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REDUCED ORDER FILTERING FOR STOCHASTIC DISCRETE-TIME SYSTEMS WITH UNKNOWN INPUTS

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Abstract: This paper presents a reduced order unbiased minimum variance estimator for stochastic discrete-time time-varying systems with unknown inputs. The necessary and sufficient conditions for the existence of the obtained filter are given. Stability and convergence conditions are developed for the time-invariant case. *Copyright ©2002 IFAC*

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1. INTRODUCTION

The problem of state estimation of dynamical systems has been studied extensively over the last three decades. It is well known that the Luenberger observer can be used in a deterministic context and the Kalman-Bucy filter theory can be applied to stochastic processes (Kwakernaak and Sivan, 1972; O'Reilly, 1983; Lewis, 1986; Middleton and Goodwin, 1990; Kučera, 1991; Kamen and Su, 1999). These estimators have been extended to deterministic systems with unknown inputs (see (Hou and Müller, 1992; Darouach *et al.*, 1994) and references therein). However for the stochastic systems with unknown inputs, only few results exist on the design of the optimal filtering. In (Kitanidis, 1987; Darouach and Zasadzinski, 1997), full-order unbiased minimum variance filters for discrete-time stochastic systems in presence of unknown inputs in the model have been developed. The optimality of these filters has been recently proved in (Kerwin and Prince, 2000).

In many practical problems we only need a partial state or linear functional of the state. In (Nagpal *et al.*, 1987; Soroka and Shaked, 1988; Nakamizo, 1997), a reduced order filtering algorithm has been developed for standard stochastic systems where the inputs are known.

In this paper, we propose a new method to design a reduced order filter for stochastic discrete-time time-varying systems with unknown inputs. The latters affect the model and also the outputs of the system. The necessary and sufficient conditions for the existence of the filter are given, they have been obtained from the unbiasedness constraints. The convergence and the stability conditions are presented for the time-invariant case.

2. REDUCED ORDER UNKNOWN INPUT FILTER

2.1 *The time varying case*

Consider the following stochastic discrete-time time-varying systems with unknown inputs

$$x(t+1) = A(t)x(t)+B(t)u(t)+F(t)d(t)+w(t) \quad (1a)$$
$$y(t) = C(t)x(t) + G(t)d(t) + v(t) \quad (1b)$$
$$z(t) = L(t)x(t) \quad (1c)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the measurement output, $u(t) \in \mathbb{R}^m$ is the known input, $d(t) \in \mathbb{R}^q$ is the unknown input and $z(t) \in \mathbb{R}^r$ is the vector to be estimated, with $r \leqslant n$. $A(t)$, $C(t)$, $B(t)$, $F(t)$, $G(t)$ and $L(t)$ are real matrices of appropriate dimensions. Without loss of generality, it is assumed that rank $L(t) = r$. $w(t)$, $v(t)$ and x_0 are zero-mean random vectors obeying to : $\mathcal{E}\left\{\begin{bmatrix} x_0 \\ v(i) \\ w(i)\end{bmatrix}\begin{bmatrix} x_0^T & v^T(j) & w^T(j)\end{bmatrix}\right\} = \begin{bmatrix} P_0 & 0 & 0 \\ 0 & Q(i)\delta_{ij} & 0 \\ 0 & 0 & R(i)\delta_{ij}\end{bmatrix}$ where $\mathcal{E}\{.\}$ is the expectation operator, δ_{ij} is the Kronecker delta function, $P_0 \geqslant 0$, $Q(i) \geqslant 0$ and $R(i) > 0$.

Let us consider the following functional r^{th}-order filter

$$\widehat{z}(t+1)=N(t+1)\widehat{z}(t)+E(t+1)u(t)+J(t+1)y(t+1) \quad (2)$$

where $\widehat{z}(t) \in \mathbb{R}^r$ is the estimate of $z(t)$, matrices $N(t)$, $E(t)$ and $J(t)$ are of appropriate dimensions. The estimation error is

$$e(t) = z(t) - \widehat{z}(t) = L(t)x(t) - \widehat{z}(t).$$

The problem is to find the matrices $N(t)$, $E(t)$ and $J(t)$ so that $\widehat{z}(t)$ is unbiased, i.e.

$$\mathcal{E}\{e(t)\} = 0 \quad (3)$$

and

$$\mathcal{J} = \operatorname{tr}(P(t)) \quad (4)$$

is minimum, where $P(t)$ is the estimation error covariance matrix defined by

$$P(t) = \mathcal{E}\left\{e(t)e^T(t)\right\}. \quad (5)$$

The dynamic of the estimation error is given by

$$\begin{aligned} e(t+1) =& N(t+1)e(t)+(\Psi(t+1)A(t)-N(t+1)L(t))x(t) \\ &- J(t+1)G(t+1)d(t+1) + (\Psi(t+1)B(t) - E(t+1))u(t) \\ &+ \Psi(t+1)F(t)d(t) + \Psi(t+1)w(t) - J(t+1)v(t+1) \end{aligned} \quad (6)$$

where

$$\Psi(t) = L(t) - J(t)C(t).$$

For $\widehat{z}(t)$ to be an unbiased estimate of $z(t)$ we must have the following constraints

$$0 = \Psi(t+1)A(t) - N(t+1)L(t) \quad (7)$$

$$0 = \Psi(t+1)F(t) \quad (8)$$

$$0 = J(t+1)G(t+1) \quad (9)$$

$$0 = \Psi(t+1)B(t) - E(t+1). \quad (10)$$

Now since $L(t)$ is a full row rank matrix, let $T_1(t) \in \mathbb{R}^{(n-r)\times n}$ be a matrix such that

$$\begin{bmatrix} L(t) \\ T_1(t)\end{bmatrix}^{-1} = \begin{bmatrix} H_1(t) & E_1(t)\end{bmatrix}.$$

Then, equation (7) is equivalent to

$$(\Psi(t+1)A(t) - N(t+1)L(t))\begin{bmatrix} H_1(t) & E_1(t)\end{bmatrix} = 0$$

which yields

$$N(t+1) = \Psi(t+1)A(t)H_1(t) \quad (11)$$

$$0 = \Psi(t+1)A(t)E_1(t). \quad (12)$$

From (8), (9), (12) and by using the definition of $\Psi(t)$, we obtain the following equation

$$J(t+1)\Theta(t+1) = \Omega(t+1) \quad (13)$$

where

$$\Theta(t+1) = \begin{bmatrix} C(t+1)A(t)E_1(t) & C(t+1)F(t) & G(t+1)\end{bmatrix}$$

$$\Omega(t+1) = \begin{bmatrix} L(t+1)A(t)E_1(t) & L(t+1)F(t) & 0\end{bmatrix}.$$

Before giving the necessary and sufficient conditions for the existence of the filter (2) for $z(t)$, define the following matrix

$$\Xi(t+1) = \begin{bmatrix} C(t+1)A(t) & C(t+1)F(t) & G(t+1) \\ L(t) & 0 & 0\end{bmatrix},$$

then we obtain the following lemma.

Lemma 1. The reduced order unbiased filter (2) exists if and only if

$$\operatorname{rank}\begin{bmatrix} \Xi(t+1) \\ \hline L(t+1)A(t) \;\; L(t+1)F(t) \;\; 0\end{bmatrix} = \operatorname{rank}\Xi(t+1). \quad (14)$$

■

If equations (7)-(10) are satisfied, the filter dynamics error (6) can be written as

$$e(t+1) = N(t+1)e(t) + \Psi(t+1)w(t) - J(t+1)v(t+1). \quad (15)$$

Now under condition (14) the general solution of equation (13) is

$$J(t+1) = J_1(t+1) + K(t+1)J_2(t+1) \quad (16)$$

where

$$J_1(t+1) = \Omega(t+1)\Theta^{\dagger}(t+1) \quad (17)$$

and $K(t) \in \mathbb{R}^{r\times(p-r_\theta)}$ is an arbitrary matrix and

$$J_2(t+1) = \alpha(t+1)\left(I_p - \Theta(t+1)\Theta^{\dagger}(t+1)\right) \quad (18)$$

where $\Theta^{\dagger}(t+1)$ is any generalized inverse matrix of $\Theta(t+1)$ and $\alpha(t) \in \mathbb{R}^{(p-r_\theta)\times p}$ is an arbitrary full row rank matrix such that $J_2(t)$ is of full row rank and $\begin{bmatrix} \Theta^{\dagger}(t) \\ J_2(t)\end{bmatrix}$ is of full column rank with $\operatorname{rank}\Theta(t) = r_\theta$.

Remark 2. The parameter matrix $\alpha(t)$ is of full row rank and must be chosen such that $\begin{bmatrix} \Theta^{\dagger}(t) \\ J_2(t)\end{bmatrix}$ is of full column rank, one choice can be done as follows : $\alpha(t) = \begin{bmatrix} 0 & I_{(p-r_\theta)}\end{bmatrix}U^T(t)$ where $\Theta(t) = U(t)\begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0\end{bmatrix}V^T(t)$ is the singular value decomposition of the matrix $\Theta(t)$, $U(t)$ and $V(t)$ are orthogonal matrices of appropriate dimensions. In this case $J_2(t) = \alpha(t)$. ■

Inserting (16) into (11) gives

$$N(t+1) = \overline{\Omega}(t+1) - K(t+1)\overline{\Gamma}(t+1) \quad (19)$$

where

$$\overline{\Omega}(t+1) = (L(t+1)A(t) - J_1(t+1)C(t+1)A(t))H_1(t) \quad (20)$$

$$\overline{\Gamma}(t) = J_2(t+1)C(t+1)A(t)H_1(t). \quad (21)$$

Then the filter (2) becomes

$$\begin{aligned}\widehat{z}(t+1) =& \overline{\Omega}(t+1)\widehat{z}(t) + \overline{E}(t+1)u(t) + J_1(t+1)y(t+1) \\ &+K(t+1)\left(J_2(t+1)y(t+1) - \overline{\Gamma}(t+1)\widehat{z}(t) - \overline{\overline{E}}(t+1)u(t)\right)\end{aligned}$$

where

$$\overline{E}(t+1) = L(t+1)B(t) - J_1(t+1)C(t+1)B(t) \quad (22)$$

$$\overline{\overline{E}}(t+1) = J_2(t+1)C(t+1)B(t), \quad (23)$$

and equation (15) becomes

$$e(t+1) = \left(\overline{\Omega}(t+1) - K(t+1)\overline{\Gamma}(t+1)\right) e(t) - (J_1(t+1) + K(t+1)J_2(t+1))\, v(t+1) + (L(t+1) - J_1(t+1)C(t+1) - K(t+1)J_2(t+1)C(t+1))\, w(t). \quad (24)$$

From (24), the error covariance propagates as

$$P(t+1) = \left(\overline{\Omega}(t+1) - K(t+1)\overline{\Gamma}(t+1)\right) P(t) \left(\overline{\Omega}(t+1) - K(t+1)\overline{\Gamma}(t+1)\right)^T + \overline{Q}(t+1) - S(t+1)K^T(t+1) - K(t+1)S^T(t+1) + K(t+1)\Phi(t+1)K^T(t+1) \quad (25)$$

with the initial condition $P(0) = L(0)P_0L^T(0)$ and

$$\overline{Q}(t+1) = (L(t+1) - J_1(t+1)C(t+1))\, Q(t)\, (L(t+1) - J_1(t+1)C(t+1))^T + J_1(t+1)R(t+1)J_1^T(t+1)$$

$$S(t+1) = (L(t+1) - J_1(t+1)C(t+1))\, Q(t)C^T(t+1) \times J_2^T(t+1) - J_1(t+1)R(t+1)J_2^T(t+1)$$

$$\Phi(t+1) = J_2(t+1)\Big(C(t+1)Q(t)C^T(t+1) + R(t+1)\Big) \times J_2^T(t+1).$$

From (25), the minimization of (4) leads to the following equation

$$K(t+1)\left(\overline{\Gamma}(t+1)P(t)\overline{\Gamma}^T(t+1) + \Phi(t+1)\right) = \overline{\Omega}(t+1)P(t)\overline{\Gamma}^T(t+1) + S(t+1). \quad (26)$$

The optimal gain $K(t+1)$ exists if and only if

$$\operatorname{rank}\begin{bmatrix}\overline{\Omega}(t+1)P(t)\overline{\Gamma}^T(t+1) + S(t+1)\\ \overline{\Gamma}(t+1)P(t)\overline{\Gamma}^T(t+1) + \Phi(t+1)\end{bmatrix} = \operatorname{rank}\left(\overline{\Gamma}(t+1)P(t)\overline{\Gamma}^T(t+1) + \Phi(t+1)\right) \quad (27)$$

or equivalently

$$\ker\left(\overline{\Gamma}(t+1)P(t)\overline{\Gamma}^T(t+1) + \Phi(t+1)\right) \subset \ker\left(\left(\overline{\Omega}(t+1)P(t)\overline{\Gamma}^T(t+1) + S(t+1)\right)^T\right). \quad (28)$$

In this case the general solution of (26) is given by (Kučera, 1991; Saberi *et al.*, 1995)

$$K(t+1) = \left(\overline{\Omega}(t+1)P(t)\overline{\Gamma}^T(t+1) + S(t+1)\right)\left(\overline{\Gamma}(t+1)P(t)\overline{\Gamma}^T(t+1) + \Phi(t+1)\right)^\dagger + \overline{K}(t+1)$$

where
$\operatorname{span}(\overline{K}^T(t+1)) \subset \ker(\overline{\Gamma}(t+1)P(t)\overline{\Gamma}^T(t+1) + \Phi(t+1))$. Using (28) and inserting this solution into (25) gives the following generalized Riccati difference equation (GRDE)

$$P(t+1) = \overline{\Omega}(t+1)P(t)\overline{\Omega}^T(t+1) - \Big(\overline{\Omega}(t+1)P(t)\overline{\Gamma}^T(t+1) + S(t+1)\Big)\Big(\overline{\Gamma}(t+1)P(t)\overline{\Gamma}^T(t+1) + \Phi(t+1)\Big)^\dagger \times \Big(\overline{\Omega}(t+1)P(t)\overline{\Gamma}^T(t+1) + S(t+1)\Big)^T + \overline{Q}(t+1).$$

Remark 3. When $r_\theta = p$ we obtain $J_2(t) = 0$, $J_1(t+1) = \Omega(t+1)\Theta^\dagger(t+1)$ and the filter becomes
$\widehat{z}(t+1) = \overline{\Omega}(t+1)\widehat{z}(t) + E(t+1)u(t) + J_1(t+1)y(t+1)$

with

$$P(t+1) = \overline{\Omega}(t+1)P(t)\overline{\Omega}^T(t+1) + \overline{Q}(t+1)$$

$$\overline{Q}(t+1) = (L(t+1) - J_1(t+1)C(t+1))\, Q(t)\, (L(t+1) - J_1(t+1)C(t+1))^T + J_1(t+1)R(t+1)J_1^T(t+1).$$

■

Remark 4. When $L(t) = I_n$, $F(t) = 0$ and $G(t) = 0$, we obtain the full-order Kalman filter. ■

Remark 5. If matrices $G(t) = 0$, $L(t) = I_n$ and $\Phi(t+1)$ is nonsingular, we obtain the results presented in (Kitanidis, 1987; Darouach and Zasadzinski, 1997). ■

2.2 *The time invariant case*

In this section we consider the case where the system matrices are constant i.e. $A(t) = A$, $B(t) = B$, $F(t) = F$, $C(t) = C$, $G(t) = G$, $L(t) = L$, $Q(t) = Q$, $R(t) = R$ and

$$\begin{bmatrix} L \\ T_1 \end{bmatrix}^{-1} = \begin{bmatrix} H_1 & E_1 \end{bmatrix}.$$

Then the filter (2) becomes

$$\widehat{z}(t+1) = \overline{\Omega}\widehat{z}(t) + \overline{E}u(t) + J_1y(t+1) + K(t+1)\left(J_2y(t+1) - \overline{\Gamma}\widehat{z}(t) - \overline{\overline{E}}u(t)\right) \quad (29)$$

where

$$\begin{aligned}\overline{\Omega} &= LAH_1 - J_1CAH_1\\ \overline{\Gamma} &= J_2CAH_1\\ \overline{E} &= LB - J_1CB\\ \overline{\overline{E}} &= J_2CB\end{aligned}$$

with $J_1 = \Omega\Theta^\dagger$ and $J_2 = \alpha(I_p - \Theta\Theta^\dagger)$ such that $\alpha \in \mathbb{R}^{(p-r_\theta)\times p}$ is a full row rank matrix and $\left[\begin{smallmatrix}\Theta^\dagger\\ J_2\end{smallmatrix}\right]$ is of full column rank, where $r_\theta = \operatorname{rank}\Theta$ and

$$\begin{aligned}\Theta &= \begin{bmatrix} CAE_1 & CF & G\end{bmatrix}\\ \Omega &= \begin{bmatrix} LAE_1 & LF & 0\end{bmatrix}.\end{aligned}$$

The error covariance propagates as

$$P(t+1) = \left(\overline{\Omega} - K(t+1)\overline{\Gamma}\right) P(t) \left(\overline{\Omega} - K(t+1)\overline{\Gamma}\right) + \overline{Q} - SK^T(t+1) - K(t+1)S^T + K(t+1)\Phi K^T(t+1) \quad (30)$$

with the initial condition $P(0) = LP_0L^T$ and where

$$\begin{aligned}\overline{Q} &= (L - J_1C)\, Q\, (L - J_1C)^T + J_1RJ_1^T\\ S &= (L - J_1C)\, QC^TJ_2^T - J_1RJ_2^T\\ \Phi &= J_2\left(CQC^T + R\right)J_2^T.\end{aligned}$$

In the sequel we consider two cases. The first one deals with a discrete Riccati difference equation (DRDE) when Φ is nonsingular, while the second case treats a generalized discrete algebraic Riccati equation (GDARE) when $(\overline{\Gamma}P\overline{\Gamma}^T + \Phi)$ is nonsingular.

2.2.1. *The case where Φ is nonsingular*

In this section we assume that Φ (or $J_2(CQC^T + R)J_2^T$) is a positive definite matrix, in this case the optimal gain $K(t+1)$ is given by

$$K(t+1) = \left(\overline{\Omega}P(t)\overline{\Gamma}^T + S\right)\left(\overline{\Gamma}P(t)\overline{\Gamma}^T + \Phi\right)^{-1}$$

and we can define the following matrices

$$\begin{aligned}\overline{\Omega}_s &= \overline{\Omega} - S\Phi^{-1}\overline{\Gamma}\\ K_s(t+1) &= K(t+1) - S\Phi^{-1}\\ Q_s &= \overline{Q} - S\Phi^{-1}S^T.\end{aligned}$$

Equation (29) can be also written as the following DRDE

$$\begin{aligned}P(t+1) = &\left(\overline{\Omega}_s - K_s(t+1)\overline{\Gamma}\right) P(t) \left(\overline{\Omega}_s - K_s(t+1)\overline{\Gamma}\right)^T\\ &+ Q_s + K_s(t+1)\Phi K_s^T(t+1).\end{aligned}$$

In this case the filter (29) becomes

$$\begin{aligned}\widehat{z}(t+1) = &\overline{\Omega}_s\widehat{z}(t) + \overline{E}_s u(t) + J_{1s}y(t+1)\\ &+ K_s(t+1)\left(J_2 y(t+1) - \overline{\Gamma}\widehat{z}(t) - \overline{\overline{E}}u(t)\right)\end{aligned} \quad (31)$$

where

$$\begin{aligned}J_{1s} &= J_1 + S\Phi^{-1}J_2\\ \overline{E}_s &= LB - J_{1s}CB\end{aligned}$$

Remark 6. The necessary condition for $\Phi > 0$ is that α to be of full row rank matrix. ■

The ARDE associated with the above DRDE is

$$P = \left(\overline{\Omega}_s - K_s\overline{\Gamma}\right) P \left(\overline{\Omega}_s - K_s\overline{\Gamma}\right)^T + Q_s + K_s\Phi K_s^T.$$

The necessary and sufficient conditions for the stability and convergence of the obtained filter, when Φ is nonsingular matrix, can be derived from the standard results on the stability and convergence of the Kalman filter (de Souza *et al.*, 1986; Lewis, 1986; Middleton and Goodwin, 1990). Before giving the conditions of stability and convergence of the filter we can give the following lemmas.

Lemma 7. The pair $(\overline{\Gamma}, \overline{\Omega}_s)$ or (the pair $(\overline{\Gamma}, \overline{\Omega})$) is detectable if and only if

$$\operatorname{rank}\begin{bmatrix}\lambda L - LA & -LF & 0\\ CA & CF & G\end{bmatrix} = \operatorname{rank}\Xi, \forall\lambda\in\mathbb{C}, |\lambda| \geqslant 1 \quad (32)$$

where $\Xi = \left[\begin{smallmatrix}CA & CF & G\\ L & 0 & 0\end{smallmatrix}\right]$. ■

Lemma 8. The pair $(\overline{\Omega}_s, Q_s^{1/2})$ has no unreachable mode on the stability boundary if and only if

$$\operatorname{rank}\begin{bmatrix}AH_1 - \exp(j\omega)H_1 & -F & Q^{1/2} & 0\\ -\exp(j\omega)CH_1 & G & 0 & R^{1/2}\end{bmatrix} = n + p, \quad \forall\omega\in[0, 2\pi]. \quad (33)$$

■

From (de Souza *et al.*, 1986; Lewis, 1986; Middleton and Goodwin, 1990) and the above results we have the following theorem.

Theorem 9. Assume that Φ is a positive definite matrix. Then there exists an unbiased stable reduced order filter (29) if and only if

(i) $\operatorname{rank}\begin{bmatrix}\Xi\\ \hline LA & LF & 0\end{bmatrix} = \operatorname{rank}\Xi$,

(ii) $\operatorname{rank}\begin{bmatrix}\lambda L - LA & -LF & 0\\ CA & CF & G\end{bmatrix} = \operatorname{rank}\Xi, \forall\lambda\in\mathbb{C}, |\lambda|\geqslant 1$,

(iii) $\operatorname{rank}\begin{bmatrix}AH_1 - \exp(j\omega)H_1 & -F & Q^{1/2} & 0\\ -\exp(j\omega)CH_1 & G & 0 & R^{1/2}\end{bmatrix} = n + p$, $\forall\omega\in[0, 2\pi]$. ■

2.2.2. *The case where* $(\overline{\Gamma}P\overline{\Gamma}^T + \Phi)$ *is nonsingular*

In this section we consider the following time-invariant filter

$$\begin{aligned}\widehat{z}(t+1) = &\overline{\Omega}\widehat{z}(t) + \overline{E}u(t) + J_1 y(t+1)\\ &+ K(k+1)\left(J_2 y(t+1) - \overline{\Gamma}\widehat{z}(t) - \overline{\overline{E}}u(t)\right)\end{aligned} \quad (34)$$

and the associated GDARE

$$P = \overline{\Omega}P\overline{\Omega}^T - \left(\overline{\Omega}P\overline{\Gamma}^T + S\right)\left(\overline{\Gamma}P\overline{\Gamma}^T + \Phi\right)^{\dagger}\left(\overline{\Omega}P\overline{\Gamma}^T + S\right)^T + \overline{Q}$$

with

$$\ker\left(\overline{\Gamma}P\overline{\Gamma}^T + \Phi\right) \subset \ker\left(\overline{\Omega}P\overline{\Gamma}^T + S\right).$$

Define the following rational matrix

$$\mathcal{F}(z) = \left[\overline{\Gamma}\left(z^{-1}I_r - \overline{\Omega}\right)^{-1} \ I_{p-r_\theta}\right] \begin{bmatrix}\overline{Q} & S\\ S^T & \Phi\end{bmatrix} \begin{bmatrix}\left(zI_r - \overline{\Omega}^T\right)^{-1}\overline{\Gamma}^T\\ I_{p-r_\theta}\end{bmatrix}.$$

It is easy to see that $\overline{Q}, S$ and Φ satisfy the condition of positive semi-definiteness as follows

$$\begin{bmatrix}\overline{Q} & S\\ S^T & \Phi\end{bmatrix} = \begin{bmatrix}M_1\\ M_2\end{bmatrix}\begin{bmatrix}M_1^T & M_2^T\end{bmatrix}$$

where

$$\begin{bmatrix}M_1\\ M_2\end{bmatrix} = \begin{bmatrix}L - J_1 C & -J_1\\ J_2 C & J_2\end{bmatrix}\begin{bmatrix}Q^{1/2} & 0\\ 0 & R^{1/2}\end{bmatrix}.$$

Matrix $\left[\begin{smallmatrix}L - J_1 C & -J_1\\ J_2 C & J_2\end{smallmatrix}\right]$ is of full row rank, this can be seen from

$$\begin{aligned}\operatorname{rank}\begin{bmatrix}L - J_1 C & -J_1\\ J_2 C & J_2\end{bmatrix} &= \operatorname{rank}\begin{bmatrix}L - J_1 C & -J_1\\ J_2 C & J_2\end{bmatrix}\begin{bmatrix}I_n & 0\\ -C & I_p\end{bmatrix}\\ &= \operatorname{rank}\begin{bmatrix}L & -J_1\\ 0 & J_2\end{bmatrix}\begin{bmatrix}I_n & 0 & 0\\ 0 & \Theta & I_p - \Theta\Theta^{\dagger}\end{bmatrix}\\ &= \operatorname{rank}\begin{bmatrix}L & -\Omega & 0\\ 0 & 0 & J_2\end{bmatrix} = r + p - r_\theta\end{aligned}$$

since J_2 is of full row rank.

Definition 10. The normal rank of the matrix $\mathcal{F}(z)$ is the rank of $\mathcal{F}(z_0)$ where z_0 is any complex number which is not a zero of $\mathcal{F}(z)$. ■

From (Kučera, 1991; Saberi *et al.*, 1995), we have the following lemma.

Lemma 11. $\mathcal{F}(z)$ has full normal rank if and only the matrix $(\overline{\Gamma}P\overline{\Gamma}^T + \Phi)$ is invertible. ■

Under the assumption that $\mathcal{F}(z)$ has full normal rank, the GDARE becomes

$$P = \overline{\Omega}P\overline{\Omega}^T - \left(\overline{\Omega}P\overline{\Gamma}^T + S\right)\left(\overline{\Gamma}P\overline{\Gamma}^T + \Phi\right)^{-1}\left(\overline{\Omega}P\overline{\Gamma}^T + S\right)^T + \overline{Q}$$

and the gain matrix K is then given by
$$K = \left(\overline{\Omega} P \overline{\Gamma}^T + S\right)\left(\overline{\Gamma} P \overline{\Gamma}^T + \Phi\right)^{-1}.$$

We can introduce the following definition for the strong stabilizability of the quadruple $(\overline{\Omega}, M_1, \overline{\Gamma}, M_2)$.

Definition 12. The quadruple $(\overline{\Omega}, M_1, \overline{\Gamma}, M_2)$ is said to be strongly stabilizable if the pair $(\overline{\Omega} - \Lambda\overline{\Gamma}, M_1 - \Lambda M_2)$ is stabilizable for every matrix Λ. ■

We have the following lemma.

Lemma 13. The quadruple $(\overline{\Omega}, M_1, \overline{\Gamma}, M_2)$ is strongly stabilizable if and only if
$$\text{rank}\begin{bmatrix} AH_1 - \lambda H_1 & -F & Q^{1/2} & 0 \\ -\lambda CH_1 & G & 0 & R^{1/2} \end{bmatrix} = n + p,$$
$$\forall \lambda \in \mathbb{C}, |\lambda| \geqslant 1.$$
■

From (Kučera, 1991; Saberi *et al.*, 1995) and the above results we have the following theorem.

Theorem 14. The gain matrix
$$K = \left(\overline{\Omega} P \overline{\Gamma}^T + S\right)\left(\overline{\Gamma} P \overline{\Gamma}^T + \Phi\right)^{-1}$$
is the unique gain which stabilizes the filter (34) and minimizes (4) if and only if

(i) $\mathcal{F}(z)$ has full normal rank,

(ii) $\text{rank}\begin{bmatrix} \Xi \\ \hline LA \;\; LF \;\; 0 \end{bmatrix} = \text{rank}\,\Xi$,

(iii) $\text{rank}\begin{bmatrix} \lambda L - LA & -LF & 0 \\ CA & CF & G \end{bmatrix} = \text{rank}\,\Xi, \forall \lambda \in \mathbb{C}, |\lambda| \geqslant 1$,

(iv) $\text{rank}\begin{bmatrix} AH_1 - \lambda H_1 & -F & Q^{1/2} & 0 \\ -\lambda CH_1 & G & 0 & R^{1/2} \end{bmatrix} = n + p$, $\forall \lambda \in \mathbb{C}, |\lambda| \geqslant 1$. ■

Now from the definitions of $\overline{\Omega}$, J, E and $\overline{\Gamma}$, the filter (34) can be written as
$$\hat{z}(t+1) = LAH_1\hat{z}(t) + LBu(t) + K\nu_r(t+1) + \mu(t+1)$$

where
$$\begin{aligned} \nu_r(t+1) &= J_2 y(t+1) - \overline{\Gamma}\hat{z}(t) - \overline{\overline{E}}u(t) \\ \mu(t+1) &= J_1\left(y(t+1) - CAH_1\hat{z}(t) - CBu(t)\right). \end{aligned}$$

In the following we shall study the properties of the sequences $\nu_r(t)$ and $\mu(t)$.

2.2.3. *Properties of the sequences $\nu_r(t)$ and $\mu(t)$*
In this section we discuss interesting properties of the sequences $\nu_r(t)$ and $\mu(t)$.

From the definitions of $\overline{\Gamma}$ and $\hat{z}(t)$ we obtain
$$\nu_r(t+1) = \overline{\Gamma}e(t) + \begin{bmatrix} J_2C & J_2 \end{bmatrix}\begin{bmatrix} w(t) \\ v(t+1) \end{bmatrix}$$

Then $\nu_r(t)$ is a zero mean sequence.

Now by using (24) we obtain the following recurrence
$$\begin{bmatrix} \nu_r(t+1) \\ e(t+1) \end{bmatrix} = \begin{bmatrix} 0 & \overline{\Gamma} \\ 0 & \overline{\Omega} - K\overline{\Gamma} \end{bmatrix}\begin{bmatrix} \nu_r(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} J_2C & J_2 \\ L - (J_1 + KJ_2)C & -(J_1 + KJ_2) \end{bmatrix}\begin{bmatrix} w(t) \\ v(t+1) \end{bmatrix}$$

and for the covariance matrix
$$\vartheta(t) = \mathcal{E}\left\{\begin{bmatrix} \nu_r(t) \\ e(t) \end{bmatrix}\begin{bmatrix} \nu_r^T(t) & e^T(t) \end{bmatrix}\right\} = \begin{bmatrix} \vartheta_{11}(t) & \vartheta_{12}(t) \\ \vartheta_{12}(t)^T & \vartheta_{22}(t) \end{bmatrix}$$

we have
$$\begin{aligned} \vartheta(t+1) = {} & \begin{bmatrix} 0 & \overline{\Gamma} \\ 0 & \overline{\Omega} - K\overline{\Gamma} \end{bmatrix}\vartheta(t)\begin{bmatrix} 0 & \overline{\Gamma} \\ 0 & \overline{\Omega} - K\overline{\Gamma} \end{bmatrix}^T \\ & + \begin{bmatrix} J_2C & J_2 \\ L - (J_1 + KJ_2)C & -(J_1 + KJ_2) \end{bmatrix}\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \\ & \times \begin{bmatrix} J_2C & J_2 \\ L - (J_1 + KJ_2)C & -(J_1 + KJ_2) \end{bmatrix}^T \end{aligned}$$

which leads to
$$\vartheta_{12}(t+1) = S^T - (\Phi + \overline{\Gamma}\vartheta_{22}(t)\overline{\Gamma}^T)K^T + \overline{\Gamma}\vartheta_{22}(t)\overline{\Omega}^T.$$

As in (Kwakernaak and Sivan, 1972), by using the optimal solution of the gain matrix K and the fact that $\vartheta_{22}(t) = P(t)$, we obtain $\vartheta_{12}(t+1) = 0$ and $\vartheta_{11}(t+1) = \overline{\Gamma}P(t)\overline{\Gamma}^T + \Phi$, which prove that the sequence $\nu_r(t)$ is white.

Now we have
$$\begin{aligned} \mu(t+1) &= J_1\left(y(t+1) - CAH_1\hat{z}(t) - CBu(t)\right) \\ &= J_1CAH_1e(t) + \begin{bmatrix} LAE_1 & LF \end{bmatrix}\begin{bmatrix} T_1x(t) \\ d(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} J_1C & J_1 \end{bmatrix}\begin{bmatrix} w(t) \\ v(t+1) \end{bmatrix} \end{aligned}$$

with
$$\mathcal{E}\{\mu(t+1)\} = \begin{bmatrix} LAE_1 & LF \end{bmatrix}\begin{bmatrix} T_1x(t) \\ d(t) \end{bmatrix}$$

and
$$\mathcal{E}\left\{\mu(t+1)\mu^T(t+1)\right\} = J_1\left(C\left(AH_1PH_1^TA^T + Q\right)C^T + R\right)J_1^T.$$

One can see that $\mathcal{E}\{\mu(t+1)\}$ is function of the state $x(t)$ and the unknown input $d(t)$. If $L = I_n$, then $\mathcal{E}\{\mu(t+1)\} = Fd(t)$, which is only function of $d(t)$. This result shows that $\mu(t+1)$ is a pseudo-innovation sequence, since it is not white. This sequence can be used to detect a possible failure represented by $d(t)$ if $L = I_n$.

3. CONCLUSION

In this paper, a new functional reduced order filtering design method for stochastic discrete-time time-varying systems with unknown inputs is proposed. The obtained results generalize those presented in (Kitanidis, 1987; Nagpal *et al.*, 1987; Darouach and Zasadzinski, 1997). The conditions for the existence of the filter are given, its properties in the time-invariant case are studied and the necessary and sufficient conditions for the convergence and stability are derived.

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