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EXTREMUM SEEKING LOOP FOR A CLASS OF PERFORMANCE FUNCTIONS [1]

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Abstract: The task of an extremum seeking controller is to drive or to command a system in order to extremize the value of a performance function that only depends on the present output of that system. Here, an approach for designing such a controller for cases in which the performance function is fully determined by the value of a multi-dimensional parameter is presented. The designed controller will be indeed adaptive; it will estimate the (unknown) value of the parameter, determining the performance function, and accordingly it will issue a command to the system in order to drive it to an output value that extremize the performance function. If a dither function can be found to satisfy some persistent exitation condition, then the output of the system will converge to a neighborhood of the extremizing output value.

Keywords: Optimal Regulators, Output Regulation, Minimization, Adaptive Control, Parameter Estimation, Linear Estimation, Nonlinear Control

1. INTRODUCTION

Usually a control system is designed to track a known setpoint. For an extremum-seeking (also known as peak-seeking) controller the set-point is not known apriori. The desired operating command that it should issue to the system is found by optimizing, on-line, the value of some performance function. The command (issue by the controller) that result in the extremization of the performance function is the desired set-point or command for the system.

Investigation of this class of problems dates back at least to 1922 (Leblanc, 1922). A subsequent flurry of interest arose in the 1950s and 1960s (Morosanov, 1957; Ostrovskii, 1957). A recent rejuvenation of the field has been witnessed in the form of applications to pressure-maximizing compressors, drag-reducing flight formations (D. Chichka, 1999; R.N. Banavar, 2000), and efficient fuel-burning in IC engines (B. Wittenmark, 1995; M. Krstić, 1997; H-H. Wang, 1998). The approaches reported by these authors separate the problem by a timescale, assuming that the system dynamics (i.e. the dynamics of the closed-loop system composed by the plant and the stabilizing main controller) are fast with respect to the dynamics of the peak-seeking controller. The approach presented here differs from that approach in the following manner:
It is assumed that the performance function is fully determined by the value of a multi-dimensional parameter and moreover it has some additional properties to be specified later on.
It is assumed that a dither function can be found to satisfy some persistent exitation condition rather than assuming a time scale separation.

Section 2 describes the dynamical system and the class of performance functions under consideration. We also properly define the problem of designing a peak-seeking control-law. In section 3 a peak-seeking control-law and in fact a methodology for designing one is presented. The behavior of the closed-loop system is analized in this section. A numerical example

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is presented in section 4 and conclusions are given in section 5.

2. STATEMENT OF THE PROBLEM AND DESCRIPTION OF THE CLASS OF PERFORMANCE FUNCTIONS

The system under consideration is described by:

$$\begin{aligned} \dot{y} &= A\,y \;+\; B\,u \quad , \; y(0) = y_0 \\ x &= C\,y \\ z &= F(x;\theta^*) \end{aligned} \tag{1}$$

where $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{p\times n}$.

The parameter $\theta^* \in \Omega \subset \mathcal{R}^q$ is given (constant, but unknown).

In addition it will also be assumed that (A, B, C) are such that the following condition is satisfied:

$(\mathcal{C}1)$ There exist matrices $K_1 \in \mathcal{R}^{m\times n}$, $K_2 \in \mathcal{R}^{m\times p}$:

$$\mathcal{A} \overset{\text{def}}{=} \begin{pmatrix} A + BK_1 & BK_2 \\ C & 0 \end{pmatrix} \quad \text{is Hurwitz.}$$

A characterization of the triplets (A, B, C) satisfying condition $(\mathcal{C}1)$ is given in (J.L. Speyer, 2000) via a Lemma that we state here for completeness.

Lemma 1 ((J.L. Speyer, 2000)). *A triplet* (A, B, C) *satisfies condition* $(\mathcal{C}1)$ *if and only if the following is satisfied:*

(i) (A, B) *is stabilizable, and*
(ii) $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ *is full row rank.*

The class of performance functions under consideration is described by:

$$F(x;\theta) \;=\; f(\langle g(x), \theta\rangle) \;\;,\; \theta \in \Omega \subset \mathcal{R}^q$$

where:

- The function $g : \mathcal{R}^p \to \mathcal{R}^q$ has continuous first partial derivatives.
- The function $f : \mathcal{R} \to \Sigma \subset \mathcal{R}$ is a homeomorphism.
- The set $\Omega \subset \mathcal{R}^q$ is closed and convex.
- For each $\theta \in \Omega$, $F(x;\theta)$ has an unique minimizer[2] $x_{min}(\theta)$, and moreover the function $x_{min} : \Omega \to \mathcal{R}^p$ has continuous first partial derivatives.

[2] Or for each $\theta \in \Omega$, $F(x;\theta)$ has an unique maximizer . . . .

The problem is the one of designing a (peak-seeking) controller that operating on the signals (z, x, y) will guarantee the stability of the closed-loop system and moreover ensure that $x(t)$ converges to some neighborhood of $x_{min}(\theta^*)$. In developing the design of such a controller we only assume knowledge of $g,\ f,\ \Omega,\ x_{min}$ and some $K_1,\ K_2$ for which the triplet (A, B, C) obey condition $(\mathcal{C}1)$.

An example of a family of performance function that can be represented in the aforementioned manner is the following:

$$G(x;(M,b,c)) \overset{\text{def}}{=} \frac{1}{2}\langle Mx, x\rangle + \langle b, x\rangle + c$$

with $M \in \mathcal{R}^{n\times n}$, $M = M^*$, satisfying $\lambda_1 I \le M \le \lambda_2 I$ for some given $0 < \lambda_1 < \lambda_2$.
The above family of functions can be expressed in the following manner:

$$G(x;(M,b,c)) \;=\; F(x;\theta) \;=\; f(\langle g(x), \theta\rangle)$$

where,

$$\begin{aligned} f(v) &= v\,, \\ g(x) &= (\frac{1}{2}x_1^2\, x_1x_2 \ldots x_1x_p \frac{1}{2}x_2^2\, x_2x_3 \ldots x_2x_p \ldots \\ &\quad x_1\, x_2 \ldots x_p\; 1)^*, \\ \theta &= (m_{1,1}\, m_{1,2} \ldots m_{1,p} m_{2,2}\, m_{2,3} \ldots m_{2,p} \ldots \\ &\quad b^*\; c)^*, \end{aligned}$$

with the set $\Omega = \{\theta \in \mathcal{R}^q \,:\, \lambda_1 I \le M \le \lambda_2 I\,\}$. In this case $x_{min}(\theta) = -M^{-1}b$.

3. DESIGN OF A PEAK-SEEKING CONTROLLER

Here, a control-law for the problem stated above is presented. In doing so, a loop transformation is first performed via,

$$u(t) = \; K_1 y(t) \;+\; K_2 \int_0^t (x(\tau) - r(\tau))\, d\tau \quad .$$

The resulting system is described by:

$$\begin{aligned} \dot{\xi}(t) &= \mathcal{A}\,\xi(t) \;+\; \mathcal{B}\,r(t) \quad , \; \xi(0) = \xi_0 \\ x(t) &= \mathcal{C}\,\xi(t) \\ z(t) &= F(x(t);\theta^*) \end{aligned} \tag{2}$$

where,

$$\mathcal{C} \overset{\text{def}}{=} \begin{pmatrix} C & 0 \end{pmatrix} \;,\; \mathcal{B} \overset{\text{def}}{=} \begin{pmatrix} 0 \\ -I \end{pmatrix} \;,\; \xi_0 \overset{\text{def}}{=} \begin{pmatrix} y_0 \\ 0 \end{pmatrix}.$$

Notice that the above LTI system represented by the triplet $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ has transfer function $\mathcal{H}(s) \overset{\text{def}}{=} \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B}$ which satisfies $\mathcal{H}(0) = I$, implying that $x(t)$ will track $r(t)$ provided it is ‘slow’ enough.

It seems therefore natural to consider the following adaptive control-law in order to tackle the problem in hand:

$$\dot{\hat{\theta}}(t) = \alpha\,[\,f^{-1}(z(t)) - \langle g(x(t)), \hat{\theta}(t)\rangle\,]\,g(x(t))\ ,$$
$$\hat{\theta}(0) = \hat{\theta}_0\ ,$$
$$r(t) = x_{min}(\,\mathcal{P}_\Omega(\hat{\theta}(t))\,) + d(t) \quad (3)$$

where, $\alpha > 0$ is an estimation gain, $d : \mathcal{R}^+ \to \mathcal{R}^p$ is a dither function which is chosen to be a member of $\mathcal{L}_\infty[0, +\infty)$, and $\mathcal{P}_\Omega : \mathcal{R}^q \to \Omega$ is the projection operator onto Ω.

With respect to the closed-loop system described by (2)-(3), the following straight-forward result stated here as a proposition is in order.

Proposition 1. *Under the above assumptions, for every initial condition* $(y_0, \hat{\theta}_0) \in \mathcal{R}^n \times \Omega$ *the differential equation (2)-(3), that describes the behavior of the closed-loop system, has an unique solution* $(\xi(t), \hat{\theta}(t))$ *on* $[0, +\infty)$, *and moreover* $(\xi, \hat{\theta}) \in \mathcal{L}_\infty[0, +\infty)$.

Further, if a dither function d ($d \in \mathcal{L}_\infty[0, +\infty)$ *) can be chosen to satisfy the following persistant exitation (PE) condition;*

(PE) *There exist* $\gamma > 0,\ T > 0,\ t_p \geq 0$: $\int_t^{t+T} [g(x(\tau))\, g^*(x(\tau))]\, d\tau \geq \gamma I\ ,\ \forall t \geq t_p$.

Then,

$$\lim_{t\to+\infty} \|\,\hat{\theta}(t) - \theta^*\| = 0,$$

$$(\textit{in fact } \|\,\hat{\theta}(t) - \theta^*\| \to 0 \textit{ exponentially})$$

$$\textit{and} \quad \lim_{t\to+\infty} \|x(t) - x_{min}(\theta^*) - (h * d)(t)\| = 0,$$

where, $h = \mathcal{L}^{-1}\{\mathcal{H}\}$.

Proof:. Via the change of variable $e \stackrel{\text{def}}{=} \theta^* - \hat{\theta}$, the differential equation (2)-(3) can be equivalently written in the following form:

$$\dot{\xi} = \mathcal{A}\,\xi + \mathcal{B}\,[x_{min}(\mathcal{P}_\Omega(\theta^* - e)) + d(t)]\quad ,$$
$$\xi(0) = \xi_0$$
$$\dot{e} = -\alpha\,[g(\mathcal{C}\,\xi)\,g^*(\mathcal{C}\,\xi)]\,e\quad ,\ e(0) = \theta^* - \hat{\theta}_0$$

Under the above assumptions, the right-hand side satisfies the Carathéodory conditions and also it is locally Lipschitz in (ξ, e) (recall that the operator $\mathcal{P}_\Omega$ is non-expansive). Therefore (Hale, 1969) the above differential equation has an unique solution $(\xi(t), e(t))$ on some interval. That this solution can be indefinitely extended follows from Maximal Interval of Existance Theorem (Hale, 1969) and from the fact that $(\xi(t), e(t))$ will be allways confined to be in a compact set. Indeed, $\|e(t)\| \leq \|e(0)\|$ together with the facts that $d \in \mathcal{L}_\infty[0, +\infty)$, $x_{min}(\mathcal{P}_\Omega(\cdot))$ is continuous and $\mathcal{A}$ is Hurwitz, implies that also $\|\xi(t)\| \leq k$ for some finite k.
The above reasoning also shows that $(\xi, e) \in \mathcal{L}_\infty[0, +\infty)$.
The continuity of g, implies that $g(\mathcal{C}\,\xi) \in \mathcal{L}_\infty[0, +\infty)$ which together with the (PE) condition implies (see (K.S. Narendra, 1989) for details) that $\|e(t)\| \to 0$ exponentially. The last result follows from the facts that $x_{min}(\mathcal{P}_\Omega(\cdot))$ is continuous, $\theta^* \in \Omega$, and $\mathcal{H}(0) = I$. □

Some remarks are in need at this point.

Remarks. ***1.*** *It is important to stress here that the proposed control-law does not use directly the data* (A, B, C) *that represent the system (1). Indeed, the above result remains valid for any* (A, B, C) *provided that* $\mathcal{A}$ *remains Hurwitz for the* K_1, K_2 *used by the controller.*
In fact, regarding the robustness of the above proposed controller, it is important to mention that defining $\gamma \stackrel{\text{def}}{=} \|\begin{pmatrix} I & 0 \end{pmatrix}(sI - \mathcal{A})^{-1}\begin{pmatrix} I \\ 0 \end{pmatrix}\|_\infty$, *the following can be claimed:*

- *If* A *in (1) is perturbed by* $\Delta \in \mathcal{R}^{n\times n}$: $\|\Delta\| < \frac{1}{\gamma}$ *then, regarding the behavior of the new closed-loop system, all the assertions in Proposition 1 remain valid (it is just needed the change of* h *by* $\tilde{h} = \mathcal{L}^{-1}\{\tilde{\mathcal{H}}\}$, *with* $\tilde{\mathcal{H}} = \mathcal{C}(sI - \tilde{\mathcal{A}})^{-1}\mathcal{B}$, $\tilde{\mathcal{A}} = \mathcal{A} + \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}$; *it is well-known (see (D. Hinrichsen, 1986) for instance) that* $\tilde{\mathcal{A}}$ *is Hurwitz).*
- *If* A *in (1) is perturbed by* $\Delta : \mathcal{R}^n \times \mathcal{R}^+ \to \mathcal{R}^{n\times n}$: $\beta = \sup_{y\in\mathcal{R}^n,\, t\geq 0}\{\Delta(y,t)\} < \frac{1}{\gamma}$, *and assume the function* Δ *is smooth enough, then, regarding the behavior of the new closed-loop system, all the assertions with exeption of the last one remain valid. The last one should be replaced by*
  $\lim_{t\to\infty} \|x(t) - \mathbb{B}_{x_{min}(\theta^*)}(\kappa)\| = 0$, *with* κ, *the radius of the ball, proporsional to* $\|d\|$ *and* β.
  Of course in such a case the proof of the proposition should be modified. In this case it is needed the use a common or joint Lyapunov function (D. Hinrichsen, 1986), to do the analysis of the dynamics described by
  $\dot{\xi} = \tilde{\mathcal{A}}\,\xi + \mathcal{B}\,[x_{min}(\mathcal{P}_\Omega(\theta^* - e)) + d(t)]$,
  $\xi(0) = \xi_0$,
  and also it is necessary to resort to the Comparison Principle (Hale, 1969; Vidyasagar, 1993).

2. *It was purposely avoided, in the proposition above, the issue of existance of a dither function d satisfying the (PE) condition and also the issue of devising one such a d. Obviously (PE) can not be achieved if there is overparametrization in the representa-*

tion of $F(x;\theta)$; this implies that the (PE) condition strongly depends on the function $g(\cdot)$ (not only on the function x). If a periodic function x exist such that the scalar functions, components of $g(x)$, are linearly independent, then (see (Chen, 1984)) such a function will satisfy the (PE) condition. Then, the question is whether or not it is possible to generate such an output x by means of a dither d. With regard to that question recall that since $\mathcal{H}(0) = I$, it may be possible to find d such the output of the system will approximate a 'very slow' signal.
The above comment suggest a method for devising dither functions that satisfy the (PE) condition. Furthermore, this comment may be the basis of future research in the analysis of adaptive peak-seeking feedback systems that take into consideration the issues left out in the proposition above.

4. AN EXAMPLE

As an example it was considered the system (1) with,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -9 & -5 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},$$

and $F(x;\theta^*) = G(x;(1,1,1))$.
The set Ω was chosen to be,
$\Omega = \{\theta \in \mathcal{R}^3 : 0.1 \leq \theta_1 \leq 5\}$.
In the control-law it was used $K_1 = 0$, $K_2 = -1.5$, $\alpha = 0.5$. The dither function was chosen as $d(t) = \text{square}(2\pi f t)$, $t \geq 0$ with $f = 0.1\,Hz$ ($\text{square}(\cdot) = \text{sign}(\sin(\cdot))$).
This numerical example was run with initial conditions

$$y_0 = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \hat{\theta}_0 = \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix};$$

the results are depicted in Figures 1-2.
As observed in Figure 1 the rate of convergence of the estimated parameter to the true value is very low. An explanation for that effect can be given by noting that the output can be roughly approximated by $x(t) \approx -1 + \mathrm{a}\sin(2\pi f t)$, $(\mathrm{a} \approx 0.7)$ for 'big' t (this can not be seen from Figure 2). Substitution of this signal into $g = (\frac{1}{2}x^2\ x\ 1)^*$ shows that the three functions are linearly independent (see (Chen, 1984)) only by the 'small' term $\frac{1}{4}\mathrm{a}^2 \cos(4\pi f t)$. This reasoning suggest that by increasing a, it may be possible to increase the rate of convergence. This is really the case; in fact by increasing the amplitude of the dither from 1 to 2 the rate of convergence dramatically increase to the point that at 500 sec better results are achieved, than previously at 5000 sec. It was also included in Figure 3 some plots showing the results of such a numerical simulation.

5. CONCLUSIONS

An adaptive control-law -and indeed and approach for devicing one- was presented here which is suitable

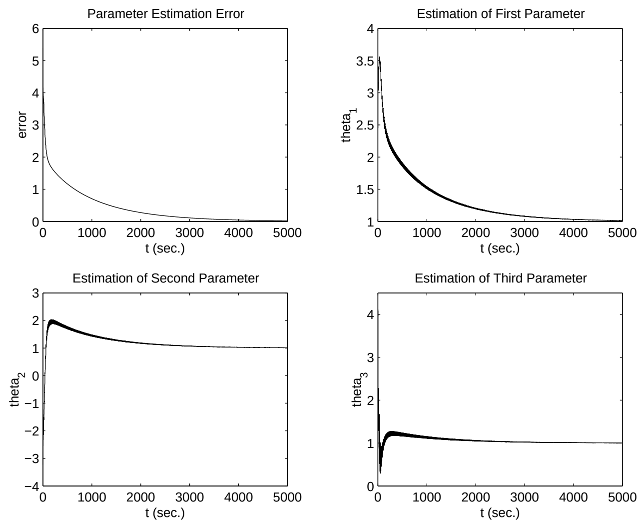

Fig. 1. Evolution of the Estimated Parameter.

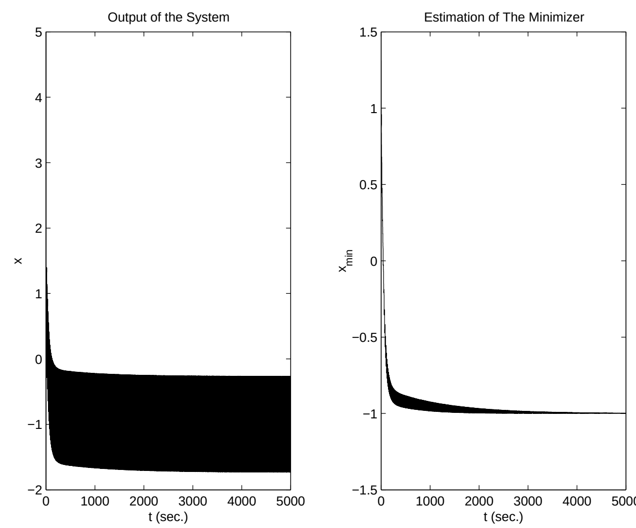

Fig. 2. Evolution of $x(t)$ and $x_{min}(\mathcal{P}_\Omega(\theta(t)))$.

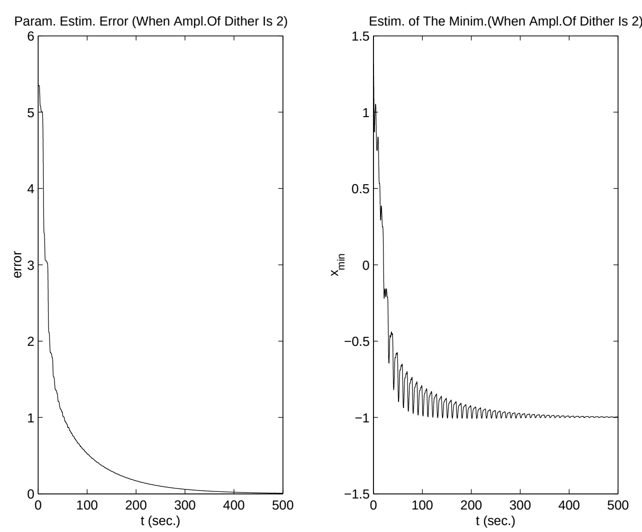

Fig. 3. Evolution of $\|e(t)\|$ and $x_{min}(\mathcal{P}_\Omega(\theta(t)))$ when the ampl. of the dither is duplicated.

for a particular class of systems requiring set-points that extremize a performance funtion. The approach presented for designing such a controller, is intended for cases in which the performance function is fully determined by the value of a multi-dimensional parameter. The designed controller has the capability of estimating the (unknown) value of the parameter, determining the performance function, provided a dither function can be found to satisfy a persistent exitation condition; then the output of the system will converge to a neighborhood of the extremizing output value.

Neither the issue of noisy measurements nor the issue of robustness in the representation of the performance function were addressed in the present work, however the control-law intrinsicaly poses some robustness properties that take care of uncertanty in the parameters that describe the dynamical system.

An numerical example was presented in order to illustrate the methodology and, most important, to face the delicate issue (left out in the analysis of the behavior of the closed-loop system) of devising an appropiated dither function that ensure the success of the estimator.

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