratic Lyapunov function for (17) and using the matrix inequality (6) it can be easily shown that the system (17) is GES about $\delta\xi = 0$ (Boyd *et al.* 1994).

Choose any trajectory, $\hat{\xi}(\cdot)$, of (14). Then from the discussion above it is clear that every other solution converges exponentially to $\hat{\xi}(\cdot)$. So the system (14) is GUAS about the bounded trajectory $\hat{\xi}(\cdot)$. Consequently, using Lemma 6, we can conclude that there exists an equilibrium state ξ^* for each constant w. Indeed, since this equilibrium point also constitutes a trajectory of the system, it is a GES equilibrium point. Then using Lemma 1, we can conclude the proof of Theorem 2. □

We now present a proof of Theorem 4.

Proof (Theorem 4) Letting $\xi = (x, x_I)$, $e = (e_1, e_2)$, $e_1 = x - \hat{x}$, $e_2 = w - \hat{w}$ and recognizing that the exogenous input is constant, we can describe the closed loop system as follows

$$\begin{aligned} \dot{\xi} &= \hat{A}_{cl}\xi + \hat{B}_p\Delta(q) + \hat{B}_w w + \hat{B}_e e \\ q &= \hat{C}_{q_{cl}}\xi + D_{qw}w + \hat{D}_{qe}e , \end{aligned} \tag{18}$$

where $\hat{A}_{cl}, \hat{C}_{q_{cl}}, \hat{B}_w$ are given in (15) and $\hat{D}_{qe} = [-D_{qu}K_P \;\; 0]$,

$$\hat{B}_e = \begin{bmatrix} -B_u K_P & 0 \\ -C_{z_{cl}} & -D_{zw} \end{bmatrix} , \quad C_{z_{cl}} = C_z + D_{zu}K_P ,$$

and

$$\dot{e} = (\tilde{A} + L\tilde{C})e + \tilde{B}_p\tilde{p} \quad \text{where } \tilde{p} = \Delta(q) - \Delta(\hat{q})$$
$$\tilde{q} = \tilde{C}_e e \quad \text{and} \quad \|\tilde{p}\| \le \sigma\|\tilde{q}\|, \tag{19}$$

where

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \qquad \hat{q} = C_{q_{cl}}\hat{x} + D_{qu}K_I x_I + D_{qw}\hat{w}.$$

Therefore the closed loop system is a cascade system where the dynamics for e (error dynamics) has stability properties independent of the dynamics of ξ. Clearly $e \equiv 0$ is always a solution to the error dynamics. Actually, when L is designed using the LMI (11) then the error dynamics are GES about the origin for all solutions.

Any trajectory of the closed loop system (18)-(19) corresponding to zero error dynamics, $(\xi(\cdot), 0)$, satisfy the following differential equations,

$$\dot{\xi} = \hat{A}_{cl}\xi + \hat{B}_p\Delta(q) + \hat{B}_w w$$
$$q = \hat{C}_{q_{cl}}\xi + D_{qw}w. \tag{20}$$

Consequently, differential equations (20) describing zero error dynamics is the same equations as in (14) because the controller gain matrices are obtained by solving the same LMI (6). As a consequence of Theorem 2, (20) has an equilibrium state ξ^*. Therefore for the overall closed loop system (18)-(19), $(\xi^*, 0)$ is an equilibrium state. Now defining $\delta\xi \equiv \xi - \xi^*$, the dynamics of the closed loop system about this equilibrium state can be described by

$$\delta\dot{\xi} = \hat{A}_{cl}\delta\xi + \hat{B}_p\delta p + \hat{B}_e e \quad \text{where}$$
$$\delta p = \Delta(\hat{C}_{q_{cl}}\xi + D_{qw}w + \hat{D}_{qe}e) - \Delta(\hat{C}_{q_{cl}}\xi^* + D_{qw}w)$$
$$\delta q = \hat{C}_{q_{cl}}\delta\xi + \hat{D}_{qe}e$$
$$\|\delta p\| \le \sigma\|\delta q\|, \tag{21}$$

cascaded by the system of differential equations given in (19) for the error, e, dynamics.

Let $V(\delta\xi) = \delta\xi^T X \delta\xi$ where $X = Q^{-1} > 0$ with Q is the solution of LMI (6). Since the inequalities (6) and (16) are equivalent, there exists some positive scalar ε_1 such that

$$\dot{V} - 2\delta\xi^T X\hat{B}_e e + \\ + \left[\sigma^2\|\hat{C}_{q_{cl}}\delta\xi\|^2 - \|\delta p\|^2\right] \le -\varepsilon_1\|\delta\xi\|^2$$

where

$$\|\hat{C}_{q_{cl}}\delta\xi\|^2 = \left\{\|\hat{C}_{q_{cl}}\delta\xi\| + \|\hat{D}_{qe}e\|\right\}^2 - \\ - 2\|\hat{C}_{q_{cl}}\delta\xi\|\|\hat{D}_{qe}e\| - \|\hat{D}_{qe}e\|^2.$$

Therefore there exist positive scalars β_1, β_2 such that

$$\dot{V} - \beta_1\|\delta\xi\|\|e\| - \beta_2\|e\|^2 + \\ + \left[\sigma^2\|\delta q\|^2 - \|\delta p\|^2\right] \le -\varepsilon_1\|\delta\xi\|^2,$$

which implies that

$$\dot{V} \le \beta_1\|\delta\xi\|\|e\| + \beta_2\|e\|^2 - \varepsilon_1\|\delta\xi\|^2.$$

Also, (11) implies that there is a scalar $\varepsilon_2 > 0$ such that for $W(e) = e^T Pe$ we have

$$\dot{W} \le -\varepsilon_2\|e\|^2.$$

Therefore, for any scalar $\gamma > 0$, $W_\gamma = \gamma e^T Pe$ satisfies $\dot{W}_\gamma \le -\gamma\varepsilon_2\|e\|^2$. Now consider a combined Lyapunov function for the overall closed loop system, $U = V + W_\gamma$ where $\gamma > 0$ is chosen such that $\gamma\varepsilon_2 > \beta_2$ and $\varepsilon_1(\gamma\varepsilon_2 - \beta_2) - \beta_1^2/4 > 0$. Then there exists a scalar $\varepsilon_3 > 0$ such that

$$\dot{U} \le -(\varepsilon_1\|\delta\xi\|^2 - \beta_1\|\delta\xi\|\|e\| + (\gamma\varepsilon_2 - \beta_2)\|e\|^2) \\ \le -\varepsilon_3(\|\delta\xi\|^2 + \|e\|^2).$$

Consequently the closed loop system is GES about the equilibrium point $(\xi^*, 0)$. Therefore, simply by using Lemma 3, we can conclude the proof of Theorem 4. □

6. AN ILLUSTRATIVE EXAMPLE

The results presented earlier will be applied to a single link manipulator with a flexible shaft. As shown in Figure 1, the single link manipulator consists of the manipulator arm which is connected to a motor via a flexible joint. The motor supplies a control torque to the arm via the flexible shaft. The flexible shaft is modeled with an inertia element connected to the manipulator arm via a torsional spring.

The equations of motion for single link manipulator with a flexible joint are given by

$$\begin{aligned} I\ddot{\theta} - mgl\sin\theta + k(\theta - \phi) &= w_1 \\ J\ddot{\phi} - k(\theta - \phi) &= u \\ y = \begin{bmatrix} \theta \\ w_2 \end{bmatrix} \text{ and } z &= \theta - w_2, \end{aligned} \tag{22}$$

where g is the gravitational acceleration constant, θ is the angle of the arm, ϕ is the angle of the motor, l is the distance from the shaft center to the center of mass of the arm, I is the moment of inertia of the arm, J is the moment of inertia of the motor plus shaft, k is the rotational spring constant of the shaft, w_1 is a disturbance torque, w_2 is a reference signal and u is the control input. A state-space description of this system can be written as

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= (mgl/I)\Delta(q) + (k/I)(x_2 - x_1) + (w_1/I) \\ \dot{x}_4 &= (k/J)(x_1 - x_2) + (u/J) \\ q &= x_1 \end{aligned}$$
$$y = \begin{bmatrix} x_1 \\ w_2 \end{bmatrix} \text{ and } z = x_1 - w_2,$$

where $\Delta(q) = \sin q$.

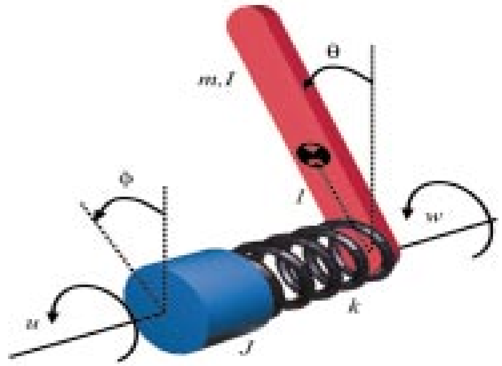

Fig. 1. Single link manipulator

The matrices describing this system can be given as,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{I} & \frac{k}{I} & 0 & 0 \\ \frac{k}{J} & -\frac{k}{J} & 0 & 0 \end{bmatrix}, \; B_p = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{I} \\ 0 \end{bmatrix}, \; B_u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J} \end{bmatrix},$$

$$B_w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{I} & 0 \\ 0 & 0 \end{bmatrix}, \quad C_q = C_z = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

$$C_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; D_{yu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; D_{yw} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$D_{qw} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \; D_{zw} = \begin{bmatrix} 0 & -1 \end{bmatrix}, \quad D_{qu} = D_{zu} = 0.$$

The only nonlinearity is $\sin x_1$ term which is clearly a globally Lipschitz nonlinearity with $\sigma = 1$.

A simulation is presented for this system where $w_1 = 2$ and $w_2 = \frac{\pi}{6}$. The results show that asymptotic tracking is achieved, the states of the closed loop system are bounded and the control input is also bounded, see Figure 2.

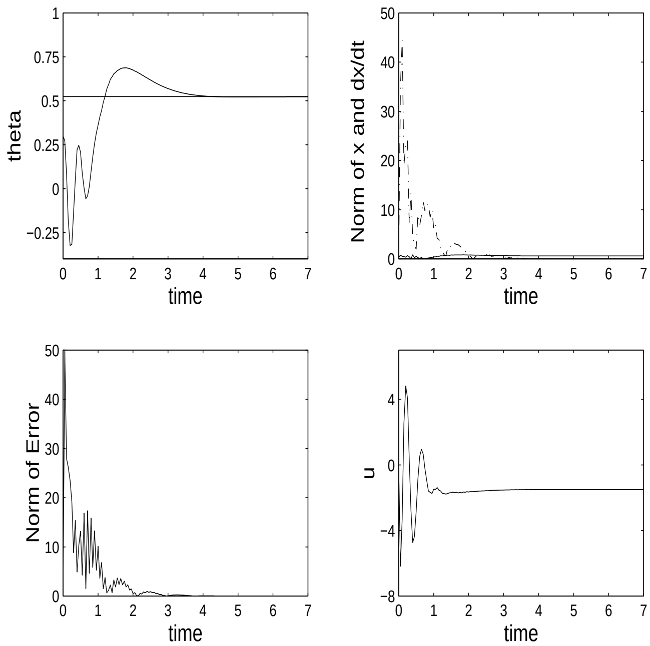

Fig. 2. Simulation Results

7. REFERENCES

Açıkmeşe, A. B. (2001). Robust tracking and disturbance rejection for uncertain/nonlinear systems. *PhD Thesis in preperation, Purdue University*.

Arcak, M. and P. Kokotovic (2001). Observer-based control of systems with slope-restricted nonlinearities. *IEEE Transactions on Automatic Control* **AC-46**(7), 1146–1150.

Balakrishnan, V. and L. Kashyap (1999). Robust stability and performance analysis of uncertain systems using linear matrix inequalities. *Journal of Optimization Theory and Applications* **100**(3), 457–478.

Benson, R.W. and W.E. Schmitendorf (1997). Augmented system approach to measurement-based robust tracking. *Dynamics and Control* **7**, 315–325.

Boyd, S., L. El Ghaoui, L., E. Feron and V. Balakrishnan (1994). *Linear Matrix Inequalities in System and Control Theory*. SIAM.

Byrnes, C.I. and A. Isidori (1998). Output regulation for nonlinear systems: an overview. *Proceedings of 37th IEEE Conference on Decision and Control* pp. 3069–3074.

Davison, E.J. (1976). The robust control of a servomechanism problem for linear time-invariant multivariable systems. *IEEE Transactions on Automatic Control* **AC-21**(1), 25–34.

Desoer, C. A. and C. Lin (1985). Tracking and disturbance rejection of mimo nonlinear systems with PI controllers. *IEEE Transactions on Automatic Control* **AC-30**(9), 861–867.

Schmitendorf, W.E. and B.R. Barmish (1986). Robust asymptotic tracking for linear systems with unknown parameters. *Automatica* **22**(3), 355–360.