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FRAMEWORK FOR STUDYING LIMITATIONS OF ACHIEVABLE PERFORMANCE IN CONTROL OF NONLINEAR COMBUSTION PROCESSES

Prashant G. Mehta * **Andrzej Banaszuk** **
Clas A. Jacobson ***

* *Center for Applied Mathematics, 657 Rhodes, Cornell University, Ithaca NY 14853, 607 255 8272,* mehta@cam.cornell.edu
** *United Technologies Research Center, East Hartford, CT 06108, USA 860 610 7381,* banasza@utrc.utc.com, http://abanaszukresearch.homestead.com/homepage_research
*** *United Technologies Research Center, East Hartford, CT 06108, USA, 860 610 7652,* jacobsca@utrc.utc.com

Abstract:
In this paper we propose a framework for studying fundamental limitations in the control of thermoacoustic instabilities described by nonlinear models. More generally, the framework applies to general interconnections of lightly damped linear oscillators and nonlinear static nonlinearities of saturation type driven by broad-band Gaussian disturbances. The key concept is to replace the static nonlinearities with their corresponding random input describing functions. This allows the application of classical fundamental limitations theory to the combustion models. In this paper we interpret the limitations results in the particular case of a model of a combustion process controlled by on-off actuators.We also formulate conservation principles to understand the dynamics associated with this combustion model.

Keywords: Fundamental limits, nonlinear control, describing functions, combustion

1. INTRODUCTION

In recent years, spurred by the need for cleaner combustion for gas turbine engines generating power, there has been a surge of activity in the area of active control of combustion. The need for active control arises due to the occurence of so-called combustion instabilities which arise due to the destabilizing feedback coupling between acoustics and heat release. These instabilities lead to large pressure oscillations in the combustor which cause increased environmental noise and decreased combustor durability (Seume *et al.*, 1997).

The combustion is typically modeled as a so-called thermoacoustic loop with a linear acoustic system, a fuel transport delay and a nonlinear static feedback (so-called heat release nonlinearity). In most combustion literature, the combustion is modeled as a limit-cycling system. More recently, there have been attempts to model turbulent combustion as thermoacoustic loop driven by a noise model of turbulence (Lieuwen and Zinn, 2000) (Mezic and Banaszuk, 2000). In both limit cycling and noise driven cases, there is significant interest in describing the qualitative and quantitative thermoacoustic dynamics that can be expected for the reasonable choices of heat release nonlinearities.

Experiments show that active control using fuel modulation is an effective way of reducing the level of pressure oscillations in combustors (Seume *et al.*, 1997) (Bloxsidge *et al.*, 1987) (Fleifil *et al.*, 1997) However, it has been observed that the achieved reduction of pressure oscillation varied between experiments from 6dB to 20dB. Moreover, in some cases the attenuation of oscillations at the primary frequency is accompanied by excitation of oscillations in some other frequencies (Bloxsidge *et al.*, 1987) (Fleifil *et al.*, 1997) This phenomenon is refered to as *secondary peaks* or *peak splitting*. An explanation of different attenuation levels and peak-splitting phenomena has been presented in (Banaszuk *et al.*, 1999*a*) in linear actuator case and in (Banaszuk *et al.*, 1999*b*) in nonlinear actuator case. The nonlinear case is handled using random-input describing functions. Results of analysis using random-input describing functions are in excellent agreement with results of model simulations and experiments.

In this paper we formulate the results for the specific case of a lightly damped oscillator with relay nonlinearity in a feedback loop driven by broad-band Gaussian noise. However, we expect that the results can be extended to more general interconnections of lightly damped linear oscillators and nonlinear static nonlinearities of saturation type driven by broad-band Gaussian disturbances. The key concept is to replace the static nonlinear functions with the corresponding random input describing functions, which allows application of classical fundamental limitations theory. This idea is straighforward and obviously appealing. However, there are obvious concerns about the validity of approximation, the well-posedness of the approach and the interpretation of results. The results of the case studies presented in (Banaszuk *et al.*, 1999*b*) and in the present paper are encouraging so far.

The paper is organized as follows. In section 2, we describe the structure of the thermoacoustic loop and review the describing function framework that allows us to analyze the general case of noise driven limit cycling thermoacoustic system. We also propose a framework for studying the fundamental limits in the control of combustion systems described by nonlinear models. In section 3, we describe the effect of noise on the presence and amplitude of the limit cycle for such systems. In particular, we demonstrate the stabilization that results due to the presence of noise. Finally, we present the conclusions in section 4.

2. FRAMEWORK FOR STUDYING LIMITATIONS OF ACHIEVABLE CONTROL PERFORMANCE USING NONLINEAR COMBUSTION MODELS

A simple model ((Peracchio and Proscia, 1998) (Banaszuk *et al.*, 1999*a*)) of a premixed combustion process is an interconnection of a linear lightly damped oscillator, delay, and a saturating nonlinearity in a feedback loop. Figure 1 gives the block diagram schematic of the combustion model. The oscillator represents the acoustic waves in the combustion chamber. The delay represents the time it takes the fuel and air to mix in a premixing nozzle, transport to the flame front, and react. The saturating nonlinearity represents the nonlinear dependence of a heat release rate on the fuel to air ratio at the flame front. The feedback loop models the effect of oscillations of air mass flow in the nozzle on the fuel to air ratio and hence on the heat release rate. Only the output of the oscillator (pressure) is accessible for measurement. Depending upon the experimental conditions, a strong broad band disturbance (representing turbulent velocity fluctuations) may be driving the system at the input of the linear oscillator. The control input (representing fuel modulation) adds to the disturbance input.

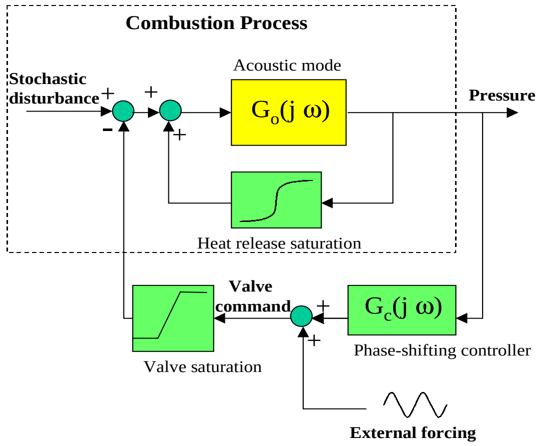

Fig. 1. Nonlinear model's block diagram

We use the random input describing functions (Gelb and Velde, 1968) to analyze the thermoacoustic system described above. We assume that the random turbulent velocity fluctuations driving the thermoacoustic loop may be modeled as a Gaussian process. Assume that the signal at the input of a static nonlinear function $h(\cdot)$ is of the form

$$u(t) = A\sin(\omega_0 t + \theta) + p(t), \qquad (1)$$

where $p(t)$ is a zero-mean Gaussian random variable with standard deviation σ and $Asin(\omega_0 t + \theta)$ is a sinusoidal signal with a random phase θ uniformly distributed over $[0, 2\pi]$. In a random input describing function analysis, the output of the nonlinear function $y(t) = h(u(t))$ is approximated by

$$y_a(t) = N_A(A,\sigma)A\sin(\omega_0 t + \theta) + N_R(A,\sigma)p(t). \qquad (2)$$

The describing functions are chosen to minimize the variance of the approximation error and are given by (Gelb and Velde, 1968)

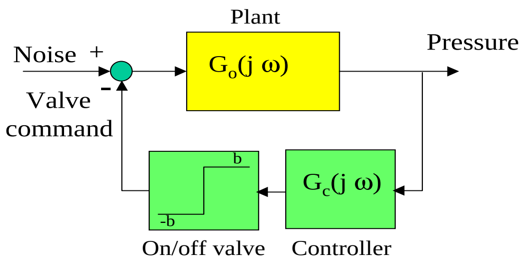

Fig. 2. Model of the controlled combustion process with on-off valves.

$$N_R(A,\sigma) = \frac{1}{\sigma^2}E[h(u(0))p(0)] = \frac{1}{(2\pi)^{\frac{3}{2}}\sigma^3}\int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dr h(A\sin(\theta)+r)r\exp(-\frac{r^2}{2\sigma^2}), \quad (3)$$

$$N_A(A,\sigma) = \frac{2}{A}E[h(u(0))\sin(\theta)] = \frac{2}{(2\pi)^{\frac{3}{2}}\sigma A} \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dr h(A\sin(\theta)+r)\sin(\theta)\exp(-\frac{r^2}{2\sigma^2}). \quad (4)$$

In the presence of Gaussian noise $r_i(t)$ (with power spectral density $\Phi_{ii}(j\omega)$) modeling the disturbance input (see Figure 1), the forced response of the thermoacoustic feedback loop admits a quasilinear description in terms of the describing function gains. The corresponding equations for self-excited oscillations (due to the presence of limit cycle) together with driving noise are

$$1 + N_A(A,\sigma)G_0(i\omega_0) = 0 \quad (5)$$

$$\Phi_{pp}(i\omega) = |\frac{G_0(i\omega)}{1+N_R(A,\sigma)G_0(i\omega)}|^2\Phi_{ii}(i\omega) \quad (6)$$

$$\sigma^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{pp}(i\omega)d\omega, \quad (7)$$

where σ^2 is the variance and Φ_{pp} the power spectral density of the pressure output $p(t)$ (see Figure 1).

We discuss the extensions of the fundamental limitations results in the simplest case of a linear stable combustion model (this represents operation at a certain high fuel/air ratio condition). Here, the effect of nonlinearity in heat release function is neglected and its linearization is incorporated in the transfer function $G_0(j\omega)$. The resulting model is represented by block diagram shown at Figure 2. We focus on the effect of nonlinear actuator characteristic on the achievable control performance. Specifically we focus on on-off actuators. Using random input describing functions, the Fourier transform of the Gaussian component of combustor pressure can be represented by the formula

$$p_g(j\omega) = G_0(j\omega)S(j\omega, A, \sigma)n_i(j\omega), \quad (8)$$

where $n_i(j\omega)$ is the Fourier transform of the driving disturbance and

$$S(j\omega, A, \sigma) = \frac{1}{1+G_0(j\omega)N_R(A,\sigma)G_c(j\omega)} \quad (9)$$

is a *modified sensitivity function* that depends on the magnitude of the limit cycle A and standard deviation σ of the Gaussian component at the input of the relay nonlinear element representing the actuator characteristic. So far, this is a merely formal expression, as we have not shown that the modified sensitivity function represents a stable system and that there are values of σ and A for which the limit cycle and Gaussian process balance is achieved in the control loop. Results of analysis presented in following sections will show that there are values of σ and A for which the limit cycle and Gaussian process balance is achieved and that the modified sensitivity function is indeed stable for all values of input noise level. Therefore, we can extend fundamental limitation results to the case of linear stable combustion process controlled with on-off actuators. A case study for this particular case has been carried out in (Banaszuk *et al.*, 1999*b*).

The above methodology for extending fundamental limitations is applicable to the more general case of control of nonlinear thermoacoustic loop with static nonlinearities both in the heat release path as well as the control path (due to saturating actuators). This provides a conceptual framework to interpret the fundamental limitations in the control of nonlinear combustion processes and thus explain effects such as peak splitting observed in the control of combustion. The key concept is to replace the static nonlinearity with the corresponding describing function gain. The formal method described above applies if one can show that the sensitivity function remains well-posed for the Gaussian driver. This is done for the special case of Figure 2 in the following section.

3. CONSERVATION RELATIONS IN DYNAMICS OF COMBUSTION

In this section, we discuss the well-posedness of the sensitivity function for the case of linear stable combustion model ($G_0(i\omega)$) driven by Gaussian noise and controlled with an on-off actuator modeled as a relay nonlinearity

$$\begin{aligned} h(u) &= b, \qquad u > 0 \\ &= -b, \qquad u < 0. \end{aligned} \quad (10)$$

For relay nonlinearity, the amplitude of the limit cycle can be found from solving the loop equation (5)

$$1 + N_A(A,\sigma)G_0(j\omega) = 0. \quad (11)$$

The following theorem summarizes the results for the relay nonlinearity in the limiting cases of $\sigma \to 0$ and $\sigma \to \infty$.

Theorem 3.1. For the relay nonlinearity,

$$N_A^0 \equiv \lim_{\sigma \to 0} N_A(A, \sigma) = \frac{4b}{\pi A} \tag{12}$$

$$N_R^0 \equiv \lim_{\sigma \to 0} N_R(A, \sigma) = \frac{4b}{2\pi A} = \frac{N_A^0}{2} \tag{13}$$

and there is an amplitude-independent bound

$$|N_A| \leq 2\sqrt{\frac{2}{\pi}} \frac{b}{\sigma} \tag{14}$$

for the describing function gain corresponding to the limit cycle.

We provide the proof in the Appendix. The significance of the above results is that for a relay nonlinearity

(1) in the limit $\sigma \to 0$, the limit cycling system admits a gain margin of 2 (with respect to the Gaussian noise) at the frequency of limit cycle,
(2) there exists σ_H such that $\forall \sigma \geq \sigma_H$, the loop can not support a limit cycle and
(3) in the limit $\sigma \to \infty$, the loop behaves as the open loop system ($G_0(i\omega)$).

To see (1) above, note that at the frequency ω_0 of the limit cycle,

$$N_A^0 G_0(i\omega_0) = -1, \tag{15}$$

so using (13), one obtains

$$N_R^0 G_0(i\omega_0) = -\frac{1}{2}. \tag{16}$$

To see (2) above, note that the limit cycle equation (11) ceases to have a solution for a choice of

$$\sigma \geq \sigma_H \equiv 2\sqrt{\frac{2}{\pi}} b \sup_{\omega} |G_0(i\omega)|. \tag{17}$$

We thus have a conservation principle (in the limits $\sigma \to 0, \sigma \to \infty$) for the thermoacoustic loop with the choice of relay heat release nonlinearity :

(1) For $\sigma \to 0$, the appearence of the limit cycle stabilizes the loop with respect to the noise thereby yielding a bounded input-output response (for the Gaussian noise driver) as solution of equation (6) and
(2) for $\sigma \to \infty$, large noise stabilizes the loop with respect to the limit cycle thereby causing the limit cycle to disappear and system to behave as a stable noise driven system.

Before we present numerical results for the intermediate values of σ for the relay heat release nonlinearity, we state a conservation equation for the relay nonlinear function $h(.)$:

Theorem 3.2. For the relay nonlinear function with inputs $A, \sigma \neq 0$

$$\frac{d(AN_A)}{d\sigma} + \frac{2\sigma}{A} \frac{d(\sigma N_R)}{d\sigma} = 0. \tag{18}$$

The proof is again deferred to the Appendix. Here, we discuss the implications of the above result. Using

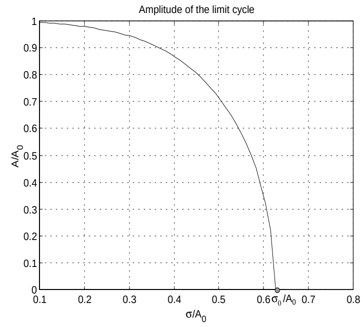

Fig. 3. The amplitude of the limit cycle in the presence of the Gaussian noise

(15), a limit cycle with harmonic component at frequency ω_0 leads to the sinusoidal describing function gain for the heat release nonlinearity in the thermoacoustic feedback loop given by

$$N_A(\sigma) = N_A^0 = -\frac{1}{G_0(i\omega_0)}. \tag{19}$$

We thus obtain for the thermoacoustic loop

$$N_A^0 \frac{dA^2}{d\sigma} = -4\sigma \frac{d(\sigma N_R)}{d\sigma}, \tag{20}$$

i.e., any increase in the quantity σN_R (variance of the signal at the nonlinearity output driving the linear system $G_0(i\omega)$ in the thermoacoustic feedback loop) is balanced by decrease in A^2, the amplitude of the limit cycle.

We next provide complete results for the case of relay heat release nonlinearity indicating the tradeoff between limit cycle amplitude and noise variance. In the absence of noise, the closed loop system exhibits a limit cycle with amplitude A_0, which results from the solution of the equation (15). The presence of Gaussian noise driving the system has the effect of supressing the self-excited oscillations. In particular, with a relay nonlinearity, the amplitude of the self-excited oscillations is easily shown to satisfy the equation

$$A(\sigma) = \int_{-A/\sigma}^{+A/\sigma} \sqrt{1 - \frac{\sigma^2 s^2}{A^2}} e^{-s^2/2} ds \frac{A_0}{\sqrt{2\pi}}. \tag{21}$$

It then follows that $A(\sigma) \leq A_0$. Figure 3 plots the numerically computed solution of the integral equation (21). Thus, the presence of noise ($\sigma > 0$) leads to a reduction in the amplitude of this limit cycle and at a critical positive value of $\sigma = \sigma_0$, the limit cycle disappears ($A(\sigma_0) = 0$) and the random input describing function gain

$$N_R = \sqrt{\frac{2}{\pi}} \frac{b}{\sigma} \quad \forall \sigma > \sigma_0. \tag{22}$$

For the values of $\sigma < \sigma_0$, the gain $N_R(\sigma)$ was numerically computed using the relationships (3) and (4). Figure 4 plots the gains $N_R(\sigma)$ and $N_A(\sigma)$. For the values of σ where limit cycle is present,

$$N_A = N_A^0 \quad \forall \sigma < \sigma_0, \tag{23}$$

with a transition at σ_0, the critical σ where the limit cycle ceases to exist. N_R monotonically increases

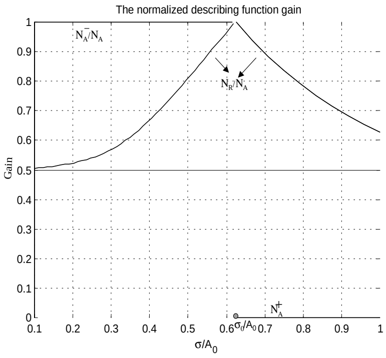

Fig. 4. Describing function gains in the presence of Gaussian noise

between 0 and σ_0 and decreases for values of $\sigma > \sigma_0$. We have

$$\begin{aligned} N_R(A(0),0) &= \frac{N_A^0}{2}, \\ N_R(A(\sigma_0),\sigma_0) &= N_A^0, \\ N_R(A(\sigma),\sigma) &< N_A^0 \qquad \forall \sigma \neq \sigma_0. \end{aligned} \tag{24}$$

The last inequality ensures that the feedback interconnection of $G_0(j\omega)$ and $N_R(A(\sigma),\sigma)$ is linearly stable for all $\sigma \neq \sigma_0$. The second equality in (24) implies that the largest loop gain occurs at the critical value σ_0 where the loop is arbitraily close to destabilization (eigenvalues on the imaginary axis). For values of σ away from σ_0, the eigenvalues move in to the LHP thereby ensuring asymptotic stability for all $\sigma \neq \sigma_0$.

We have thus shown that the modified sensitivity function (9) is a stable for the Gaussian noise driver and thus the extensions of the fundamental limitation result discussed in section 2 are valid.

4. CONCLUSION

In this paper, we have shown the utility of conservation relations in understanding both the dynamics and control of combustion instabilities. The dynamics are understood as a trade-off between the variance of the pressure output of the thermoacoustic loop and the amplitude of the limit cycle present. The random input describing function framework allows one to extend the linear fundamental limitations to the control of combustion. In particular, this allows one to understand the so-called *peak-splitting* phenomenon for the combustion with static nonlinearities.

5. APPENDIX

Proof of Theorem 3.1 The proof of equations (12) and (13) are obtained by direct computation using equations (3) and (4). To prove the amplitude independent bound (14), we express the equation (4) as

$$\begin{aligned} N_A = {} & \frac{2}{(2\pi)^{3/2}A}\left[\int_0^{2\pi}\int_{|s|>\frac{A}{\sigma}} + \int_0^{2\pi}\int_{-\frac{A}{\sigma}}^{+\frac{A}{\sigma}}\right] \\ & h(Asin(\theta)+\sigma s)e^{-s^2/2}sin(\theta) \\ = {} & \frac{2}{(2\pi)^{3/2}A}\int_0^{2\pi}\int_{-\frac{A}{\sigma}}^{\frac{A}{\sigma}} h\, e^{-s^2/2}sin(\theta), \end{aligned} \tag{25}$$

where the first part is zero because

$$h(Asin(\theta)+\sigma s) = \pm b \tag{26}$$

for a relay independent of θ and $\int_0^{2\pi} sin(\theta) = 0$. The equation (14) then follows by

$$\begin{aligned} |N_A| &\le \frac{2}{(2\pi)^{3/2}A}\int_0^{2\pi}\int_{-\frac{A}{\sigma}}^{\frac{A}{\sigma}} |h\, e^{-s^2/2}sin(\theta)| \\ &\le \frac{2}{(2\pi)^{3/2}A}2\pi\frac{2A}{\sigma}b = 2\sqrt{\frac{2}{\pi}}\frac{b}{\sigma}. \end{aligned} \tag{27}$$

■

Proof of Theorem 3.2 If $\sigma \neq 0$, on multiplying both sides of the equation (3) by σ and differentiating with respect to σ, one obtains

$$\begin{aligned} \frac{d(\sigma N_R)}{d\sigma} = {} & \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_0^{2\pi} h'(Asin\theta+\sigma s) \\ & \left[\frac{dA}{d\sigma}sin\theta + s\right] se^{-s^2/2} \end{aligned} \tag{28}$$

and for a relay nonlinearity, the right hand side gives the boundary integral

$$\begin{aligned} \frac{d(\sigma N_R)}{d\sigma} &= \frac{2b}{(2\pi)^{3/2}}\int_{Asin\theta+\sigma s=0}\left[\frac{dA}{d\sigma}sin\theta + s\right] se^{-s^2/2} \\ &= \left[1-\frac{\sigma}{A}\frac{dA}{d\sigma}\right]\frac{2b}{(2\pi)^{3/2}}\int_{-\frac{A}{\sigma}}^{\frac{A}{\sigma}} s^2e^{-s^2/2}. \end{aligned} \tag{29}$$

The proof is then completed by differentiating (4) and computations similar to ones provided above yield the equation

$$\frac{d(AN_A)}{d\sigma} = -\frac{2\sigma}{A}\left[1-\frac{\sigma}{A}\frac{dA}{d\sigma}\right]\frac{2b}{(2\pi)^{3/2}}\int_{-\frac{A}{\sigma}}^{\frac{A}{\sigma}} s^2e^{-s^2/2}. \tag{30}$$

Using equations (29) and (30), we obtain the equation (18)

$$\frac{d(AN_A)}{d\sigma} + \frac{2\sigma}{A}\frac{d(\sigma N_R)}{d\sigma} = 0. \tag{31}$$

thus proving the result. ■

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