15th Triennial World Congress, Barcelona, Spain

SIMULTANEOUS LINEAR QUADRATIC POLE PLACEMENT (LQPP) CONTROL DESIGN

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Abstract: The pole placement (PP) technique for design of a linear state feedback control system requires specification of *all* the closed-loop pole locations even though only a few poles dominate the system's transient response characteristics. The linear quadratic regulator (LQR) method, on the other hand, optimizes the system transient response and does not directly impose the location of the dominant poles. This paper presents a new optimal linear quadratic pole placement (LQPP) technique that simultaneously assigns some poles to exact desired dominant locations (partial pole placement) and adjusts the rest of the poles to optimize an LQ performance index (parametric optimization). As a result, a designer can fashion his control design to benefit from the PP and LQ techniques. The paper only addresses the LQPP technique for a single-input full-state feedback control system. *Copyright © 2002 IFAC*

Keywords: Pole assignment, linear output feedback, linear quadratic regulator, simultaneous stabilization, optimization

1. INTRODUCTION

An intricate task in designing a linear quadratic regulator (LQR) is selection of suitable weighting matrices for its performance measure. An LQR design may be considered satisfactory when computer simulation of the closed-loop control system response meets a certain transient requirement. A pole-placement (PP) design technique, on the other hand, directly defines the transient stability for the system by specifying its closed-loop pole locations. Although only a few closed-loop poles often dominates the transient response characteristic, the PP technique calls for specification of all the system poles. The intricate task here is where to place the less dominating poles. Poles placed at inappropriate locations can result in high feedback gains.

Intriguing relationships between the LQ weight selection and pole location have prompted extensive studies in the literature. Several authors have investigated methods for designing a LQR feedback control so that the eigenvalues of the closed-loop system lie within a certain region of the numerical complex plane. Their methods ensure the closed-loop poles lie in the vicinity of a newly shifted left half-plane (Amin, 1985; Alexandridis, *et al.*, 1987; Sugimoto, *et al.*, 1989; Eastman, *et al.*, 1984), inside a disk/region (Kawasaki, *et al.*, 1983; Furuta, *et al.*, 1987; Moheimani, *et al.*, 1996; Wittenmark, *et al.*, 1987), or inside a vertical strip Shieh, *et al.*, 1986; Koshkouei, *et al.*, 1999).These modified LQ methods, some of which are successive algorithms, shift *all* the poles into a given region; the placement of the poles therefore are usually regionally restrictive and inexact in location.

Others have investigated an inverse problem approach to the PP and LQ design (Fujii, *et al.*, 1984; Fujii, 1987; Sugimoto, 1998). These include using asymptotic properties of the LQR to assign poles, and factorization and transformation to assign *n-m* of the closed-loop poles where *n* and *m* being the number of system states and inputs. The asymptotic and *n-m* requirements make the design somewhat restrictive.

This article presents a new optimal linear quadratic pole placement (LQPP) technique for a single-input state feedback control system that ***simultaneously*** assign some closed-loop poles to exact desired dominant locations (partial pole placement) and adjust the rest of the poles to optimize an LQ performance index (parametric optimization). The partial pole placement assigns n_1 ($< n$) poles, and the parametric optimization uses the remaining n_2 ($= n - n_1$) poles to optimize a static output feedback optimal control problem (Levine, *et al.*, 1970; Cheok, *et al.*, 1988; Cheok, *et al.*, 1985; Cheok *et al.*, 1986a, b). An illustration is provided as a motivation, followed by a formalization of the LQPP technique.

2. ILLUSTRATION

Consider a state-variable representation of a dc motor positioning system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_1 \end{bmatrix} u + \begin{bmatrix} 0 \\ g_1 \\ 0 \end{bmatrix} w$$

$$\triangleq \mathbf{A}\,\mathbf{x} + \mathbf{B}\,u + \mathbf{G}\,w$$

where $a_1 = -0.82900$, $a_2 = 0.56400$, $a_3 = -10.960$, $a_4 = -29.280$, $b_1 = 499.98$ and $g_1 = -9.9910$. x_1 and x_2 are the position and speed of output shaft, x_3 is the motor current, u is the voltage input to the motor driver circuit and w is an external disturbance torque. The poles of the system are at 0, -1.0480e+000 and -2.9061e+001 (eigenvalues of **A**). The pair [**A**, **B**] is completely controllable. Assume that all the states are accessible for measurement. The goal is to design a simple linear state-feedback servo-positioning controller of the form

$$u = -\mathbf{K}\,\mathbf{x} + K_r\, r$$

for the above system. **K** is a 1×3 state feedback gain matrix, K_r is a scalar feedforward gain and *r* a scalar reference input. In this example, K_r can be set to the first element of **K**, i.e., K_r = **K**(1). Three control designs are compared below; the third technique being the new method proposed in this paper.

Pole Placement (PP) Design. Specifying the desired closed-loop system poles as, say,[$-10 \pm j10$, -100], we determine the gain **K** using a standard pole placement technique [See Matlab Control Toolbox].

Linear Quadratic (LQ) Design. If we choose to minimize a quadratic performance index $J = \int_0^\infty \left(\mathbf{x}'\,\mathbf{Q}\,\mathbf{x} + \mathbf{u}'\,\mathbf{R}\,\mathbf{u}\right) dt$ with **Q** = diag([10000 D_{eq} R_a]) and **R** = 1, we determine the optimal gain **K** using the standard LQR technique [Matlab].

Linear Quadratic Pole Placement (LQPP) Design. In the third case, we propose to specify two poles, say, $-10 \pm j10$, as the desired dominant closed-loop system poles (partial pole placement) and use the third excess pole to optimize the same quadratic performance index *J* (parametric optimization). Using the LQPP technique described in Section 3 and a Matlab program, we determine the gain **K.**

Results: Table 1 shows the eigenvalues of the closed-loop system matrix [**A-BK**], and the feedback gain **K** for each of the designs. It also compares the performance index $J = \int_0^\infty \left(\mathbf{x}'\,\mathbf{Q}\,\mathbf{x} + \mathbf{u}'\,\mathbf{R}\,\mathbf{u}\right) dt$ computed from the transient responses shown in Figure 1, for the three cases with the initial condition set to $\mathbf{x}(0) = [\,1 \quad 0 \quad 0\,]'$ and $r(t) = 0$ and $w(t) = 0$.

Comparison of Closed-Loop Poles: In the PP design, $-10 \pm j10$ are selected to be the dominant closed-loop poles and –100 is a selected non-dominant pole whose contribution quickly vanishes as the system response approaches steady state. In the LQ design, -6.1767±j6.1484 become the dominant poles and -371.27 is a non-dominant pole as consequence of the optimization. For the LQPP case, -10±10 are retained as the assigned dominant poles; the third pole, –27.264, is the optimizing pole which also turns out to be relatively non-dominant.

Comparison of Feedback Gains. The LQPP gains are generally smaller among the designs. This can be expected since the LQPP is an optimal version of PP and so employs smaller gains to avoid excessive overshoots and undershoots. Comparison of LQPP and PP gains to LQR gain is less obvious since the LQR does not take desired poles into account. It turns out that the LQR requires larger gains as it attempts to optimize the performance.

Comparison of Transient Responses: The response of LQPP is very similar to that of PP as expected because of the partial pole placement by LQPP guarantees that the desired dominant poles are the same as those of PP. The response of LQ is optimal but is different from that of LQPP and PP.

Comparison of Performance Index: The index J for LQ is the smallest as expected since it is the optimal result. The J for LQPP is the next smallest as it minimizes the index under partial pole placement constraint. And J for PP is the largest of the three as it does not consider any optimization

The example highlights the features and potential benefits of the simultaneous LQPP design. It gives a designer more authority to fashion the outcome of the control design.

3. FORMALIZATION OF LQPP TECHNIQUE

3.1 Statement of LQPP Problem

Consider a single-input linear dynamic system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad \mathbf{x}(0) = \mathbf{x}_0$$

where $\mathbf{x} \in \Re^n$ is the state vector and $u \in \Re^1$ is the scalar input. Assume that the pair [**A, B**] is controllable, and the initial condition has a zero mean, i.e., $E[\mathbf{x}_0] = 0$ and a covariance $\mathbf{X}_0 = E\left[\mathbf{x}_0\mathbf{x}_0'\right]$. The objective of the LQPP is to find a optimal linear state feedback control, $u = -\mathbf{K}\,\mathbf{x}$, such that it simultaneously places n_1 of the closed-loop system poles at desired location $\{\lambda_1, \;\cdots\;, \lambda_{n_1}\}$ and use the unspecified remaining n_2 poles $\{\lambda_{n1+1}, \cdots, \lambda_n\}$ to minimize the expected (mean) quadratic performance index

$$J = E\left\{\int_0^\infty \left(\mathbf{x}'\mathbf{Q}\mathbf{x} + u'\mathbf{R}u\right)dt\right\}.$$

That is

$$\lambda\left(\mathbf{A}\text{-}\mathbf{BK}\right) = \left\{\underbrace{\left(\lambda_1, \;\cdots\;, \lambda_{n1}\right)}_{\text{specified desired poles}}, \underbrace{\left(\lambda_{n1+1}, \;\cdots\;, \lambda_n\right)}_{\text{arguments to minimize } J}\right\}$$

3.2 Summary of LQPP Solution

The optimal LQPP gain is given by

$$\mathbf{K} = -\left[\mathbf{d} - \mathbf{a} + \mathbf{p}_1\,\mathbf{L}\right]\mathbf{T}^{-1}$$

where each of the terms are described below.

Partial Pole Placement: a is the system characteristic vector defined by

$$\mathbf{a} = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \in \Re^{1\times n}$$

$$\left|\lambda\mathbf{I}_n - \mathbf{A}\right| = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$$

$\mathbf{T}^{-1}$ is the top-companion controllable canonical form transformation matrix ($\mathbf{x} = \mathbf{T}\,\mathbf{x}_c$) computed from

$$\mathbf{T} = \mathbf{W}\,\mathbf{W}_c^{-1} \quad \text{or} \quad \mathbf{T}^{-1} = \mathbf{W}_c\;\mathbf{W}^{-1}$$

$$\mathbf{W} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

$$\mathbf{W}_c = \begin{bmatrix} \mathbf{B}_c & \mathbf{A}_c\mathbf{B}_c & \cdots & \mathbf{A}_c^{n-1}\mathbf{B}_c \end{bmatrix}$$

$$\mathbf{A}_c = \begin{bmatrix} & & -\mathbf{a} & & \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \quad \mathbf{B}_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \\ 0 \end{bmatrix} = \mathbf{T}^{-1}\mathbf{B}$$

d id the desired characteristics vector given by

$$\mathbf{d} = \begin{bmatrix} \mathbf{p}_0 & 0_{1\times n_2} \end{bmatrix} \in \Re^{1\times n}, \qquad n = n_1 + n_2$$

$$\mathbf{p}_0 = \begin{bmatrix} d_1 & \cdots & d_{n_1} \end{bmatrix} \in \Re^{1\times n_1}$$

$$\left(\lambda - \lambda_1\right)\cdots\left(\lambda - \lambda_{n_1}\right) = \lambda^{n_1} + d_1\lambda^{n_1\text{-}1} + \cdots + d_{n_1}$$

and the matrix **L** (a Toeplitz matrix) is

$$\mathbf{L} = \begin{bmatrix} 1 & \mathbf{p}_0 & 0 & \\ 0 & \ddots & \ddots & 0 \\ & 0 & 1 & \mathbf{p}_0 \end{bmatrix} \in \Re^{n_2\times n}$$

Parametric Optimization: $\mathbf{p}_1 \in \Re^{1\times n_2}$ is an optimal gain that satisfies the extremum condition given by the following coupled algebraic Lyapunov equations:

$$\mathbf{p}_1 = \mathbf{R}^{-1}\mathbf{B}_1'\,\mathbf{V}\,\mathbf{M}\,\mathbf{C}_1'\left(\mathbf{C}_1\mathbf{M}\,\mathbf{C}_1'\right)^{-1}$$

$$0 = (\mathbf{A}_1 - \mathbf{B}_1\mathbf{p}_1\mathbf{C}_1)'\,\mathbf{V} + \mathbf{V}(\mathbf{A}_1 - \mathbf{B}_1\mathbf{p}_1\mathbf{C}_1) + \mathbf{Q} + \mathbf{C}_1'\,\mathbf{p}_1'\mathbf{R}\,\mathbf{p}_1\mathbf{C}_1$$

$$0 = (\mathbf{A}_1 - \mathbf{B}_1\mathbf{p}_1\mathbf{C}_1)\mathbf{M} + \mathbf{M}(\mathbf{A}_1 - \mathbf{B}_1\mathbf{p}_1\mathbf{C}_1)' + \mathbf{X}_0$$

where $\mathbf{A}_1 = \mathbf{A} - \mathbf{B}\left[\mathbf{d} - \mathbf{a}\right]\mathbf{T}^{-1}$ and $\mathbf{C}_1 = \mathbf{L}\mathbf{T}^{-1}$.

3.3 Derivation of LQPP Technique

$\mathbf{p}_1$ represents the unspecified system characteristics:

$$\mathbf{p}_1 = \begin{bmatrix} e_1 & \cdots & e_{n_2} \end{bmatrix} \in \Re^{1\times n_2}$$

$$\left(\lambda - \lambda_{n_1+1}\right)\cdots\left(\lambda - \lambda_n\right) = \lambda^{n_2} + e_1\lambda^{n_2\text{-}1} + \cdots + e_{n_2}$$

$\mathbf{p}_1$ is not known at this time and will be used to minimize the quadratic performance index J. Transformation of the closed-loop system matrix **A-BK** to its canonical controllable form yields

$$\mathbf{A} - \mathbf{BK} \;\Leftrightarrow\; \mathbf{A}_c - \mathbf{B}_c\mathbf{K}_c = \begin{bmatrix} & -\mathbf{a} - \mathbf{K}_c & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

where $\mathbf{K}_c = \mathbf{KT} = \begin{bmatrix} k_{c1} & \cdots & k_{cn} \end{bmatrix}$. The characteristic polynomial of the canonical controllable form closed-loop system matrix can be equated to the combined characteristics of the specified and unspecified pole locations of the system as follows:

$$\left|\mathbf{A}_c - \mathbf{B}_c\mathbf{K}_c\right| = \lambda^n + (a_1 + k_{c1})\lambda^{n-1} + \cdots + (a_n + k_{cn})$$
$$= \left(\lambda^{n_1} + d_1\lambda^{n_1-1} + \cdots + d_{n_1}\right)\left(\lambda^{n_2} + e_1\lambda^{n_2-1} + \cdots + e_{n_2}\right)$$

Evaluating the above polynomial products in terms of vector algebra yields (Chen 1984)

$$\begin{bmatrix}1 & \mathbf{a}\end{bmatrix} + \begin{bmatrix}0 & \mathbf{K}_c\end{bmatrix} = \begin{bmatrix}1 & \mathbf{p}_0\end{bmatrix}\begin{bmatrix}1 & \mathbf{p}_0 & & \\ & \ddots & \ddots & \\ & & 1 & \mathbf{p}_0\end{bmatrix}_{(n_2+1)\times(n+1)}$$

which reduces to

$$\mathbf{a} + \mathbf{K}_c = \begin{bmatrix}\mathbf{p}_0 & \mathbf{0}_{1\times n_2}\end{bmatrix} + \mathbf{p}_1\begin{bmatrix}1 & \mathbf{p}_0 & & \\ & \ddots & \ddots & \\ & & 1 & \mathbf{p}_0\end{bmatrix}_{n_2\times n} = \mathbf{d} + \mathbf{p}_1\mathbf{L}$$

Therefore, $\mathbf{K}_c = \mathbf{d} - \mathbf{a} - \mathbf{p}_1\mathbf{L}$ and hence

$$\mathbf{K} = \mathbf{K}_c\mathbf{T}^{-1} = \left[\mathbf{d} - \mathbf{a} - \mathbf{p}_1\mathbf{L}\right]\mathbf{T}^{-1}$$

At this stage, $\mathbf{d}$, $\mathbf{a}$, $\mathbf{L}$ & $\mathbf{T}^{-1}$ have been defined. The next step is to determine $\mathbf{p}_1$. It turns out that $\mathbf{p}_1$ can be posed as a constant gain output feedback optimal control problem [Athens, Cheok]. Substituting $u = -\mathbf{Kx}$ into $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u$ results in

$$\dot{\mathbf{x}} = \underbrace{\left[\mathbf{A} - \mathbf{B}\left[\mathbf{d} - \mathbf{a}\right]\mathbf{T}^{-1}\right]}_{\mathbf{A}_1}\mathbf{x} - \left[\mathbf{B}\right]\mathbf{p}_1\underbrace{\left[\mathbf{L}\mathbf{T}^{-1}\right]}_{\mathbf{C}_1}\mathbf{x}$$

which can then be expressed as an output feedback control problem:

$$\dot{\mathbf{x}} = \mathbf{A}_1\,\mathbf{x} + \mathbf{B}\,u_1, \quad \mathbf{y}_1 = \mathbf{C}_1\,\mathbf{x}, \quad u_1 = -\,\mathbf{p}_1\,\mathbf{y}_1$$

Since the closed-loop system trajectory for the system is given by $\mathbf{x}(t) = e^{(\mathbf{A}_1 - \mathbf{B}\mathbf{p}_1\mathbf{C}_1)t}\mathbf{x}_0$, the integral quadratic performance index can be reduced to an algebraic index

$$J = E\left\{\int_0^\infty \left(\mathbf{x'Qx} + \mathbf{u'Ru}\right)dt\right\} = trace\left[\mathbf{VX}_0\right]$$

$$\mathbf{V} = \int_0^\infty e^{(\mathbf{A}_1 - \mathbf{B}\mathbf{p}_1\mathbf{C}_1)'t}\left(\mathbf{Q} + \mathbf{C}_1'\mathbf{p}_1'\mathbf{R}\mathbf{p}_1\mathbf{C}_1\right)e^{(\mathbf{A}_1 - \mathbf{B}\mathbf{p}_1\mathbf{C}_1)t}dt$$

V is the solution of the algebraic Lyapunov equation

$$0 = \left(\mathbf{A}_1 - \mathbf{B}\mathbf{p}_1\mathbf{C}_1\right)'\mathbf{V} + \mathbf{V}\left(\mathbf{A}_1 - \mathbf{B}\mathbf{p}_1\mathbf{C}_1\right) + \mathbf{Q} + \mathbf{C}_1'\mathbf{p}_1'\mathbf{R}\mathbf{p}_1\mathbf{C}_1$$

which is the constraint that the performance index J is subject to. To alleviate the constraint, we introduce a Lagrange multiplier matrix $\mathbf{M}$ with a trace operation to define an equivalent unconstrained algebraic performance index: $J = tr\left[\mathbf{VX}_0\right] +$

$$tr\left[\mathbf{M}\left[\left(\mathbf{A}_1 - \mathbf{B}\,\mathbf{p}_1\mathbf{C}_1\right)'\mathbf{V} + \mathbf{V}\left(\mathbf{A}_1 - \mathbf{B}\,\mathbf{p}_1\mathbf{C}_1\right) + \mathbf{Q} + \mathbf{p}_1'\mathbf{R}\mathbf{p}_1\right]\right]$$

Proceeding to minimize J with respect to $\mathbf{p}_1$, $\mathbf{M}$ and $\mathbf{V}$ yields the extremum condition. In other words, the optimal gain $\mathbf{p}_1$ must satisfy the coupled algebraic Lyapunov type equations.

3.4 Computation of optimal gain $\mathbf{p}_1$

Because of the complexity, it is not possible to find a closed form solution to the coupled algebraic Lyapunov equations.. Various iterative numerical techniques (Levine, *et al.*, 1970; Cheok, *et al.*, 1988; Cheok, *et al.*, 1985; Cheok *et al.*, 1986a; Cheok, *et al.*, 1986b) have been successfully employed to find a convergent solution for $\mathbf{p}_1$.

4. CONCLUSION

The simultaneous LQPP design yields a system performance that is a compromise between the strictly PP and LQ designs. The example shows how it potential benefits from PP and LQ and yet employs generally smaller feedback gains. The LQPP design procedure is straightforward although it requires solving a set of nonlinear coupled algebraic Lyapunov equations. The idea of simultaneously partial pole assignment for dominant response behavior and optimizing a performance index makes sense and good design. Extension of the results to multi-input systems is currently being investigated.

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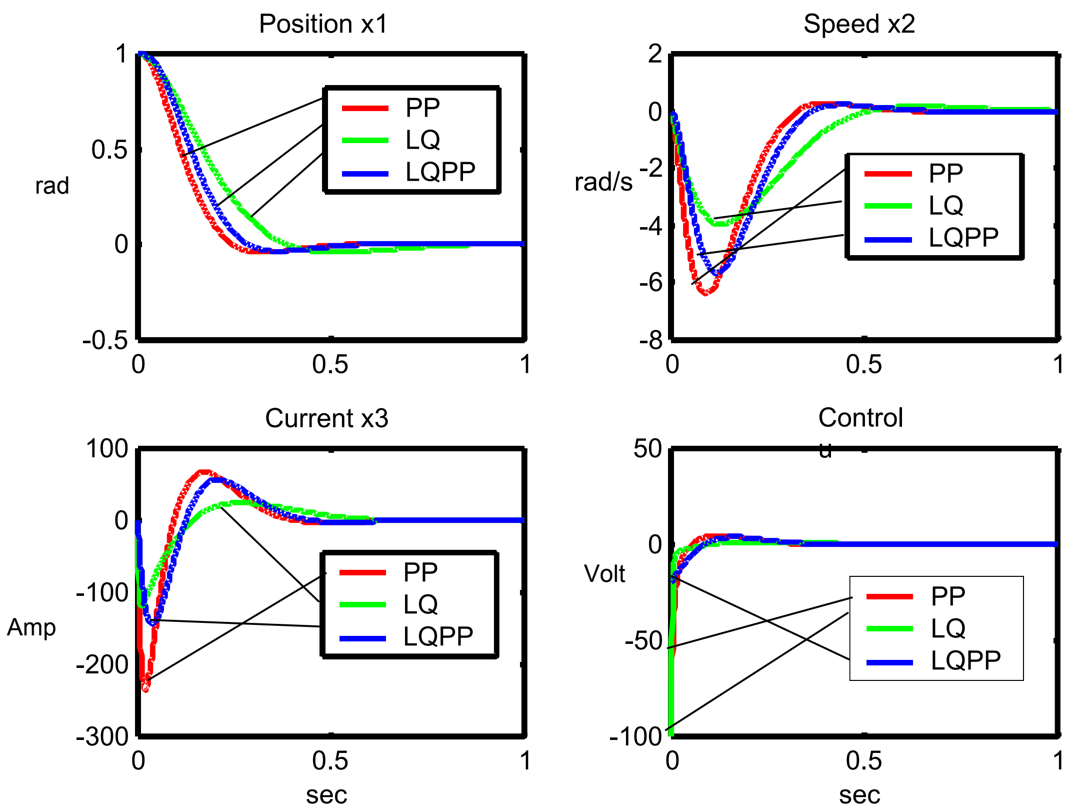

Figure 1. Transient responses for the PP, LQ and LQPP design

Table 1. Results of PP, LQ and LQPP designs

	Closed-loop poles	**Feedback gain K**	**Performance index *J***
PP	-10.000±j10.000, -100.00	[70.922 7.4292 0.17988]	2.3705e+006
LQ	-6.1767±j6.1484, -371.27	[100.000 15.3962 0.70704]	1.7064e+006
LQPP	-10.000±j10.000, -27.264	[19.336 2.4844 3.43108]	1.9564e+006