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ON A DETECTABILITY CONCEPT OF DISCRETE-TIME INFINITE MARKOV JUMP LINEAR SYSTEMS [⋆]

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Abstract: This paper introduces a concept of detectability for discrete-time infinite Markov jump linear systems that relates the stochastic convergence of the output with the stochastic convergence of the state. It is shown that the new concept generalizes a known stochastic detectability concept and, in the finite dimension scenario, it is reduced to the weak detectability concept. It is also shown that the detectability concept proposed here retrieves the well known property of linear deterministic systems that observability is stricter than detectability.

Keywords: Stochastic jump processes, Markov parameters, Markov models, systems concepts, observability.

1. INTRODUCTION

This paper is concerned with the discrete-time Infinite Markov jump linear system (MJLS) defined in a fixed stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_k), P)$ by

$$\Psi : \begin{cases} x(k+1) = A_{\theta(k)}x(k), & k \geq 0, \\ y(k) = C_{\theta(k)}x(k), & x(0) = x_0, \theta(0) = \theta_0 \end{cases} \tag{1}$$

where x and y are the state and the output variables, respectively. The mode θ is the state of an underlying discrete-time Markov chain $\Theta = \{\theta(k); k \geq 0\}$ taking values in $S = \{1, 2, \ldots\}$ and having a stationary transition probability matrix $\mathbb{P} = [p_{ij}]$, $i, j \in \mathbb{Z}$. $\theta_0 \in S$ is a random variable for which $\mu_i = P(\theta_0 = i)$, $i \in S$, and x_0 is a second order random variable. It is assumed that matrices A_i and C_i, $i \in S$, belong respectively to the collections of real matrices $A = (A_1, A_2, \ldots)$, $\dim(A_i) = n \times n$, and $C = (C_1, C_2, \ldots)$, $\dim(C_i) = q \times n$, for which $\sup_{i \in S} \|A_i\| < \infty$ and $\sup_{i \in S} \|C_i\| < \infty$. We also assume that $x(k)$ and $\theta(k)$ are observed at each time instant k.

When one deals with system Ψ, the usual detectability concept is the stochastic detectability (S-detectability), which is a dual concept of stochastic stabilizability; see (Costa and Fragoso, 1995) in the same setting of this paper, or (Fragoso and Baczynski, 2001) in the continuous time case, or (Costa, 1995) and (Morozan, 1995) in the finite dimension case. However, the S-detectability concept presents the drawbacks pointed out in the sequel.

Consider the weak observability (W-observability) concept that follows from the extension of the finite state space case, see Section 4. It appears in (Costa and do Val, n.d.*b*), (Costa and do Val, 2001) and (Morozan, 1995), and it is more general than other observability concepts for MJLS, like the ones appear-

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ing in (Ji and Chizeck, 1990). We show by means of an example in Section 5 that S-detectability does not generalize W-observability. This suggests that S-detectability is conservative, recalling that detectability generalizes observability in the context of linear deterministic systems, linear time-varying systems, see e.g.(Anderson and Moore, 1981), or even in the context of MJLS in finite state space, see (Costa and do Val, n.d.*b*), (Costa and do Val, 2001), and (do Val and Costa, 2002). Moreover, W-observability does not generalize S-detectability as well, and the concepts are not comparable; this sometimes compelled authors to consider both concepts, like in (Morozan, 1995). We also mention that some basic properties of the usual detectability concept in the linear deterministic setting are not retrieved by the S-detectability concept; along this line, S-detectability does not assure that non-observed trajectories are stable in the stochastic sense and it does not relate convergence (in the stochastic sense) of the state and output trajectories.

In the finite space state case, the above criticism was overcome by the introduction of the weak detectability (W-detectability) concept in (Costa and do Val, n.d.*b*) and (Costa and do Val, 2001), or in (Costa and do Val, n.d.*a*) for the continuous-time case. The concept generalizes the S-detectability concept and it reproduces geometric and qualitative properties of the deterministic concepts within the MJLS setting.

This paper extends the weak detectability concept to the infinite Markov state space case. The new concept, which is referred to as W_S-detectability, relates the stochastic convergence of the output with the stochastic convergence of the state. It is shown that the new concept generalizes the S-detectability concept and, in the finite dimension scenario, it is reduced to the concept of weak detectability. It is also shown that W_S-detectability generalizes W-observability.

The paper is organized as follows. In Section 2 we present basic concepts. In Section 3 we introduce the concept of W_S-detectability and we present the comparison with the S-detectability concept; the W_S-detectability concept in finite state space is studied in Section 3.2. The concept of W-observability is studied in Section 4 and Section 5 presents examples showing that a non S-detectable system can be W_S-detectable.

2. NOTATION AND BASIC RESULTS

Let $\mathbb{R}^n$ represent the linear space of all n-dimensional vectors. Let $R^{r,n}$ (respectively, R^n) represent the normed linear space formed by all $r \times n$ real matrices (respectively, $n \times n$) and R^{n0} (R^{n+}) the closed convex cone $\{U \in R^n : U = U' \geq 0\}$ (the open cone $\{U \in R^n : U = U' > 0\}$) where U' denotes the transpose of U; $U \geq V$ ($U > V$) signifies that $U - V \in R^{n0}$ ($U - V \in R^{n+}$). Let $H_1^{r,n}$ ($H_\infty^{r,n}$) denote the linear space formed by sequences of matrices $H = \{H_i; i \in S\}$ such that $\sum_{i\in S} ||H_i|| < \infty$ ($\sup_{i\in S} ||H_i|| < \infty$); also, $H^n \equiv H^{n,n}$. We denote by H_1^{n0} (H_1^{n+}) the set H^n when it is made up by $H_i \in R^{n0}$ ($H_i \in R^{n+}$) for all $i \in S$ and similarly for H_∞^{n0} and H_∞^{n0}. For $H \in H_1^{r,n}$ we define the inner product

$$< H, V >= \sum_{i\in S} tr\{H_i' V_i\}$$

and the norm

$$\|H\|_1 = \sum_{i\in S} \|H_i\| \tag{2}$$

and for $H \in H_\infty^{r,n}$ we define $\|H\|_\infty = \sup_{i\in S} \|H_i\|$.

Remark 1. Notice that for $H \in H_1^{n0}$ we have that $||H||_1 \leq < H, I > \leq n||H||_1$. Indeed,

$$\begin{aligned}\|H\|_1 &= \sum_{i\in S} \|H_i\| \leq \sum_{i\in S} tr(H_i) = < H, I > \\ &= tr\Big(\sum_{i\in S} H_i\Big) \leq n \Big\|\sum_{i\in S} H_i\Big\| \leq n \sum_{i\in S} \|H_i\| = n\|H\|_1\end{aligned}$$

Let us define the operators $E : H_1^n \to H_1^n$, and $T : H^n \to H^n$ as

$$\begin{aligned} E_i(U) &= \sum_{j\in S} p_{ij} U_j \\ T_i(U) &= A_i' E_i(U) A_i, \quad i = 1, 2, \ldots \end{aligned} \tag{3}$$

and $L : H_1^n \to H_1^n$, the dual of operator T, as

$$L_i(U) = \sum_{j\in S} p_{ji} A_j U_j A_j', \quad i \in S$$

It is shown in (Costa and Fragoso, 1995) that the limits in (3) are well defined. We denote $T^0(U) = U$, and for $k \geq 1$, we can define $T^k(U)$ recursively by $T^k(U) = T(T^{k-1}(U))$ and similarly for L. Notice that T and L are linear. We also define the following linear system related to system Ψ:

$$\Phi : \begin{cases} X_i(k+1) = L_i(X(k)), k \geq 0 \\ X(0) = X \in H_1^{n0} \end{cases} \tag{4}$$

The relationship between systems Ψ and Φ is presented in the following proposition. The result is adapted from (Costa and Fragoso, 1995).

Proposition 1. Consider systems Ψ and Φ. In connection with the initial condition (x_0, θ_0), define $X \in H_1^n$ as $X_{\theta_0} = x_0 x_0'$ and $X_i = 0, i \neq \theta_0$. Then,

$$X_i(k) = E_{x_0,\mu_0}\{x(k)x(k)' 1_{\theta(k)=i}\} \tag{5}$$

Notice that with this result we can write, for instance, $E_{x_0,\theta_0}\{|x(k)|^2\} =< X(k), I >$. We introduce the functional

$$W^N(X) = \sum_{k=0}^{N} < X(k), C'C > \tag{6}$$

whenever $X(0) = X$, and

$$W(X) = \lim_{N\to\infty} W^N(X) \tag{7}$$

2.1 Stochastic Detectability

Definition 1. We say that $(A,\mathbb{P})$ is stochastically stable (S-stable) if for each $X \in H_1^{n0}$,

$$\sum_{k=0}^{\infty} \|X(k)\|_1 < \infty$$

Remark 2. Notice from Remark 1 that the condition of S-stability is equivalent to $\sum_{k=0}^{\infty} < X(k), I >< \infty$ and, in view of Proposition 1, we have that $\sum_{k=0}^{\infty} E\{|x(k)|^2\} < \infty$ for each initial condition x_0 and θ_0.

Remark 3. Consider a linear operator $\mathcal{R} : H_1^n \to H_1^n$, let $r_\sigma(\mathcal{R})$ denote the spectral radius of $\mathcal{R}$; it is known that $r_\sigma(\mathcal{R}) < 1$ if and only if $\sum_{k=0}^{\infty} ||\mathcal{R}^k(H)|| < \infty$. Then, $(A,\mathbb{P})$ is S-stable if and only if $r_\sigma(\mathcal{L}) < 1$, $\forall H \in H_1^n$.

Definition 2. We say that $(A,C,\mathbb{P})$ is stochastically detectable (S-detectable) if there exists $L \in H_\infty^{q,n}$ for which $(A+LC,\mathbb{P})$ is S-stable.

3. WEAK DETECTABILITY

Notice that the functional in (7) has the physical interpretation of the accumulated energy of the output process y in the sense that

$$W(X) = \lim_{N\to\infty} E\Big\{\sum_{k=0}^{N} x(k)' C'_{\theta(k)} C_{\theta(k)} x(k)\Big\} \qquad (8)$$
$$= \lim_{N\to\infty} E\Big\{\sum_{k=0}^{N} |y(k)|^2\Big\}$$

whenever $X_{\theta_0} = x_0 x_0'$ and $X_i = 0, i \neq \theta_0$. Then, the weak detectability concept relates the energy of the output and the trajectory, as follows.

Definition 3. (W_S-detectability). We say that $(A,C,\mathbb{P})$ is W_S-detectable provided that

$$\sum_{k=0}^{\infty} \|X(k)\|_1 < \infty \qquad (9)$$

whenever $W(X) < \infty$.

Remark 4. In view of (8), the condition $W(X) < \infty$ has the interpretation of stochastic convergence of the output. Then, the W_S-detectability concept relates stochastic convergence of the state and the output.

Remark 5. We use the subscript S in the definition above to emphasize the condition (9) which comes from the stochastic stability condition. Variants of the concept of detectability arise with different conditions on the trajectory. For instance, when one replaces (9) by the weaker condition $\lim_{k\to\infty} \|X(k)\|_1 = 0$, which is a mean square condition, then we denote W_{MS}-detectability. Of course, one has that W_{MS}-detectability is weaker than W_S-detectability.

3.1 W_S-detectability and S-detectability

In this section we examine the relationship between S-detectability and W_S-detectability to show that the former implies the latter. We show by means of examples, in Section 5, that the reverse implication fails.

Theorem 1. Suppose $(A,\mathbb{P})$ is S-stable. Then $(A + GD, D, \mathbb{P})$ is W_S-detectable for each $G \in H_\infty^{n,q}$ and $D \in H_\infty^{q,n}$.

In the proof of the above theorem, $X(\cdot)$ will refer to the system Φ trajectories and likewise, $\widehat{X}(\cdot)$ will refer to:

$$\widehat{\Phi}: \begin{cases} x(k+1) = (A_{\theta(k)} + G_{\theta(k)} D_{\theta(k)}) x(k), k \geq 0 \\ y(k) = D_{\theta(k)} x(k), \\ x(0) = x_0,\ \theta(0) = \theta_0 \end{cases}$$

We need the following preliminary result. Let $\varepsilon > 0$ be such that $(1+\varepsilon^2) r_\sigma(\mathcal{L}) < 1$ (recall that $r_\sigma(\cdot)$ is the spectral radius) and for $H \in H_1^{n0}$ we define the operator $\mathcal{L}_G : H_1^{n0} \to H_1^{n0}$ as

$$\mathcal{L}_{Gi}(H) = (1+1/\varepsilon^2) \sum_{j\in S} p_{ji} G_j D_j H_j D_j' G_j'$$

Lemma 1. The series $\sum_{k=0}^{\infty} \mathcal{L}_G(\hat{X}(k))$ converges provided that $\widehat{W}(X) < \infty$.

PROOF. We start evaluating, for $H \in H_1^{n0}$,

$$||\mathcal{L}_G(H)||_1 = \sum_{i\in S} \|\mathcal{L}_{Gi}(H)\| \qquad (10)$$
$$= \sum_{i\in S} \Big\|(1+1/\varepsilon^2) \sum_{j\in S} p_{ji} G_j D_j H_j D_j' G_j'\Big\|$$
$$\leq (1+1/\varepsilon^2) ||G||_\infty^2 \sum_{j\in S}\sum_{i\in S} p_{ji} \|D_j H_j D_j'\|$$
$$= \alpha \sum_{j\in S} \|D_j H_j D_j'\| = \alpha \|DHD'\|_1$$

where $\alpha = (1+1/\varepsilon^2)\,\|G\|_\infty^2$. Employing (10) we can write, for $T_1 < T_2$,

$$\Big\|\sum_{k=0}^{T_1} \mathcal{L}_G(\hat{X}(k)) - \sum_{k=0}^{T_2} \mathcal{L}_G(\hat{X}(k))\Big\|_1 \leq \sum_{k=T_1}^{T_2} \|\mathcal{L}_G(\hat{X}(k))\|_1$$
$$\leq \sum_{k=T_1}^{T_2} \alpha \|D\hat{X}(k)D'\|_1 \qquad (11)$$

Recalling that $\widehat{W}(X) < \infty$, we obtain

$$\sum_{k=T_1}^{T_2} < D\hat{X}(k)D', I > \to 0 \text{ as } T_1, T_2 \to \infty$$

and Remark 1 provides that

$$\sum_{k=T_1}^{T_2} ||D\hat{X}(k)D'||_1 \to 0 \text{ as } T_1, T_2 \to \infty \qquad (12)$$

From (11), and (12) we obtain

$$\Big\|\sum_{k=0}^{T_1} \mathcal{L}_G(\hat{X}(k)) - \sum_{k=0}^{T_2} \mathcal{L}_G(\hat{X}(k))\Big\|_1 \to 0$$

as $T_1, T_2 \to \infty$.

Proof of Theorem 1

We shall show that $\sum_{k=0}^{\infty} \left\|\hat{X}(k)\right\|_1 < \infty$ provided that

$$\widehat{W}(X) = \sum_{k=0}^{\infty} < \hat{X}(k), D'D >< \infty$$

Let us define $L_\varepsilon : H_1^{n0} \to H_1^{n0}$ by $L_\varepsilon = (1+\varepsilon^2)L$, that is, $L_\varepsilon(H) = \sum_{j\in S} p_{ji}(1+\varepsilon^2)A_jH_jA_j'$. We also define the series $M(k)$, $k \geq 0$, with $M(k) \in H_1^{n0}$ by

$$\begin{cases} M(k+1) = L_\varepsilon(M(k)) + L_G(\hat{X}(k)) \\ M(0) = \hat{X}(0) \end{cases}$$

Note that

$$\begin{aligned} M(m) = &L_\varepsilon^m(\hat{X}(0)) + L_\varepsilon^{m-1}(L_G(\hat{X}(0))) \\ &+ L_\varepsilon^{m-2}(L_G(\hat{X}(1))) + \ldots + L_G(\hat{X}(m-1)) \end{aligned} \tag{13}$$

and we can write

$$\begin{aligned} \sum_{k=0}^{\infty} < M(k), I >= &\sum_{k=0}^{\infty} < L_\varepsilon^k(\hat{X}(0)), I > \\ &+ \sum_{m=0}^{\infty}\sum_{k=0}^{m-1} < L_\varepsilon^k\left(L_G(\hat{X}(m-k-1))\right), I > \end{aligned} \tag{14}$$

Lemma 1 allows us to define $M = \sum_{m=0}^{\infty} L_G(\hat{X}(m))$. Then, for the second term in the right hand side of (14) we evaluate

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{k=0}^{m-1} < L_\varepsilon^k\left(L_G(\hat{X}(m-k-1))\right), I > \\ &= \sum_{k=0}^{\infty}\sum_{m=k+1}^{\infty} < L_\varepsilon^k\left(L_G(\hat{X}(m-k-1))\right), I > \\ &= \sum_{k=0}^{\infty} < L_\varepsilon^k\left(\sum_{m=0}^{\infty} L_G(\hat{X}(m))\right), I > \\ &= \sum_{k=0}^{\infty} < L_\varepsilon^k(M), I > \end{aligned} \tag{15}$$

From (14) and (15) we obtain

$$\sum_{k=0}^{\infty} < M(k), I > = \sum_{k=0}^{\infty} < L_\varepsilon^k(\hat{X}(0)+M), I > \tag{16}$$

and one has that $\sum_{k=0}^{\infty} ||L_\varepsilon^k(\hat{X}(0)+M)||_1$ converges (recall that $r_\sigma(L_\varepsilon) < 1$ and see Remark 3) and from Remark 1 we conclude that

$$\sum_{k=0}^{\infty} ||M(k)||_1 < \infty \tag{17}$$

Now we show by induction that

$$\hat{X}(k) \leq M(k) \tag{18}$$

Indeed, for $k = 0$ we defined $M(0) = \hat{X}(0)$; assuming $\hat{X}(k) \leq M(k)$ one can check that

$$\begin{aligned} \hat{X}(k+1) &= \sum_{j\in S} (A_j - G_jD_j)\hat{X}(k)(A_j - G_jD_j)' \\ &\leq L_\varepsilon(\hat{X}(k)) + L_G(\hat{X}(k)) \\ &\leq L_\varepsilon(M(k)) + L_G(\hat{X}(k)) = M(k+1) \end{aligned}$$

and the induction is complete. Finally, from (17) and (18) we obtain

$$\sum_{k=0}^{\infty} ||\hat{X}(k)||_1 \leq \sum_{k=0}^{\infty} ||M(k)||_1 < \infty$$

Theorem 2. If $(A, C, \mathbb{P})$ is S-detectable then $(A, C, \mathbb{P})$ is W_S-detectable.

PROOF. Since $(A, C, \mathbb{P})$ is S-detectable, from definition there exists $L \in M^{n,q}$ such that $(A+LC, \mathbb{P})$ is S-stable and, from Theorem 1, $(A+LC+GD, D, \mathbb{P})$ is W_S-detectable for each $G \in M^{s,q}$ and $D \in M^{q,s}$. The proof is completed by retrieving the original system Φ with the choice $D = C$ and $G = -L$.

3.2 W_S-detectability in Finite State Space

In this section we study the concept of W_S-detectability of Markov jump linear systems in finite state space, $S = \{1, \ldots, N\}$. We show that the concept of W_S-detectability and the concept of W-detectability presented in (Costa and do Val, 2001) are equivalent.

Definition 4. (W-detectability). We say that $(A, C, \mathbb{P})$ is W-detectable when there exist integers N_d, $k_d \geq 0$ and scalars $0 \leq \delta < 1$, $\gamma > 0$ such that $W^{N_d}(X) \geq \gamma\|X\|$ whenever $\|X(k_d)\|_1 \geq \delta\|X\|_1$

We shall need the following preliminary results, which are adapted from (Costa and do Val, 2001, Lemmas 7 and 8).

Proposition 2. (i) $W(X) = 0$ if $W^{n^2N}(X) = 0$;

(ii) $(A, C, \mathbb{P})$ is W-detectable if and only if $||X(k)||_1 \to 0$ as $k \to \infty$ whenever $W^{n^2N}(X) = 0$.

Ji et al. in (Ji *et al.*, 1991) have shown that the S-stability concept is equivalent to other second moment stability concepts, such as MS-stability and exponential stability. The next result follows.

Proposition 3. (i) $||X(k)||_1 \to 0$ as $k \to \infty$ if and only if the series $\sum_{k=0}^{\infty} ||X(k)||_1$ converges.

(ii) If $\sum_{k=0}^{\infty} ||X(k)||_1$ diverges, then $\|X(k)\|_1 \geq \rho\xi^k\|X(0)\|_1$ for some $0 < \rho \leq 1$ and $\xi \geq 1$.

Remark 6. In finite space state, equivalence between the concepts of W_S-detectability and W_{MS}-detectability follows from the equivalence among the second moment stability concepts.

The main result of the section is as follows.

Lemma 2. Assume that $S = \{1, \ldots, N\}$. Then, $(A, C, \mathbb{P})$ is W_S-detectable if and only if $(A, C, \mathbb{P})$ is W-detectable.

PROOF.

Necessity. Assume that $W^{n^2 N}(X) = 0$. Proposition 2 (i) yields $W(X) = 0$, and the W_S-detectability hypothesis provides that $\lim_{N\to\infty}\sum_{k=0}^{N}||X(k)||_1 < \infty$ which means that $||X(k)||_1 \to 0$ as $k\to\infty$. Proposition 2 (ii) concludes the proof.

Sufficiency. Assuming $(A,C,\mathbb{P})$ is W-detectable, we show that $W(X) = \infty$ whenever the series $\sum_{k=0}^{\infty}||X(k)||_1$ diverges. In this situation, from Proposition 3 we have that there exists $0 < \rho \le 1$ and $\xi \ge 1$ for which

$$\|X(k)\|_1 \ge \rho\xi^k \|X(0)\|_1 \tag{19}$$

Let us define the sequence $\mathcal{N} = \{n_0, n_1, \ldots\}$ where $n_0 = 0$ and each n_m, $m = 1,2,\ldots$, is the smallest integer such that $n_m \ge n_{m-1}+1$ and n_m satisfies

$$\|X((n_m+1)k_d)\|_1 \ge \delta\|X(n_m k_d)\|_1$$

If the number of elements of $\mathcal{N}$ is finite, one can check that

$$\lim_{t\to\infty}\|X(t\,k_d)\|_1 = 0$$

which contradicts the initial hypothesis that the series $\sum_{k=0}^{\infty}||X(k)||_1$ converges, see also Proposition 3. Then we conclude that $\mathcal{N}$ has infinitely many elements, and we can take a subsequence from $\mathcal{N}$ with infinitely many elements, $\mathcal{N}' = \{n_{m_0}, n_{m_1}, \ldots\}$, where $n_{m_0} = m_0 = 0$ and each m_t, $t = 1,2,\ldots$, is the smallest integer such that $n_{m_t} \ge n_{m_{t-1}} + \max\{1, (N_d/k_d)\}$. We can write:

$$\begin{aligned} W^N(X) &= \sum_{k=0}^{N-1} < X(k), Q + C'C > \\ &\ge \sum_{t=0}^{t'}\sum_{k=0}^{N_d} < X(n_{m_t}k_d + k), C'C > \\ &\ge \sum_{t=0}^{t'} \gamma\|X(n_{m_t}k_d)\| \ge \gamma\rho\|X(0)\|\,t' \end{aligned}$$

where t' is the largest integer for which $n_{m_{t'}}k_d + N_d < N$, in such a manner that $t'\to\infty$ as $N\to\infty$ and we conclude that $W(X) = W^\infty(X) = \infty$.

4. W-OBSERVABILITY CONCEPT

The following W-observability concept is adapted from the finite state space case, see (Costa and do Val, n.d.*b*), (Costa and do Val, 2001) and (Morozan, 1995).

Definition 5. Consider system Φ. We say that $(A,C,\mathbb{P})$ is W-observable when there exist a positive integer N_d and a scalar $\gamma > 0$ such that $W^{N_d}(X) \ge \gamma\|X\|_1$ for each initial condition X.

Notice that the above concept reflects the idea that there exists a number N_d for which a minimal level γ of energy is present at the output in any interval of lenght N_d.

Lemma 3. If $(A,C,\mathbb{P})$ is W-observable then $(A,C,\mathbb{P})$ is W_S-detectable.

PROOF. We shall show that $\sum_{k=0}^{\infty}||X(k)||_1 < \infty$ provided that $W(X(0)) < \infty$, assuming $(A,C,\mathbb{P})$ is W-observable. From the W-observability condition we derive, for each $k \ge 0$,

$$||X(k)||_1 \le \frac{1}{\gamma}\sum_{t=k}^{k+N_d} < X(t), C'C > \tag{20}$$

and we write

$$\begin{aligned} \sum_{k=0}^{\infty}||X(k)||_1 &\le \frac{1}{\gamma}\sum_{k=0}^{\infty}\sum_{t=k}^{k+N_d} < X(t), C'C > \\ &\le \frac{1}{\gamma}\sum_{t=0}^{N_d-1}\sum_{k=0}^{\infty} < X(k), C'C > \\ &= \frac{N_d}{\gamma}\sum_{k=0}^{\infty} < X(k), C'C > < \infty \end{aligned} \tag{21}$$

5. EXAMPLES

In this section we present examples of systems in infinite Markov space state which are W-observable or W_S-detectable but they are not S-detectable.

We shall need the following preliminary result, see (Costa and Fragoso, 1995, Lemma 2).

Proposition 4. $(A,C,\mathbb{P})$ is S-detectable if and only if there exists $L \in H_\infty^{r,n}$ and $P \in H_\infty^{n+}$ such that

$$P_i - (A_i + L_iC_i)'E_i(P)(A_i + L_iC_i) > 0 \tag{22}$$

Example 1. In this example, a change in a single Markov state induces the lost of S-detectability without affecting the W-observability. Let $n = 1$ and $C_i = 1$, $i \in S$. Notice that (22) holds with $L = -A$ and one has that $(A,C,\mathbb{P})$ is trivially S-detectable, no matter how $A = (a_1, a_2, \ldots)$ is chosen. Moreover, $W(X) \ge < X, C'C > \ge \|X\|_1$ and $(A,C,\mathbb{P})$ is W-observable. Now, assume that there exists $j \in S$ for which $a_j^2 p_{jj} > 1$ and $p_{jj} < 1$, and let $C_j = 0$. In this case, (22) provides

$$(1 - a_j^2 p_{jj})P_j - a_j^2\Big(\sum_{i\ne j} p_{ji}P_i\Big) > 0$$

which leads to $(1 - a_j^2 p_{jj})P_j > 0$ and $P_j < 0$; then, from Proposition 4 we conclude that $(A,C,\mathbb{P})$ is not S-detectable. On the other hand, we evaluate

$$\begin{aligned} W^2(X) &= \sum_{k=0}^{1} < \mathcal{L}^k(X), C'C > = < X, \sum_{k=0}^{1}\mathcal{T}^k(C'C) > \\ &\ge \sum_{i\ne j} tr(X_iC_i'C_i) + tr(X_j(\mathcal{T}_j(C'C))) \\ &= \sum_{i\ne j} tr(X_i) + tr\left(a_j^2(1-p_{jj})X_j\right) \\ &\ge \min\{1, (a_j^2(1-p_{jj}))\}\sum_{i\in S} tr(X_i) \\ &\ge \min\{1, (a_j^2(1-p_{jj}))\}||X||_1 \end{aligned}$$

and we have that $(A,C,\mathbb{P})$ is W-observable.

The conservativeness of the S-detectability in face of W_S-detectability in the infinite Markov state space case is inherited, in some extension, from the finite state space case. For instance, let us consider systems that presents Markov chains with distinct communicating classes $S_j = \{i_1, \ldots, i_{n_j}\}$, $j = 1, \ldots, N$, for which $P\{\theta(k+1) \in S_j | \theta(k) \in S_i\} = 0$ for all $i \neq j$. Let us denote such a system by Φ_c; we also denote $A^j = (A_i)$, $i \in S_j$ and similarly for C^j and $\mathbb{P}^j$. The following result holds.

Lemma 4. Consider system Φ_c. $(A, C, \mathbb{P})$ is W_S-detectable (respectively, S-detectable) if and only if $(A^j, C^j, \mathbb{P}^j)$ is W_S-detectable (S-detectable) for $j = 1, \ldots, N$.

PROOF. We only present the guidelines of the proof. Necessity is straightforward to verify.

Sufficiency. Assume that $W(X) < \infty$. Let us decompose the initial condition X as $X = X^1 + \ldots + X^N$ in such a manner that $X_i^j = 0$ for $i \neq j$, and $W(X^j) < \infty$, $j = 1, \ldots, N$. From W_S-detectability of $(A^j, C^j, \mathbb{P}^j)$, one has that $\sum_{k=0}^{\infty} ||L^k(X^j)||_1 < \infty$. The linearity of system Φ_c provides $X(k) = \sum_{j=1}^{N} L^k(X^j)$ which leads to $\sum_{k=0}^{\infty} ||X(k)||_1 = \sum_{j=1}^{N} \sum_{k=0}^{\infty} ||L^k(X^j)||_1 < \infty$ and one has that the system $(A, C, \mathbb{P})$ is W_S-detectable. As regards to S-detectability, the result follows from the fact that there is no coupling between the equations in (22) related with different clusters.

The next example relies on an example with finite state space in (Costa and do Val, 2001).

Example 2. Let $S_1 = \{1, 2\}$ and $S_2 = \{3, 4, \ldots\}$,

$$P = \begin{bmatrix} p_{11} & (1-p_{11}) & 0 & \cdots & & \\ (1-p_{22}) & p_{22} & 0 & \cdots & & \\ 0 & 0 & p_{33} & p_{34} & \cdots \\ \vdots & \vdots & p_{43} & p_{44} & \cdots \\ & & \vdots & \vdots & \end{bmatrix}$$

Let $A_1 = a_1$, $A_2 = a_2$, $C_1 = 1$, $C_2 = 0$, $p_{11} > 0$ and $p_{22} a_2^2 > 1$ and assume that the system associated with S_2 is S-detectable. For this simple example (22) provides $(1 - a_1^2 p_{11}) P_1 > a_1^2 (1 - p_{11}) P_2$ which has no positive solution, and thus the overall system is not S-detectable. On the other hand, it is shown in (Costa and do Val, 2001) that the system associated with S_1 is W-observable, and we conclude from Lemma 4 that the overall system is W_S-detectable.

6. CONCLUSIONS

In this paper we study the concept of W_S-detectability for discrete-time infinite Markov jump linear systems. The W_S-detectability concept relates stochastic convergence of the output and the state, or equivalently the finiteness of the quadratic functional $W(X)$ and stochastic stability of the system; this is an important feature in filtering and in control problems.

The paper shows that the W_S-detectability concept generalizes the previous S-detectability concept and, in the finite dimension scenario, it is reduced to the W-detectability concept. It is also shown in the paper that W_S-detectability generalizes W-observability, thus retrieving a well known property of linear deterministic systems.

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