15th Triennial World Congress, Barcelona, Spain

GLOBAL ASYMPTOTIC STABILITY OF OSCILLATIONS WITH SLIDING MODES

Jorge M. Gonçalves *

* *Caltech, Pasadena, CA 91125, USA*

Abstract: This paper explores a new methodology based on quadratic surface Lyapunov functions to globally analyze oscillations with sliding modes in relay feedback systems (RFS). The method consists in efficiently construct quadratic Lyapunov functions on switching surfaces that can be used to show that impact maps, i.e., maps from one switch to the next, are contracting. This, in turn, shows that the system is globally stable. Several classes of piecewise linear systems (PLS) were previously successfully analyzed with this methodology. In this paper, we consider PLS whose trajectories switch between subsystems of different dimensions. We present and discuss distinct relaxations leading to sufficient conditions of different conservatism and computationally complexity. The results in this paper open the door to the analysis of other, more complex classes of PLS.

Keywords: Relay, Global Stability, Quadratic Surface Lyapunov Functions, Impact Maps

1. INTRODUCTION

Many classes of PLS exhibit oscillations. In fact, the study and understanding of oscillations are of great interest in many applications. Hopping robots (Ringrose, 1997) are examples of such applications. Another application of oscillations in PLS are neural oscillators (see (Williamson, 1999) and references therein). These oscillators are models of two biological neurons in mutual inhibition which form a resonant circuit. Such oscillators can be found in numerous applications in robots when a period motion is desired. Another class of systems that exhibits limit cycle oscillations are relay feedback systems (RFS). A vast collection of applications of relay feedback can be found in the first chapter of (Tsypkin, 1984). More recent examples include the delta-sigma modulator (Ardalan and Paulos, 1987) and the automatic tuning of PID regulators (Åström, 1995). For all of these applications, it is important to show a certain design control strategy is globally stable in its domain of operation.

Few rigorous results exist for analyzing oscillations in PLS. Existence and local stability conditions for limit cycles of PLS can be found in (Gonçalves, 2000). In terms of global stability, most available results are for second-order systems using phase-plane analysis. For RFS, the problem of global stability of limit cycles has been studied for many years. (Atherton, 1975), (Tsypkin, 1984), and (Neimark, 1972) survey a number of analysis methods. In (Gonçalves *et al.*, 2001) a new methodology was introduced that efficiently globally analyzed symmetric unimodal limit cycles of RFS.

It is with the purpose of better understand oscillations and especially sliding modes in PLS that here we analyze RFS, for this is one of the simplest classes of PLS. Sliding modes in RFS has been studied by several researchers (Johansson *et al.*, 1999; di Bernardo *et al.*, 2000; Utkin, 1995). These references study and characterize different types of limit cycles with sliding modes. (Johansson *et al.*, 1999) gives necessary and sufficient conditions for the existence of fast switches and establishes a relation between such fast switches and sliding modes, showing that the difference between the number of poles and the number of zeros of the plant results in different types of sliding modes. Existence and local stability of certain classes of limit cycles with sliding modes are also studied and discussed in (Johansson *et al.*, 1999).

In (Gonçalves *et al.*, 2001), we gave efficient sufficient conditions for global asymptotic stability of symmetric unimodal limit cycles. The method was based in finding a quadratic surface Lyapunov function for the associated impact map of a RFS. The search for such function was efficiently done by solving a set of LMIs. Examples analyzed included minimum-phase systems, systems of relative degree larger than one, and of high dimension, for which no other analysis methodology could be applied. In this paper, we explore how similar ideas can be applied to globally asymptotically analyze limit cycles with sliding modes in RFS. The results here are applied to symmetric limit cycles with four switches per cycle with a sliding mode. Stability conditions for other types of limit cycles can be obtained in a similar way.

Note that when a trajectory evolves in the switching surface, such trajectory is induced by an LTI system of dimension lower than the original dimension of the system. This means that global analysis of RFS involves analyzing linear subsystems of different dimensions. Thus, this research opens the door to analysis of other, more complex classes of PLS.

2. PRELIMINARIES

2.1 Definitions

Consider a SISO LTI system satisfying the following linear dynamic equations

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$, in feedback with a relay (see figure 1)

$$u(t) \in \begin{cases} \{-1\} & \text{if } y(t) > 0 \\ [-1,1] & \text{if } y(t) = 0 \\ \{1\} & \text{if } y(t) < 0 \end{cases} \quad (2)$$

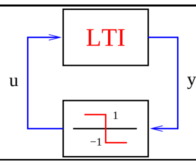

Fig. 1. Relay Feedback System

In the state space, the *switching surface* S of the RFS is the surface of dimension $n-1$ where $y = 0$. More precisely,

$$S = \{x \in \mathbb{R}^n : \ Cx = 0\}$$

Consider the following subset of S.

$$S_+ = \{x \in S : \ CAx > CB\}$$

This set is important since it characterizes those points in S that result in trajectories of the RFS that move to the region where $u = 1$ (see figure 2). Similarly, define $S_- = -S_+$.

Depending on the open loop system, a RFS may or may not have sliding modes. As explained in (Johansson *et al.*, 1999), if $CA^kB < 0$, where $k \in \{0, 1, ..., n-1\}$ is the smallest number such that $CA^kB \neq 0$, then there are no sliding modes. This case was considered in (Gonçalves *et al.*, 2001) and its results are reviewed in the next section. The main topic of this paper, that we will consider later on, is the case of sliding modes. Also, if A is invertible, assume $CA^{-1}B < 0$ or otherwise the system cannot have global limit cycles (see (Gonçalves, 2000) for details).

An interesting property of RFS is their symmetry around the origin. This property tells us that, in terms of stability analysis, symmetric limit cycles only need to be studied on half of their periods.

2.2 Symmetric Unimodal Limit Cycles

Only for this section, assume $CA^kB < 0$, where $k \in \{0, 1, ..., n-1\}$ is the smallest number such that $CA^kB \neq 0$. The simplest class of limit cycles of RFS are *symmetric unimodal limit cycles*. A limit cycle is *symmetric* if for every $x \in \gamma$ it is also true that $-x \in \gamma$ and it is *unimodal* if it only switches twice per cycle. Let $\xi(t)$ be a nontrivial periodic solution of (1)-(2) with period $2t^*$, and let γ be the limit cycle defined by the image set of $\xi(t)$. Let $x^* \in S$ be the intersection point of γ with S. Conditions for existence and local stability of symmetric unimodal limit cycles can be found in (Åström, 1995).

Define the impact map T as the map from S_+ to itself. A notion that will be useful is the notion of expected switching times. Let $x(0) = x^* + \Delta \in S_+$. Define t_Δ as the set of all times $t_i \geq 0$ such that $y(t_i) = 0$ and $y(t) \geq 0$ on $[0, t_i]$. Define also the set of *expected switching times* as

$$\mathcal{T} = \{t | \ t \in t_\Delta, \ \Delta \in S_+ - x^*\} \quad (3)$$

The next proposition says that most impact maps induced by an LTI flow between two hyperplanes can be represented as linear transformations analytically parameterized by a scalar function of the state. For simplification, we assume A is invertible. The more general case can be found in (Gonçalves, 2000).

Proposition 2.1. Let $x^*(t) = e^{At}\left(x^* - A^{-1}B\right) + A^{-1}B$. Assume $|Cx^*(t)| \geq K\|x^*(t) + x^*\|$, for some $K > 0$ and all $t \in \mathcal{T}$. Define

$$H(t) = \left(I - \frac{(x^*(t) + x^*)C}{Cx^*(t)}\right)e^{At}$$

for all $t \in \mathcal{T}$ (for $t = t^*$, H(t) is defined by the limit as $t \to t^*$). Then, for any $\Delta \in S_+ - x^*$ and $\Delta_1 \in T(\Delta)$ there exists a $t \in \mathcal{T}$ such that

$$\Delta_1 = H(t)\Delta \quad (4)$$

Such $t \in t_\Delta$ is the switching time associated with Δ_1.

The impact map is a map from S_+ to S_+ and, for each point in S_+, there is at least one associated switching time. An interesting property of this map is that the set of points in S_+ with the same switching time t forms a convex subset of a linear manifold of dimension $n-2$. Let S_t be that set, i.e., let S_t be the set of points $x^* + \Delta \in S_+$ that have t as a switching time, i.e., $t \in t_\Delta$. In other words, a trajectory starting at $x_0 \in S_t$ satisfies both $y(t) \geq 0$ on $[0, t]$, and $y(t) = 0$. It is now possible to check quadratic stability of impact maps by solving a set of LMIs.

Proposition 2.2. Consider the RFS (1)-(2). Assume there exists a symmetric unimodal limit cycle γ with period $2t^*$. If

$$P - H'(t)PH(t) > 0 \qquad \text{on } S_t - x^* \qquad (5)$$

for some $P > 0$ and for all expected switching times $t \in \mathcal{T}$, where "$D > 0$" stands for $x'Dx > 0$ for all nonzero $x \in X$, then the limit cycle is globally asymptotically stable.

With this result, a large number of examples with a unique locally stable symmetric unimodal limit cycle was successfully globally analyzed (Gonçalves *et al.*, 2001). In fact, it is still an open problem whether there exists an example with a globally stable symmetric unimodal limit cycle that could not be successfully analyzed with this methodology. Examples analyzed include minimum-phase systems, systems of relative degree larger than one, and of high dimension.

3. SLIDING MODES IN RFS

3.1 Preliminaries

In this section, we assume that $CB > 0$. This means a RFS will have sliding modes. The system dynamics are given by $\dot{x} = Ax - B$ when $Cx > 0$ and by $\dot{x} = Ax + B$ when $Cx < 0$. When the trajectory is in the switching surface, i.e. $Cx = 0$, several cases may occur. If $CAx > CB$, $u = -1$, if $CAx < -CB$, $u = 1$, and if $-CB < CAx < CB$ a trajectory cannot move to the region where $u = -1$ or $u = 1$ since the vector field on both sides of the switching surface points toward the switching surface (see figure 2). In these situations, the trajectory evolves along the switching surface with a control law $|u| \leq 1$ satisfying $Cx(t) = 0$. Taking the derivative, yields $C\dot{x} = 0$ and, after solving for u, we get $u = -CAx/(CB)$. Thus, the system dynamics in this region are given by $\dot{x} = PAx$ where $P = I - BC/(CB)$. These system dynamics are valid as long as $-CB \leq CAx(t) \leq CB$, or equivalently, that $|u(t)| \leq 1$. When $u = \pm 1$, the trajectory is in the region of the switching surface where $CAx = \mp CB$.

Let S_+ and S_- be subsets of S as defined before. Define also

$$S_0 = \{x \in S : \ -CB < CAx < CB\}$$

as the region in S of sliding modes,

$$S_{0+} = \{x \in S : \ CAx = CB\}$$

and, finally, $S_{0+} = -S_{0-}$.

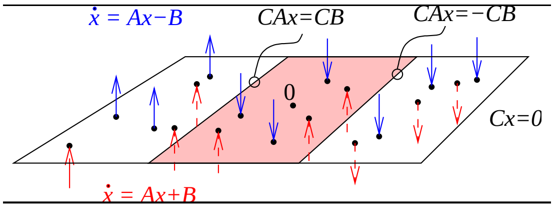

Fig. 2. Vector fields of the RFS on both sides of S, defining the subsets of S

Assume the system has a limit cycle with a sliding mode. Let $\xi(t)$ be a nontrivial periodic solution of (1)-(2) with period $2(t_1^*+t_2^*)$, and let γ be the limit cycle defined by the image set of $\xi(t)$ (see figure 3). Conditions for the existence and local stability of symmetric limit cycles with a sliding mode can be found in (Johansson *et al.*, 1999; di Bernardo *et al.*, 2000).

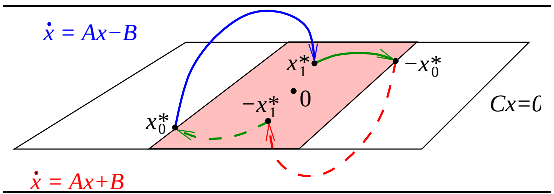

Fig. 3. Limit cycle with a sliding mode

From now on, we assume the existence of a limit cycle with a sliding mode. Such limit cycle intersects S_{0+} and S transversely at $x_0^* \in S_{0+}$ and $x_1^* \in S_0$, respectively (see figure 3). The switching time from x_0^* to x_1^* is t_1^* and from x_1^* to $-x_0^*$ is t_2^*. Assume also that all trajectories starting at S switch (this condition can be easily verified (Gonçalves, 2000)) or otherwise the limit cycle cannot be globally stable.

In this paper, we are interested in systems with a unique locally stable limit cycle with sliding modes. For such systems, the idea is to construct quadratic Lyapunov functions on the switching surface S to prove that all possible maps from one switch to the next are globally stable. This, in turn, shows that the limit cycle is globally asymptotically stable. This is the topic of the next two sections.

3.2 Global Stability

Global analysis of limit cycles with sliding modes is more complicated than the simplest form of symmetric unimodal limit cycles studied in (Gonçalves, 2000) and recalled in section 2.2. The fact that there are more than two switches per cycle requires the analysis of at least two impact maps, when comparing with only one for unimodal limit cycles. Also, a trajectory in sliding mode is induced by a system with dimension lower than the dimension of the original system. Thus, we will have to deal with the interaction of systems of different dimensions.

In this section, we explain the simplest condition to analyze RFS with sliding modes. This is based on showing that *two* impact maps are globally stable by constructing two quadratic Lyapunov functions on the switching surface of the system. In the next section, a different relaxation is given based on more than two impact maps. This alternative approach exploits the fact that the sliding mode has lower dimension and, in some cases, leads to more attractive computational solutions.

The simplest way of analyzing limit cycles with sliding modes using impact maps is by considering two impact maps (see figure 4). The first impact map leaves the switching surface at some point $x_0^* + \Delta_0 \in S_+ \cup S_{0+}$ and, induced by $\dot{x} = Ax - B$, maps to some point $x_1^* + \Delta_1 \in S_0 \cup S_{0-} \cup S_-$. If $\Delta_1 \in S_- - x_1^*$ then, due to the symmetry of the system, we can consider a new trajectory starting at $-x_1^* - \Delta_1 \in S_+$ using impact map 1 once again. On the other hand, if $\Delta_1 \in S_0 - x_1^*$, then we have a sliding mode. This leads to impact map 2 which is a map from $x_1^* + \Delta_1 \in S_0$ to $-x_0^* + \Delta_2 \in S_{0-}$ induced by $\dot{x} = PAx$. For analysis purposes, and again by the symmetry property, next we apply again impact map 1 at $x_0^* - \Delta_2 \in S_{0+}$.

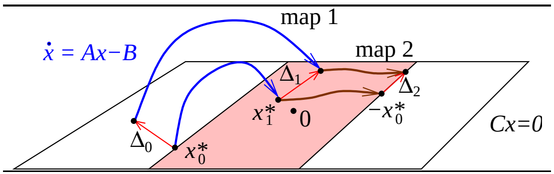

Fig. 4. Analysis using two impact maps

Define the sets of *expected switching times* $\mathcal{T}_1$ and $\mathcal{T}_2$ as the sets of all possible switching times associated with each respective impact map, as in (3). Define also $S_{t_1} \subset S_+$ and $S_{t_2} \subset S_0$ as in section 2.2. Note that both S_{t_1} and S_{t_2} are subsets of linear manifolds of dimension $n-2$.

Define two quadratic forms on S_+

$$V_1(\Delta) = \Delta' P_1 \Delta, \quad \Delta \in S_+ \cup S_{0+} - x_0^*$$
$$V_2(\Delta) = \Delta' P_2 \Delta, \quad \Delta \in S_0 - x_1^*$$

where $P_1, P_2 > 0$. Global stability of the system follows if

$$V_2(\Delta_1) < V_1(\Delta_0), \quad \forall \Delta_0 \in S_+ \cup S_{0+} - x_0^*$$
$$V_1(\Delta_2) < V_2(\Delta_1), \quad \forall \Delta_1 \in S_0 - x_1^*$$

Let $x_0^*(t)$ be the solution of $\dot{x} = Ax - B$ with initial condition x_0^* for all $t \geq 0$, and $x_1^*(t)$ the solution of $\dot{x} = PAx$ with initial condition x_1^* for all $t \geq 0$. From (Gonçalves, 2000), each impact map can be decomposed in a linear transformation analytically parameterized by the associated switching time. Define

$$H_1(t) = \left(I - \frac{(x_0^*(t) - x_1^*)C}{Cx_0^*(t)} \right) e^{At}$$

and

$$H_2(t) = \left(I - \frac{(x_1^*(t) + x_0^*)CA}{CB + CAx_1^*(t)} \right) e^{PAt}$$

Then,

$$\Delta_1 = H_1(t_1)\Delta_0$$
$$\Delta_2 = H_2(t_2)\Delta_1$$

for all $\Delta_0 \in S_+ \cup S_{0+} - x_0^*$ and $\Delta_1 \in S_0 - x_1^*$ with respective switching times $t_1 \in \mathcal{T}_1$, $t_2 \in \mathcal{T}_2$. We have then the following result.

Theorem 3.1. Consider the RFS (1)-(2) with $CB > 0$ and assume there exists a symmetric limit cycle γ with a sliding mode and a period $2(t_1^* + t_2^*)$. The limit cycle is globally asymptotically stable if there exist $P_1, P_2 > 0$ such that

$$\begin{cases} R_1(t_1) = P_1 - H_1'(t_1) P_2 H_1(t_1) > 0 & \text{on } S_{t_1} - x_0^* \\ R_2(t_2) = P_2 - H_2'(t_2) P_1 H_2(t_2) > 0 & \text{on } S_{t_2} - x_1^* \end{cases} \quad (6)$$

for all expected switching times $t_1 \in \mathcal{T}_1$ and $t_2 \in \mathcal{T}_2$.

(Gonçalves, 2000) includes a long discussion on how to relax (6) to a set of LMIs. One of the most basic relaxations is to allow both R_1 and R_2 to be positive definite over all the switching surface, i.e.,

$$\begin{cases} R_1(t_1) > 0 & \text{on } S - x_0^* \\ R_2(t_1) > 0 & \text{on } S - x_1^* \end{cases} \quad (7)$$

for all expected switching times $t_1 \in \mathcal{T}_1$ and $t_2 \in \mathcal{T}_2$.

For each t_1, t_2 these conditions are LMIs for which we can solve for $P_1, P_2 > 0$ using efficient available software. Note that each condition in (7) depends only on a single scalar parameter, i.e., R_1 depends only on t_1 and not on t_2, and, similarly, R_2 depends only on t_2. Computationally, this means that when we grid each set of expected switching times, this will only affect one condition in (7).

To reduce conservatism, extra conditions can be added to (7) using the so-called S-procedure. How this is done is explained in (Gonçalves, 2000). The cost of loss of conservatism in this case is the increase in computations. Another alternative to reduce the conservatism of (7) is to use extra impact maps and Lyapunov functions to take advantage that $x_0^* \in S_{0+}$ and the fact that the set S_{0+} is of dimension $n-2$. This is considered in the next section.

4. REDUCING CONSERVATISM

There are several ways to reduce the conservatism of conditions (7). One is the inclusion of extra conditions in (7) using the S-procedure. The S-procedure, however, is known to be conservative whenever the number of constraints is higher than 1. Also, every new constraint increases the computationally complexity. It is then important to have alternatives to reduce complexity.

One of such alternatives is to use an extra Lyapunov function and more impact maps.

If the limit cycle is transversal at every switch, then, after some switches, trajectories close to limit cycle will only switch at S_0 and S_{0-}. Thus, it makes sense to define an impact map from S_{0+} to S_0 and define a Lyapunov function in a subset of S_{0+}. Note that such Lyapunov function is defined in a set of dimension $n-2$. The addition of this impact map requires the addition of two more. So, lets define the 5 impact maps (see figure 5).

Starting with impact maps 1 and 2, they map points from S_+ to itself and from S_+ to S_0, respectively, and are defined as follows. Let $x^* \in S$ such that the solution $x^*(t)$ of $\dot{x} = Ax - B$ does not intersect S for all $t > 0$. We divide points in S_+ in two regions: the region such that the switch will occur at S_- (domain of impact map 1) and the region such that the switch will occur at S_0 (domain of impact map 2). Therefore, impact map 1 takes points $x^* + \Delta_0 \in S_+$ and, induced by $\dot{x} = Ax - B$, switches at S_- at $-x^* + \Delta_1$. Due to the symmetry of the system, this can be mapped back to S_+ at $x^* - \Delta_1$. Impact map 2 takes points $x^* + \Delta_0 \in S_+$ and, again, induced by $\dot{x} = Ax - B$ switches at S_0 at $x_1^* + \Delta_1$. Impact map 3 is similar to impact map 2 from section 3.2. Impact maps 4 and 5 are defined as impact maps 1 and 2 with the difference that their domains are subsets of S_{0+} instead of S_+.

Define the sets of *expected switching times* $\mathcal{T}_i$, $i = 1, ..., 5$, as the sets of all possible switching times associated with each respective impact map, as explained in section 2.2. Define also S_{t_i}, $i = 1, ..., 5$ as in sections 2.2 and 3.2. Next, we define three quadratic Lyapunov functions in S

$$V_+(\Delta) = \Delta' P_+ \Delta + 2\Delta' g_+ + \alpha_+, \quad \Delta \in S_+ - x^*$$
$$V_0(\Delta) = \Delta' P_0 \Delta, \quad \Delta \in S_0 - x_1^*$$
$$V_{0+}(\Delta) = \Delta' P_{0+} \Delta, \quad \Delta \in S_{0+} - x_0^*$$

where $P_+, P_0, P_{0+} > 0$. Global stability of the system follows if

$$V_+(-\Delta_1) < V_+(\Delta_0), \quad \forall \Delta_0 \in S_+ - x^*$$
$$V_0(\Delta_1) < V_+(\Delta_0), \quad \forall \Delta_0 \in S_+ - x^*$$
$$V_{0+}(\Delta_2) < V_0(\Delta_1), \quad \forall \Delta_1 \in S_0 - x_1^*$$
$$V_+(-\Delta_1) < V_{0+}(\Delta_0), \quad \forall \Delta_0 \in S_{0+} - x_0^*$$
$$V_0(\Delta_1) < V_{0+}(\Delta_0), \quad \forall \Delta_0 \in S_{0+} - x_0^*$$

Each impact map can be written as a linear transformation analytically parameterized by the associated switching time, as before. We then get $H_i(t)$, $i = 1, ...5$. Define also

$$w_i(t) = \frac{C_i e^{A_i t}}{d_i - C_i z_i^*(t)}$$

where $C_i = C$, $A_i = A$, $d_i = 0$, $i = 1, 2, 4, 5$, $C_3 = CA$, , $A_3 = PA$, $d_3 = -CB$, and $z_1^*(t) = z_2^*(t) = x^*(t)$, $z_4^*(t) = z_5^*(t) = x_0^*(t)$, $z_3^*(t) = x_1^*(t)$. We have then the following result.

Theorem 4.1. Consider the RFS (1)-(2) with $CB > 0$ and assume there exists a symmetric limit cycle γ with a sliding mode and a period $2(t_1^* + t_2^*)$. The limit cycle is globally asymptotically stable if there exist $P_+, P_0, P_{0+} > 0$, g_+, and α_+ such that

$$R_1(t_1) \stackrel{\text{def}}{=} P_+ - H_1' P_+ H_1$$
$$+2(I + H_1') g_+ w_1 > 0 \quad on\ S_{t_1} - x^*$$
$$R_2(t_2) \stackrel{\text{def}}{=} P_+ - H_2' P_0 H_2 + 2 g_+ w_2$$
$$+ w_2' \alpha_+ w_2 > 0 \quad on\ S_{t_2} - x^*$$
$$R_3(t_3) \stackrel{\text{def}}{=} P_0 - H_3' P_{0+} H_3 > 0 \ on\ S_{t_3} - x_1^*$$
$$R_4(t_4) \stackrel{\text{def}}{=} P_{0+} - H_4' P_+ H_4 + 2 H_4' g_+ w_4$$
$$- w_4' \alpha_+ w_4 > 0 \quad on\ S_{t_4} - x_0^*$$
$$R_5(t_5) \stackrel{\text{def}}{=} P_{0+} - H_5' P_0 H_5 > 0 \quad on\ S_{t_5} - x_0^*$$

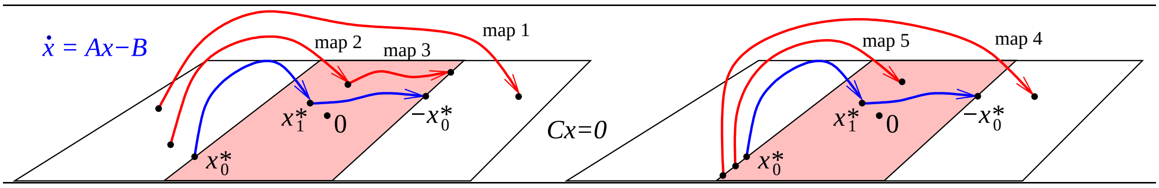

Fig. 5. Analysis with 5 impact maps

for all expected switching times $t_i \in \mathcal{T}_i$, $i = 1, \ldots, 5$ (not that the arguments t_i were removed in some expressions for simplification).

Note that each condition $R_i(t_i)$ depends only on one parameter t_i. This means that the complexity of adding extra impact maps does not grow exponentially. Note also that the P_+ and P_0 are $n-1 \times n-1$ symmetric matrices, but P_{0+} is only an $n-2 \times n-2$ symmetric matrix.

A relaxation of the conditions in theorem 4.1, similar to what was done in the previous section when (6) was relaxed to (7), can also be done here. We then obtained a set of LMIs that can be solved efficiently.

5. CONCLUSIONS

In (Gonçalves *et al.*, 2001) we analyzed symmetric unimodal limit cycles of RFS. In this paper we proposed to better understand RFS by studying oscillations with sliding modes in RFS. We presented several relaxations to obtain conditions in the form of LMIs that, when satisfied, guarantee the global asymptotic stability of limit cycles with sliding modes in RFS. The main idea was to show that impact maps associated with RFS are contracting in some sense. This is possible since such maps, although nonlinear, multivalued, and not continuous, can be represented as linear transformations analytically parametrized by the respective switching time. Quadratic surface Lyapunov functions can this way be constructed by simply solving a set of LMIs.

The results in this paper open the door to the analysis of more complex classes of hybrid systems. In particular, hybrid systems with “jumps” in the state, allowing a hybrid system not only to have a non-continuous vector field, but also allow switches between subsystems of different dimensions.

REFERENCES

Ardalan, S. H and J. J. Paulos (1987). An analysis of nonlinear behavior in delta-sigma modulators. *IEEE Transactions on Circuits and Systems* **6**, 33–43.

Åström, Karl J. (1995). Oscillations in systems with relay feedback. *The IMA Volumes in Mathematics and its Applications: Adaptive Control, Filtering, and Signal Processing* **74**, 1–25.

Atherton, D. P. (1975). *Nonlinear Control Engineering*. Van Nostrand.

di Bernardo, Mario, Karl Johansson and Francesco Vasca (2000). Self-oscillations and sliding in relay feedback systems: Symmetry and bifurcations. *International Journal of Bifurcations and Chaos*.

Gonçalves, Jorge M. (2000). Constructive Global Analysis of Hybrid Systems. PhD thesis. Massachusetts Institute of Technology. Cambridge, MA.

Gonçalves, Jorge M., Alexandre Megretski and Munther A. Dahleh (2001). Global stability of relay feedback systems. *IEEE Transactions on Automatic Control* **46**(4), 550–562.

Johansson, Karl H., Anders Rantzer and Karl J. Åström (1999). Fast switches in relay feedback systems. *Automatica*.

Neimark, Yu. I. (1972). *Methods of Pointwise Mappings in the Theory of Nonlinear Oscillations*. NAUKA. (In Russian).

Ringrose, Robert P. (1997). Self-Stabilizing Running. PhD thesis. Massachusetts Institute of Technology. Cambridge, MA.

Tsypkin, Ya.Z (1984). *Relay Control Systems*. Cambridge University Press, Cambridge, UK.

Utkin, Vadim I. (1995). *Sliding Modes in Control Optimization*. Springer-Verlag, NY.

Varigonda, Subbarao and Tryphon Georgiou (2001). Dynamics of relay relaxation oscillators. *IEEE Transactions on Automatic Control* **46**(1), 65–77.

Williamson, Matthew M. (1999). Robot Arm Control Exploiting Natural Dynamics. PhD thesis. Massachusetts Institute of Technology. Cambridge, MA.