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FREQUENCY DOMAIN IDENTIFICATION OF PARTIAL FRACTION MODELS

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Abstract: The paper constructs a method for identification of models given in partial fraction representation by using a specific discrete bi-orthogonal system of functions specified by the poles and their multiplicities. Using frequency domain data, an iteration algorithm convergent in second order is given that incorporates procedures for finding not only the pole locations, but also their corresponding multiplicity. *Copyright ©2002 IFAC.*

Keywords: System identification, orthogonal rational systems, bi-orthogonal rational systems, algorithm to compute poles.

1. INTRODUCTION

System identification based upon the *partial fraction representation* of the transfer function is recognized as a classical approach in systems science (Keviczky *et al.*, 1987). It has several applications, e.g. in identifying vibrating structures, see (Gilpin *et al.*, 1992). The recent results in the field of applying rational orthogonal bases in system identification, see (Van den Hof *et al.*, 1995), (Wahlberg, 1994), (Ninness and Goodwin, 1995), and (Schipp and Bokor, 1998), however, offer a new opportunity to revise the classical methods in order to improve their efficiency both in their modelling power and computability.

The use of rational orthogonal bases supposes some a priori knowledge upon the system poles, hence the classical methods preserve their significance in the approximate estimation of the poles, which can serve as starting point for the new ones. Particularly the *subspace methods* elaborated in the beginning of the 90's proved to be efficient for both time and frequency domain data, see (Overschee and Moor, 1991). The main drawback of both classical partial fraction identification and subspace methods, that they do not handle of poles with multiplicity larger than one, or they assume a priori fixed multiplicities.

In our paper an iterative method to identify models given in partial fraction form will be presented. This will utilizes the benefits of the use of bi-orthogonal rational systems, can handle the multiplicity of the poles, furthermore produces good convergence and easy computability.

The following notations will be used: $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denotes the unit disc, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ the unite circle; and $\mathcal{A}$ is the set of functions analytic in $\mathbb{D}$ and continuous in $\mathbb{D} \cup \mathbb{T}$. If $a \in \mathbb{C}$ then $\overline{a}$ denotes its complex conjugate. $\mathbb{N}^* := \{0, 1, 2, \dots\}$, i.e. it denotes the set of natural numbers $\mathbb{N}$ complemented with the number 0.

The basic model used in the method is the partial fraction representation of a system transfer function $f = f_{\hat{\mathbf{a}}}$ with poles $\hat{\mathbf{a}} := (\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n) \in \mathbb{C}$, where $\hat{a}_k := 1/\overline{a}_k$, $a_k \in \mathbb{D}$, and $a_k \neq a_\ell$ if $k \neq \ell$ $(k, \ell = 1, \dots, n)$. Suppose that the multiplicity of a_k is $m_k - 1$, where $m_k \geq 2, m_k \in \mathbb{N}$, and let $\mathbf{m} = (m_1, m_2, \dots, m_n)$ $(k = 1, \dots, n)$. The partial fraction representation can be written as follow:

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$$f(z) = f_{\mathbf{a}}(z) = \sum_{k=1}^{n} \sum_{\ell=0}^{m_k-2} \frac{A_{k\ell} z^\ell}{(1-\overline{a_k} z)^{\ell+1}} \qquad (1)$$

The identification problem is formulated as the determination of the multiplicities $\mathbf{m}$, and the corresponding parameters $A_{k\ell}$, $\mathbf{a}$ in (1) using a set of measured frequency response sample values of f. It is important to stress, that the multiplicities — as it will be clarified latter — are considered not to be fixed, but they will vary in the course of the procedure.

To elaborate the method a discrete scalar product is introduced, and based upon it a bi-orthogonal rational system is defined to represent the function (1). An iterative scheme will be set up, which starting from an initial pole placement converges into the true poles including their multiplicities. The method has been proved to locally converge in second order. The coefficients of the partial fraction can be computed on the basis of the bi-orthogonal representation. A numerical method can be constructed, which uses discrete values of the transfer function, hence it can be applied as a system identification method based upon frequency domain data.

The structure of the paper is the following: first a bi-orthogonal system of rational functions will be constructed, to represent the transfer function, on the basis of a discrete scalar product introduced by the discrete Cauchy formula (presented in Appendix); than an iteration method will be given with the purpose to estimate both the pole structure and the exact location of the poles; and some identification examples will be presented finally.

2. CONSTRUCTION OF A BI-ORTHOGONAL SYSTEM DEFINED BY PARTIAL FRACTION

Assume that the system of (1) has poles all with multiplicity one; in this case it can be represented on a system of rational functions φ_i as follows

$$f_{\mathbf{a}}(z) = \sum_{k=1}^{n} \frac{A_k}{1-\overline{a_k} z} = \sum_{\ell \in \mathbb{N}^*} A_\ell \varphi_\ell(z) \qquad (2)$$

where

$$\varphi_\ell(z) := \frac{1}{1-\overline{a}_\ell z}$$

In order to construct the representation, a discrete scalar product

$$[F, G] := [F, G]_N := \frac{1}{N} \sum_{z \in \mathbb{T}_N} F(z) \overline{G(z)} \qquad (3)$$

defined on the discrete group $\mathbb{T}_N$ depending on $N \in \mathbb{N}^*$ (see Appendix) will be used. This scalar product can be applied for rational systems analogously to the ordinary – continuous – scalar product, corresponding to the discrete Cauchy formula that is introduced in the Appendix.

We shall construct discrete bi-orthogonal systems depending on the parameters

$$(a_1, a_2, \ldots, a_n) \in \mathbb{D}^n,$$

where $a_i \neq a_j$, if $i \neq j$. Namely for $z \in \mathbb{C}$ and $\ell = 1, 2, \ldots, n$ set

$$\varphi_\ell(z) := \frac{1}{1-\overline{a}_\ell z}, \quad \Phi_\ell(z) := \frac{1-a_\ell^N}{\omega_\ell(a_\ell)} \omega_\ell(z). \qquad (4)$$

where

$$\omega_\ell(z) = \prod_{j=1, j\neq \ell}^{n} (z - a_j).$$

If $N \geq n$ then the systems (4) are bi-orthogonal with respect the scalar product $[\cdot,\cdot]_N$, i.e. for any couple $k, \ell \in \{1, 2, \ldots, n\}$ we have

$$[\Phi_k, \varphi_\ell]_N = \delta_k^\ell, \qquad (5)$$

where δ_k^ℓ is the Kronecker symbol.

Indeed, by (22) of Theorem 2 (in Appendix) for $1 \leq k$, $\ell \leq n$

$$[\Phi_k, \varphi_\ell] = \frac{1}{N} \sum_{\zeta \in \mathbb{T}_N} \frac{\Phi_k(\zeta)}{1-a_\ell \overline{\zeta}} = \frac{1}{N} \sum_{\zeta \in \mathbb{T}_N} \Phi_k(\zeta) \frac{\zeta}{\zeta - a_\ell}$$
$$= \frac{1}{2\pi\,\mathrm{i}} \int_{\mathbb{T}_N} \frac{\Phi_k(\zeta)}{\zeta - a_\ell}\, d\zeta = \frac{\Phi_k(a_\ell)}{1-a_\ell^N} = \delta_k^\ell$$

hence (5) is proved.

Following from the bi-orthogonality of φ_ℓ and Φ_ℓ, the coefficients A_ℓ in the representation (2) can be computed by the scalar product

$$A_\ell = [f, \Phi_\ell]_N.$$

Generalizing this construction to the case of poles with multiplicity larger than one

$$f_{\mathbf{a}}(z) = \sum_{k=1}^{n} \sum_{\ell=0}^{m_k-2} \frac{A_{k\ell} z^\ell}{(1-\overline{a}_k z)^{\ell+1}} = \sum_{(k,\ell) \in \mathcal{J}_{\mathbf{n}}} A_{k\ell} \varphi_{k\ell}(z) \qquad (6)$$

where

$$\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{D}^n \quad (a_i \neq a_j \quad \text{if} \quad i \neq j),$$

and consider the rational functions

$$\varphi_{k\ell}(z) := \frac{z^\ell}{(1-\overline{a}_k z)^{\ell+1}} \qquad (7)$$

$$(z \in \mathbb{C},\ k = 1, 2, \ldots, n, \ell = 0, 1, \ldots, m_k - 1),$$

where $m_1, m_2, \ldots, m_n \in \mathbb{N}^*$, $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, $m := m_1 + \ldots + m_n$, and

$$\mathcal{J}_{\mathbf{n}} = \{(i, j) : j \in \mathbb{N}, 0 \leq j < m_i, i = 1, \ldots, n\}. \qquad (8)$$

Obviously the numbers $\widehat{a}_k := 1/\overline{a}_k$ are the poles of $\varphi_{k\ell}$ with the multiplicity $\ell + 1$.

We show that there exists a collection of polynomials

$$\Phi_{k\ell} = \Phi_{k,\ell}^{\mathbf{m}}(\cdot, \mathbf{a}) \in \mathcal{P}_{m-1} \quad ((k, \ell) \in \mathcal{J}_{\mathbf{m}})$$

such that the systems

$$(\varphi_i, i \in \mathcal{J}_{\mathbf{m}}), \quad (\Phi_i, i \in \mathcal{J}_{\mathbf{m}})$$

are bi-orthogonal with respect the scalar product $[\cdot,\cdot]_N$, if $N \geq m$, i.e.

$$[\Phi_{k\ell},\varphi_{rs}]_N = \delta_{kr}\delta_{\ell s} \quad ((k,\ell),(r,s) \in \mathcal{J}_{\mathbf{m}}),$$

hence the coefficients $A_{k\ell}$ in the representation (6) can be expressed by the scalar product

$$A_{k\ell} = [f,\Phi_{k\ell}]_N.$$

Moreover the polynomials $\Phi_{k\ell}$ can be written in the form

$$\Phi_{k\ell} = \omega_k P_{k\ell} \quad ((k,\ell) \in \mathcal{J}_{\mathbf{m}}), \tag{9}$$

where

$$P_{k\ell} \in \mathcal{P}_{m_k-1},$$
$$\omega_k(z) := \omega_k(z,\mathbf{a}) := \prod_{i=1,i\neq k}^{n} (z-a_i)^{m_i}$$
$$(k = 1,2,\ldots,n, z \in \mathbb{C}).$$

The polynomials $P_{k\ell}$ can be expressed by the partial sums of the Taylor-series expansion

$$P_k(z) := P_k(z,\mathbf{a}) := \frac{1-z^N}{\omega_k(z,\mathbf{a})} = \sum_{j=0}^{\infty} p_{kj}(z-a_k)^j \tag{10}$$

$$(|z-a_k| < r_k,\ z \in \mathbb{D},$$
$$r_k := \min\{|a_j - a_k| : j = 1,2,\ldots,n, j \neq k\}),$$

namely

$$P_{k\ell}(z) = (z-a_k)^{\ell} \sum_{j=0}^{m_k-\ell-1} p_{kj}(z-a_k)^j \tag{11}$$
$$(z \in \mathbb{C}, (k,\ell) \in \mathcal{J}_{\mathbf{m}})$$

and by (10)

$$p_{kj} = \frac{P_k^{(j)}(a_k,\mathbf{a})}{j!} := \frac{1}{j!}\,\frac{d^j}{dz^j} P_k(z,\mathbf{a})\bigg|_{z=a_k}.$$

We prove

Theorem 1. Let $\mathbf{a} = (a_1,\ldots,a_n) \in \mathbb{D}^n$, where $a_i \neq a_j$, if $1 \leq i,j \leq n$ and $i \neq j$ and fix the vector $\mathbf{m} = (m_1,\ldots,m_n)$ with $m_j \in \mathbb{N}^*$ and the natural number $N \geq m_1 + \ldots + m_n$. Then there exists an unique system of polynomials $\Phi_{k\ell} \in \mathcal{P}_{m-1}$ $((k,\ell) \in \mathcal{J}_{\mathbf{m}})$ such that the systems $\varphi_{k\ell}$ and $\Phi_{k\ell}$ $((k,\ell) \in \mathcal{J}_{\mathbf{m}})$ are bi-orthogonal with respect the scalar product (3). Moreover the polynomials $\Phi_{k\ell}$ can be written in the form (9) and the coefficients of $P_{k\ell}$ are defined by (10) and (11).

Proof: By (22) of Theorem 2(in Appendix)

$$[\Phi_{k\ell},\varphi_{rs}]_N = \frac{1}{N}\sum_{\zeta\in\mathbb{T}_N} \frac{\Phi_{k\ell}(\zeta)\overline{\zeta}^s}{(1-a_r\bar{\zeta})^{s+1}} =$$
$$= \frac{1}{N}\sum_{\zeta\in\mathbb{T}_N} \frac{\Phi_{k\ell}(\zeta)\zeta}{(\zeta-a_r)^{s+1}} = \frac{1}{2\pi\,\mathrm{i}}\int_{\mathbb{T}_N} \frac{\Phi_{k\ell}(\zeta)}{(\zeta-a_r)^{s+1}}\,d\zeta =$$
$$= \frac{1}{s!}\,\frac{d^s}{dz^s}\frac{\Phi_{k\ell}(z)}{1-z^N}\bigg|_{z=a_r}$$

and consequently the systems in question are bi-orthogonal if and only if

$$\frac{1}{s!}\,\frac{d^s}{dz^s}\frac{\Phi_{k\ell}(z)}{1-z^N}\bigg|_{z=a_r} = \delta_k^r\delta_\ell^s \tag{12}$$

for any couple (k,ℓ), $(r,s) \in \mathcal{J}_{\mathbf{m}}$. Thus the $\Phi_{k\ell}$ $((k,\ell) \in \mathcal{J}_{\mathbf{m}})$ polynomials are the fundamental polynomials of the weighted Hermite interpolation problem

$$(\rho_N\Phi)^{(j)}(a_i) = b_{ij} \quad ((i,j) \in \mathcal{J}_{\mathbf{m}}),$$

where $\rho_N(z) := (1-z^N)^{-1}$ $(z \in \mathbb{C})$ is the weight function and b_{ij} are given numbers. From (12) it follows that $\Phi_{k\ell}$ is of the form

$$\Phi_{k\ell}(z) = P_{k\ell}(z)\prod_{j=1,j\neq k}^{n}(z-a_j)^{m_j} = P_{k\ell}(z)\omega_k(z)$$

$(z \in \mathbb{C})$, and by (12)

$$(\rho_N\omega_k P_{k\ell})^{(j)}(a_k) = \delta_j^\ell j! \quad (\ell \leq j < m_k). \tag{13}$$

This is equivalent to

$$\sum_{i=0}^{j}\binom{j}{i}(\rho_N\omega_k)^{(j-i)}(a_k)P_{k\ell}^{(i)}(a_k) = \delta_j^\ell j!$$

$(\ell \leq j < m_k)$. Thus $P_{k\ell}^{(i)}(a_k) = 0$, if $i < \ell$ and

$$\sum_{i=\ell}^{j}\frac{(\rho_N\omega_k)^{(j-i)}(a_k)}{(j-i)!}\,\frac{P_{k\ell}^{(i)}(a_k)}{i!} = \delta_j^\ell \quad (\ell \leq j < m_k). \tag{14}$$

We consider the infinite system of linear equations with respect to $p_{k0}, p_{k1},\ldots,p_{ki},\ldots$:

$$\sum_{i=0}^{j}\frac{(\rho_N\omega_k)^{(j-i)}(a_k)}{(j-i)!}p_{ki} = \delta_0^j \quad (j \in \mathbb{N}). \tag{15}$$

The coefficient of p_{kj} in j-th equation is $(\rho_N\omega_k)(a_k) \neq 0$, consequently this system has a unique solution. Comparing this with (13) and (14) we get

$$\frac{P_{k\ell}^{(i)}(a_k)}{i!} = p_{k(\ell-i)} \quad (i \geq \ell).$$

It is clear that the Taylor-coefficients of the function

$$P_k(z) := \frac{1-z^N}{\omega_k(z)} = \sum_{j=0}^{\infty}p_{kj}(z-a_k)^j \quad ((|z-a_k| < r_k))$$

satisfy (15) and Theorem 1 is proved.

To evaluate the numbers p_{kj} we introduce the function

$$S_k(z) := \frac{P_k'(z)}{P_k(z)} = \sum_{j=0}^{N-1}\frac{1}{z-\epsilon_N^j} - \sum_{j=1,j\neq k}^{n}\frac{1}{z-a_j},$$

$(|z-a_k| < r_k)$, where $\epsilon_N^j = \exp(2\pi\,\mathrm{i}j/N)$. Hence by

$$P_k^{(\ell+1)}(a_k) = \sum_{j=0}^{\ell}\binom{\ell}{j}P_k^{(j)}(a_k)S_k^{(\ell-j)}(a_k) \quad (\ell \in \mathbb{N})$$

we get the following recursion:

$$p_{k(\ell+1)} = \frac{1}{\ell+1}\sum_{j=0}^{\ell} p_{kj}s_{k(\ell-j)} \quad (\ell \in \mathbb{N}), \tag{16}$$

where

$$\begin{aligned} s_{ki} :=& \frac{S_k^{(i)}(a_k)}{i!} = \\ =& \sum_{j=0}^{N-1} \frac{(-1)^i}{(a_k - \epsilon_N^j)^{i+1}} - \sum_{j=1, j\neq k}^{n} \frac{(-1)^i}{(a_k - a_j)^{i+1}}, \end{aligned} \tag{17}$$

$(i \in \mathbb{N})$. On the basis (16) and (17) the coefficients p_{kj} can be computed.

3. ITERATION ON POLE STRUCTURE

In this section an iteration method will be constructed to compute the poles – including their multiplicities – of a rational function by applying the partial fraction model introduced by (6). This algorithm is locally convergent in second order and uses the values of the rational function only in the points of $\mathbb{T}^N$, i.e. uniformly spaced sample values of the frequency response of the underlying system.

To compute the vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{D}^n$ we use the system

$$\Phi_{k\ell}^{\mathbf{m}}(\cdot, \mathbf{a}) \quad ((k, \ell) \in \mathcal{J}_{\mathbf{m}}).$$

Let $N \geq m := m_1 + m_2 + \ldots + m_n$. Then by Theorem 1 the systems

$$\Phi_{k\ell}^{\mathbf{m}}(\cdot, \mathbf{a}), \quad \varphi_{k\ell}(\cdot, \mathbf{a}) \quad ((k, \ell) \in \mathcal{J}_{\mathbf{m}}),$$

depending on $\mathbf{a} \in \mathbb{D}^n$, are bi-orthogonal with respect to the scalar product (3). Denote

$$\begin{aligned} \phi_k(\cdot, \mathbf{a}) &:= \Phi_{k(m_k-1)}^{\mathbf{m}}(\cdot, \mathbf{a}) \\ \phi_k^-(\cdot, \mathbf{a}) &:= \Phi_{k(m_k-2)}^{\mathbf{m}}(\cdot, \mathbf{a}) \end{aligned} \tag{18}$$

and introduce

$$F_k(\mathbf{a}) := [\phi_k(\cdot, \mathbf{a}), f_{\mathbf{a}}]_N, \quad F_k^-(\mathbf{a}) := [\phi_k^-(\cdot, \mathbf{a}), f_{\mathbf{a}}]_N$$

$(k = 1, \ldots, n)$, the conjugate coefficients of the bi-orthogonal expansion of the the functions (18). This expansion will be infinite, unless it is applied on the exact pole locations. In that case the expansion will contain $m_k - 1$ nonzero terms, hence

$$F_k(\mathbf{a}) = 0, \quad F_k^-(\mathbf{a}) = \overline{\lambda}_k = \overline{A}_{k(m_k-2)} \ (\neq 0)$$

for $(k = 1, \ldots, n)$.

An iteration process can be given by introducing the functions

$$\begin{aligned} G_k(\mathbf{a}) :=& a_k + \frac{1}{m_k - 1}\frac{F_k(\mathbf{a})}{F_k^-(\mathbf{a})} \quad (k = 1, \ldots, n), \\ G :=& (G_1, \ldots, G_n). \end{aligned} \tag{19}$$

The iteration process is given by

$$\mathbf{a}^{\nu+1} := G(\mathbf{a}^\nu) \quad (\nu \in \mathbb{N}, \mathbf{a}^0 \in \mathbb{D}^n). \tag{20}$$

It can be proved that the iteration procedure (20) is locally convergent in second order, i.e. there exists an $r > 0$, $K > 0$ such that if for the initial value $\|\mathbf{a}^0 - \mathbf{a}\| < r$, then

$$\|\mathbf{a}^{\nu+1} - \mathbf{a}\| \leq K\|\mathbf{a}^\nu - \mathbf{a}\|^2 \ (\nu \in \mathbb{N}). \tag{21}$$

The proof is based upon the characteristics of the first and second order partial derivatives of function (19). Namely it can be proved, that the Jacoby matrix $\partial_\ell G_k(\mathbf{a})$ of the function is zero, moreover the second partial derivatives $\partial_{sr}G_k(\mathbf{a})$ $(s \neq r)$ are equal to 0, and $\partial_{ss}G_k(\mathbf{a})$, $\partial_{kk}G_k(\mathbf{a})$ are finite; that implies (21).

The realization of the iteration process requires an initial pole location $\mathbf{a}^0$, which is advantageous to be near enough to the true location. An efficient method to find an approximate pole location can be e.g. the subspace method (Overschee and Moor, 1994).

The input data required by the method – due to the discrete scalar product that has applied – are discrete points of the frequency response of system to be identified. These type of data can easily be acquired by FFT procedures from time-domain data, or – if it is possible to realize – by direct frequency response measurements.

4. EXAMPLES

A simulation example is presented: an iteration on a single pole has been performed for multiple cases – pure and noisy function with several initial conditions. A simulated spectral function based upon a real pole with $a = 0.9$ and multiplicity 2 has been generated. Number of sample point has been selected as 128. For the noisy cases the function has been modified with additive normally distributed pseudo-random signal. Noise variance 0.01 – 0.1 relative to the function gain has been used.

The method has been realized as a $MATLAB^{®}$ procedure. The termination of the iteration has been performed by the maximal absolute value of the difference of the subsequent pole values, as an error bound value 10^8 has been used. The iteration count has been limited to 300.

Aa the first example the iteration using the noiseless function is presented: the procedure has been started from different pole values, as $a^{(0)} = 0.5$, 0.6, 0.8, 0.95, 0.99; the iteration proved to be robustly convergent for all ones. Figure 1 presents the worst case $a^{(0)} = 0.5$. The series of subsequent pole values on the upper diagram), as well as the error values on the lower one. In the case of overestimating the pole multiplicity, the convergence go wrong. The iteration error sequence of this case is presented on Figure 2: pole multiplicity has been selected to be 3. The iteration has been stopped on 300 cycles.

In the presence of noise, the procedure keeps the convergence capabilities until some level of variance (typically 0.25 relative to the function gain), however the limit differs from the expected pole values, i.e.

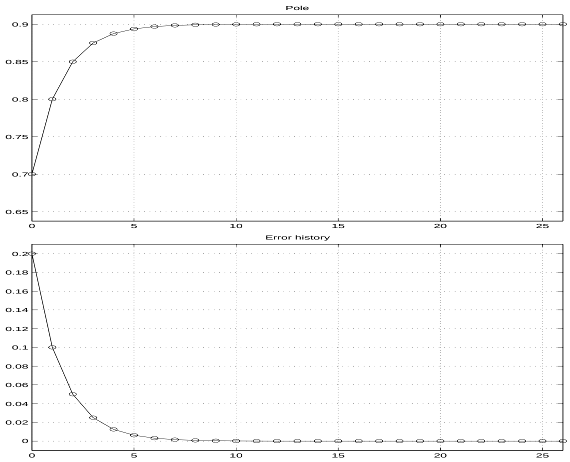

Fig. 1. Iteration on single pole: the pure case

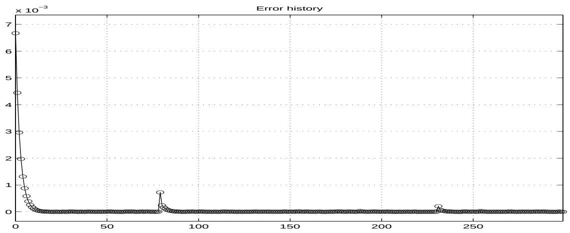

Fig. 2. Iteration with overestimated multiplicity

the method produces a random bias error. The results of two experiments are presented in Figures 3 and 5 that has been generated under noise level 0.1, and initial pole location $a = 0.85$. The Figures 4 and 4 present the Nyquist diagram and the amplitude spectrum belonging to the pure, noisy, and the estimated functions for the two realizations respectively. Both

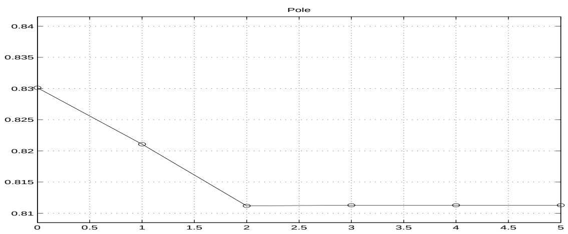

Fig. 3. Iteration on single pole: noise - realization 1.

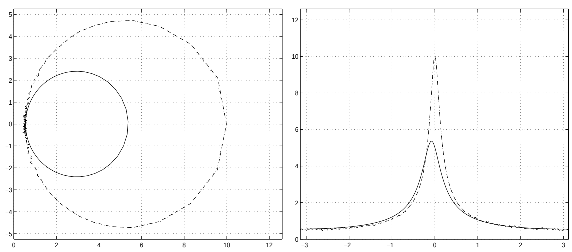

Fig. 4. Spectral functions: realization 1.

realizations show good convergence properties, however the resulted pole values are $0.8113 - 0.0636i$ and $0.8781 - 0.0625i$ respectively, instead of the expected value 0.9. The noise level applied in the experiments can be considered to be quite large compared to the level commonly achieved in the practice.

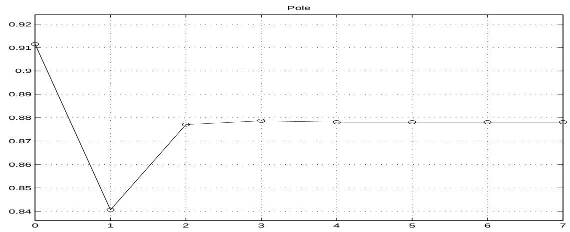

Fig. 5. Iteration on single pole: noise - realization 2.

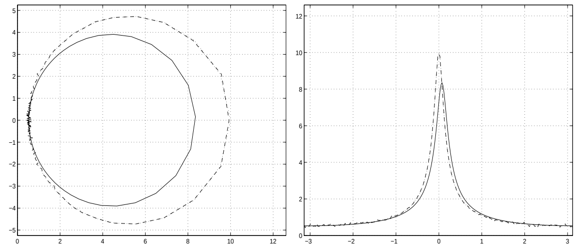

Fig. 6. Spectral functions: realization 2.

5. CONCLUSION

A discrete bi-orthogonal system has been constructed for estimating the poles – of multiplicity greater even than one – in partial fraction representation of transfer functions. Using frequency domain data, an iteration method has been given and convergence in second order has been proved. The method can be used to estimate the system poles with the purpose of identification and control design.

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APPENDIX

The discrete Cauchy formula. In section 2 the discrete analogue of the Cauchy integral formula was needed. Remember, that if $F \in \mathcal{A}$ then for any $a \in \mathbb{D}$ and $n \in \mathbb{N}$ (see e.g. (Garnett, 1981))

$$\frac{1}{2\pi\,\mathrm{i}} \int_{\mathbb{T}} \frac{F(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta = \frac{F^{(n)}(a)}{n!}.$$

Replacing $\mathbb{T}$ by the discrete group

$$\mathbb{T}_N := \{e^{2\pi\,\mathrm{i}k/N} : k = 0, 1, \ldots, N-1\} \quad (N \in \mathbb{N}^*)$$

and the integral by the sum

$$\frac{1}{2\pi\,\mathrm{i}} \int_{\mathbb{T}_N} F(\zeta)\, d\zeta := \frac{1}{N} \sum_{\zeta \in \mathbb{T}_N} F(\zeta)\zeta$$

we get a similar formula for polynomials. For function $F \in \mathcal{A}$ obviously

$$\lim_{N\to\infty} \frac{1}{2\pi\,\mathrm{i}} \int_{\mathbb{T}_N} F(\zeta)\, d\zeta = \frac{1}{2\pi\,\mathrm{i}} \int_{\mathbb{T}} F(\zeta)\, d\zeta.$$

Denote $\mathcal{P}_n$ the set of complex polynomials of degree n. Then the following analogue of the Cauchy integral formula holds.

Theorem 2. Let $n \in \mathbb{N}$, $N \in \mathbb{N}^*$ be fixed numbers and denote $P \in \mathcal{P}_{N+n-1}$ a polynomial.

i) Then for any $a \in \mathbb{D}$ we have

$$\frac{1}{2\pi\,\mathrm{i}} \int_{\mathbb{T}_N} \frac{P(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta = \frac{1}{n!} \frac{d^n}{dz^n} \frac{P(z)}{1 - z^N}\bigg|_{z=a}. \tag{22}$$

ii) Furthermore if $a_0, a_1, \ldots, a_n$ are distinct points in $\mathbb{D}$, then

$$\frac{1}{2\pi\,\mathrm{i}} \int_{\mathbb{T}_N} \frac{P(\zeta)}{(\zeta - a_0)\ldots(\zeta - a_n)}\, d\zeta = \sum_{j=0}^{n} \frac{1}{\omega_j(a_j)} \frac{P(a_j)}{1 - a_j^N}, \tag{23}$$

where

$$\omega_j(z) := \prod_{\ell=0, \ell\neq j}^{n} (z - a_\ell) \quad (z \in \mathbb{C},\, j = 1, 2, \ldots, n)$$

are the fundamental polynomials of Lagrange interpolation.

Proof: First we prove (22) for $n = 0$. To this end write $P \in \mathcal{P}_{N-1}$ in the form

$$P(z) = \sum_{j=0}^{N-1} c_j z^j \quad (z \in \mathbb{C}).$$

Observe that for $\zeta \in \mathbb{T}_N$ we have $\zeta^N = 1$ and consequently

$$\frac{\zeta}{\zeta - a} = \frac{1}{1 - a\bar{\zeta}} = \frac{1}{1 - a^N} \frac{1 - (a\bar{\zeta})^N}{1 - a\bar{\zeta}} = \\ = \frac{1}{1 - a^N} \sum_{j=0}^{N-1} a^j \bar{\zeta}^j \quad (\zeta \in \mathbb{T}_N).$$

Using the orthogonality of the discrete trigonometric system we get

$$\frac{1}{2\pi\,\mathrm{i}} \int_{\mathbb{T}_N} \frac{P(\zeta)}{\zeta - a}\, d\zeta = \frac{1}{N} \sum_{\zeta \in \mathbb{T}_N} P(\zeta) \frac{\zeta}{\zeta - a} = \\ = \frac{1}{1 - a^N} \sum_{j=0}^{N-1} c_j a^j = \frac{P(a)}{1 - a^N}$$

and for $n = 0$ (22) is proved.

To show (23) write $P \in \mathcal{P}_{n+N-1}$ in the form

$$P(z) = Q(z)(z - a_0)\ldots(z - a_n) + R(z),$$

where $R \in \mathcal{P}_n$, $Q \in \mathcal{P}_{N-2}$. Applying Lagrange interpolation formula to R we get

$$\frac{P(z)}{(z - a_0)\ldots(z - a_n)} = Q(z) + \frac{R(z)}{(z - a_0)\ldots(z - a_n)} = \\ = Q(z) + \sum_{j=0}^{n} \frac{R(a_j)}{z - a_j} = Q(z) + \sum_{j=0}^{n} \frac{P(a_j)}{z - a_j}.$$

Since $Q \in \mathcal{P}_{N-2}$, the orthogonality of the discrete trigonometric system implies

$$\int_{\mathbb{T}_N} Q(\zeta)\, d\zeta = 0,$$

and applying (22) in the case $n = 0$ we get (23).

Observe that the right hand side in (23) can be expressed by the divided differences of the function

$$G(z) := \frac{P(z)}{1 - z^N} \quad (z \in \mathbb{D}).$$

Namely (23) is equivalent to

$$\frac{1}{2\pi\,\mathrm{i}} \int_{\mathbb{T}_N} \frac{P(\zeta)}{(\zeta - a_0)\ldots(\zeta - a_n)}\, d\zeta = G(a_n, \ldots, a_1, a_0). \tag{24}$$

(Compare eg. (Henrici, 1987), pp. 247.)

Since for any $G \in \mathcal{A}$

$$G(a_n, \ldots, a_1, a_0) \to \frac{G^{(n)}(a)}{n!} \quad \text{as} \quad a_j \to a,$$

$j = 0, 1, \ldots, n$, therefore (22) follows from (24).

Taking the limit in (22) as $N \to \infty$ we obtain the continuous variant of the formula.