15th Triennial World Congress, Barcelona, Spain

STABILITY OF TWO–VARIABLE INTERVAL POLYNOMIALS VIA POSITIVITY

Dragoslav D. Šiljak * **and Dušan M. Stipanović** **

* *Santa Clara University, Santa Clara CA 95053, USA*
** *Stanford University, Stanford CA 94305, USA*

Abstract: Stability criteria are proposed for two–variable (2D) polynomials having interval parameters in polynomic uncertainty structures. Both the left–half plane and unit circle domains are considered. Save for a minor condition, the criteria reduce robust stability testing of 2D polynomials to testing positivity of only two polynomials. The appealing feature of the new robustness criteria is that positivity testing can be carried out by using the efficient Bernstein minimization algorithms.

Keywords: Two–variable polynomials, interval parameters, stability, left–half plane, unit circle, positive polynomials, Bernstein expansion, time–delay systems.

1. INTRODUCTION

Stability of two–dimensional polynomials arises in fields as diverse as 2D digital signal and image processing (Huang, 1981; Dudgeon and Mersereau, 1984; Lim, 1990) and time–delay systems (Kamen, 1982), repetitive, or multipass, processes (Rogers and Owens, 1993) and target tracking in radar systems. For this reason, there have been a large number of stability criteria for 2D polynomials, which have been surveyed and discussed in a number of papers (Jury, 1986; Premaratne, 1993; Hu, 1994; Bistriz, 1999, 2000; Mastorakis, 2000). In achieving the maximal efficiency of 2D stability tests, the reduction of algebraic complexity offered by the stability criteria in (Šiljak, 1975) has been useful. Apart from some minor conditions, the criteria convert stability testing of a 2D polynomial to testing of only two 1D polynomials, one for stability and the other for positivity.

Due to inherent uncertainty of the underlying models, it has been long recognized that in practical applications it has been necessary to test robustness of stability to parametric variations (Šiljak, 1989). Almost exclusively, the 2D robust stability tests have been based on the elegant Kharitonov solution of the stability problem involving 1D interval polynomials (Bose and Zeheb, 1986; Basu, 1990; Rajan and Reddy, 1991; Kharitonov *et al.*, 1997; Xiao *et al.*, 1999). In the context of 2D polynomials, the solution lost much of its simplicity resulting in numerically involved algorithms. This fact made the testing of 2D polynomials with interval parameters difficult, especially in the case of multiaffine and polynomic uncertainty structures.

The purpose of this paper is to present new criteria for testing stability of 2D polynomials with interval parameters, which are based on the criteria of (Šiljak, 1975) and the positivity approach to the interval parametric uncertainties advanced in (Šiljak and Stipanović, 1999). An appealing feature of the new criteria is the possibility of using the efficient Bernstein minimization algorithms (Garloff, 1986; Malan *et al.*, 1992) to carry out the numerical part of the positivity tests. Furthermore, the proposed formulation can handle the polynomic uncertainty structures having interval parameters, and can be easily extended to systems with time–delays along the lines of (Kamen, 1982).

2. STABILITY CRITERION

Let us consider a real two–variable polynomial

$$h(s,z) = \sum_{j=0}^{n}\sum_{k=0}^{m} h_{jk} s^j z^k \qquad (1)$$

where $s, z \in \mathbb{C}$ are complex variables, and for some j, k the coefficients h_{jn} and h_{mk} are not both zero. We are interested in determining conditions under which the polynomial $h(x,z)$ satisfies the stability property

$$h(s,z) \neq 0, \quad \{s \in \mathbb{C}_-^c\} \cap \{z \in \mathbb{C}_-^c\}\ , \qquad (2)$$

where $\mathbb{C}_-^c$ is the complement of $\mathbb{C}_- = \{s \in \mathbb{C} : \text{Re } s < 0\}$, the open left half of the complex plane $\mathbb{C}$. As shown by Ansell (1964), property (2) is equivalent to

$$\begin{aligned} h(s,1) &\neq 0, \qquad \forall s \in \mathbb{C}_-^c \quad (3a) \\ h(i\omega,z) &\neq 0, \qquad \forall z \in \mathbb{C}_-^c \quad (3b) \end{aligned}$$

To test (3a) we can use the standard Routh test (*e.g.*, Lehnigh, 1966). To verify (3b) we follow Ansell's approach and consider the polynomial $c(z) = h(i\omega, z)$,

$$c(z) = \sum_{k=0}^{m} c_k z^k\ , \qquad (4)$$

where

$$c_k = \sum_{j=0}^{n} h_{jk} s^j \qquad (5)$$

and $s = i\omega$. With $c(z)$ we associate the symmetric $m \times m$ Hermite matrix $C = (c_{jk})$ with elements c_{jk} defined by (*e.g.*, Lehnigh, 1966)

$$c_{jk} = 2(-1)^{(j+k)/2} \sum_{\ell=1}^{j} (-1)^{\ell} \text{Re } (c_{m-\ell+1} \ \ \bar{c}_{m-j-k+\ell})\ , \quad (j+k) \text{ even}$$

$$c_{jk} = 2(-1)^{(j+k)/2} \sum_{\ell=1}^{j} (-1)^{\ell} \text{Im } (c_{m-\ell+1} \ \ \bar{c}_{m-j-k+\ell})\ , \quad (j+k) \text{ odd} \qquad (6)$$

where the overbar denotes conjugacy and $j \leq k$. We recall that $C > 0$ if and only if $c(z) = 0$ implies $z \in \mathbb{C}_-$. Since $C = C(i\omega)$ is a real symmetric matrix, we define a real even polynomial

$$g(\omega^2) = \det\ C(i\omega) \qquad (7)$$

and replace ω^2 by ω to get a polynomial $g(\omega)$. We also define a polynomial

$$f(s) = h(s, 1) \qquad (8)$$

and state the following (Šiljak, 1975):

Theorem 1. A two–variable polynomial $h(s,z)$ has the stability property (2) if and only if

(i) $f(s)$ is $\mathbb{C}_-$–stable.
(ii) $g(\omega)$ is $\mathbb{R}_+$–positive.
(iii) $C(0)$ is positive definite.

Conditions (i) means that $f(s) = 0$ if and only if $s \in \mathbb{C}_-$, while condition (ii) is equivalent to $g(\omega) > 0$ for all $\omega \geq 0$.

In stability analysis of recursive digital filters (*e.g.*, Huang, 1981; Dudgeon and Mersereau, 1984), it is of interest to establish necessary and sufficient conditions for a polynomial $h(s,z)$ to have the stability property

$$h(s,z) \neq 0, \quad \{s \in \bar{\mathbf{K}}^0\} \cap \{z \in \bar{\mathbf{K}}^0\} \qquad (9)$$

where $\mathbf{K} = \{s \in \mathbb{C} : |s| = 1\}$ is the unit circle, and $\bar{\mathbf{K}}^0 = \mathbf{K} \cup \mathbf{K}^0$ is the closure of $\mathbf{K}^0 = \{s \in \mathbb{C} : |s| < 1\}$. By following Huang (1972), one can show that (9) is equivalent to

$$\begin{aligned} h(s,0) &\neq 0, \qquad \forall s \in \bar{\mathbf{K}}^0 \quad (10a) \\ h(e^{i\omega},z) &\neq 0, \qquad \forall z \in \bar{\mathbf{K}}^0 \quad (10b) \end{aligned}$$

Condition (10a) means that $h(s,0) = 0$ if and only if $s \in \mathbf{K}^0$. To test condition (10b), we consider

$$d(z) = z^m h(e^{i\omega}, z^{-1}) \qquad (11)$$

which we write as a polynomial

$$d(z) = \sum_{k=0}^{m} d_k z^k\ , \qquad (13)$$

with coefficients

$$d_k = \sum_{j=0}^{n} h_{j,m-k} s^k, \qquad (14)$$

and $s = e^{i\omega}$. With the polynomial $d(z)$ we associate the Schur–Cohn $m \times m$ matrix $D = (d_{jk})$ specified by

$$d_{jk} = \sum_{\ell=1}^{j} (d_{m-j+\ell} \bar{d}_{m-k+\ell} - \bar{d}_{j-\ell} d_{k-\ell}), \qquad (14)$$

where $j \leq k$ (*e.g.*, Jury, 1982). The matrix $D(e^{i\omega})$ is a Hermitian matrix and we define

$$g(e^{i\omega}) = \det\ D(e^{i\omega}), \qquad (15)$$

where $g(\cdot)$ is a self–inversive polynomial. We also define the polynomial

$$f(s) = s^n h(s^{-1}, 0) \qquad (16)$$

and state the following (Šiljak, 1975):

Theorem 2. A two–variable polynomial $h(s,z)$ has the stability property (9) if and only if

(i) $f(s)$ is $\mathbf{K}^0$–stable.
(ii) $g(z)$ is $\mathbf{K}$–positive.
(iii) $D(1)$ is positive definite.

Condition (i) means that the polynomial $f(s)$ has all zeros inside the unit circle $\mathbf{K}$. Positivity of $g(z)$ on $\mathbf{K}$ can be verified by applying the methods of (Šiljak, 1973).

Finally, we show how the mixture of the two previous stability properties can be handled using the same tools. The desired property is defined as

$$h(s,z) \neq 0, \quad \{s \in \mathbb{C}_-^c\} \cap \{z \in \bar{\mathbf{K}}^0\}\ . \qquad (17)$$

By following Ansell (1964), one can show that this property is equivalent to

$$\begin{array}{lll} h(s,0) \neq 0, & \forall s \in \mathbb{C}_-^c & (18a) \\ h(i\omega, z) \neq 0, & \forall z \in \bar{\mathbf{K}}^0 & (18b) \end{array}$$

In this case, the polynomial $d(z)$ is defined as

$$d(z) = z^m h(i\omega, z^{-1}) \ , \tag{19}$$

which is used to obtain the polynomial $g(\cdot)$ *via* equations (12)–(15). From (18a), we get the polynomial

$$f(s) = h(s,0), \tag{20}$$

then define $\mathbf{I} = \{z \in \mathbb{C} : \text{Re } z = 0\}$ and arrive at (Šiljak, 1975):

Theorem 3. A two–variable polynomial $h(s,z)$ has the stability property (17) if and only if

(i) $f(s)$ is $\mathbb{C}_-$–stable.
(ii) $g(z)$ is $\mathbf{I}$–positive.
(iii) $D(0)$ is positive definite.

We note that $\mathbf{I}$–positivity of $g(z)$ can be reformulated as $\mathbb{R}_+$–positivity (see, Šiljak and Šiljak, 1998).

3. UNCERTAIN POLYNOMIALS

We are interested in studying stability properties of uncertain two–variable polynomials with polynomic uncertainty structures. A polynomial $h(s,z;p)$ is given as

$$h(s,z;p) = \sum_{j=0}^{n} \sum_{k=0}^{m} h_{jk}(p) s^j z^k, \tag{21}$$

where $h_{jk}(p)$ are polynomials themselves in uncertain parameter vector $p \in \mathbb{R}^r$. We assume that p resides in a box

$$\mathbf{P} = \{p \in \mathbb{R}^r : \ \ p_k \in [\underline{p}_k, \bar{p}_k], k \in \mathbf{r}\} \ . \tag{22}$$

We want to investigate the robust versions of stability properties defined in the preceding section. In the case of (2), for example, we are interested in testing the robust property

$$h(s,z;p) \neq 0, \ \ \{s \in \mathbb{C}_-^c\} \cap \{z \in \mathbb{C}_-^c\} \cap \{p \in \mathbf{P}\}. \tag{23}$$

To accommodate the uncertainty in $h(s,z;p)$ we define the polynomial families $\mathcal{F} = \{f(\cdot,p) : p \in \mathbf{P}\}$, $G = \{g(\cdot,p) : p \in \mathbf{P}\}$ and state a straightforward modification of Theorem 1.

Theorem 4. An uncertain two–variable polynomial $h(s,z;p)$ has the robust stability property (23) if and only if

(i) $\mathcal{F}$ is $\mathbb{C}_-$–stable.
(ii) $\mathcal{G}$ is $\mathbb{R}_+$–positive.
(iii) $C(0,p)$ is positive definite for all $p \in \mathbf{P}$.

Robust versions of the remaining two stability properties of the preceding section can be tested by Theorem 4 *via* bilinear transformation in pretty much the same way D–stability was tested in (Šiljak and Stipanović, 1999). We also note the structural similarity of Theorem 4 with theorems on robust SPR properties (Stipanović and Šiljak, 2001), which motivates the work presented next.

Condition (i) in Theorem 4 obviously means that all zeros of $f(s,p)$ lie in $\mathbb{C}_-$ for all $p \in \mathbf{P}$. To establish this type of robust stability *via* polynomial positivity, we define the magnitude function

$$\hat{f}(s,p) = f(s,p)\overline{f(s,p)} = \sum_{k=0}^{n} \sum_{j=0}^{n} a_k(p) \bar{a}_j(p) s^l \bar{s}^j \tag{24}$$

where overbar denotes conjugation. We note immediately that the magnitude function $\hat{f}(s,p) = |f(s,p)|^2$ is nonnegative for all $s \in \mathbb{C}$. This obvious fact is essential in the following development.

Let us form a family $\hat{\mathcal{F}} = \{\hat{f}(\cdot,p) : \ p \in \mathbf{P}\}$ and use the result of (Šiljak and Stipanović, 1999) to conclude that a family $\mathcal{F}$ is $\mathbb{C}_-$–stable if and only if the corresponding family $\hat{\mathcal{F}}$ is $\mathbf{I}$–positive, and $f(s,p')$ is $\mathbb{C}_-$–stable for some $p' \in \mathbf{P}$. Furthermore, from (7) it follows that positivity of det $C(0;p)$ is included in testing condition (ii) of Theorem 4. This means that to test condition (iii) of Theorem 4, it suffices to verify that $C(0;p^{"})$ is positive definite for some $p^{"} \in \mathbf{P}$. We finally arrive at

Theorem 5. An uncertain two–variable polynomial $h(s,z;p)$ has the robust stability property 23 if and only if

(i) $\hat{\mathcal{F}}$ is $\mathbb{R}_+$–positive and $f(s;p')$ is $\mathbb{C}_-$–stable for some $p' \in \mathbf{P}$.
(ii) $\mathcal{G}$ is $\mathbb{R}_+$–positive.
(iii) $C(0;p'')$ is positive definite for some $p'' \in \mathbf{P}$.

Example 1. To illustrate the application of Theorem 5, let us use the two–variable polynomial from (Xiao *et al.*, 1999),

$$h(s,z;p) = h_{11}(p)sz + h_{10}(p)s + h_{01}(p)z + h_{00}(p) \ , \tag{25}$$

where

$$\begin{aligned} h_{11}(p) &= 0.9 - 0.1p_1 - 0.3p_2 \\ h_{10}(p) &= 0.8 - 0.5p_1 + 0.3p_2 \\ h_{01}(p) &= 1 + 0.2p_1 + 0.3p_2 \\ h_{00}(p) &= 1.6 + 0.5p_1 - 0.7p_2 \end{aligned} \tag{26}$$

and

$$\mathbf{P} = \{p \in \mathbb{R}^2; \ \ p_1 \in [-0.3, 0.4], \ \ p_2 \in [0.1, 0.5]\} \ . \tag{27}$$

To test condition (i) we compute the polynomial

$$f(s;p) = (1.7 - 0.6p_1)s + 2.6 + 0.7p_1 - 0.4p_2 \tag{28}$$

and note that, in this simple case, we do not need to compute the corresponding magnitude function $\hat{f}(\omega;p)$. Robust $\mathbb{C}_-$–stability of $f(s;p)$ follows directly from positivity of its coefficients. Indeed,

$$\begin{aligned} 1.7-0.6p_1 &\geq 1.46 \\ 2.6+0.7p_1-0.4p_2 &\geq 2.19 \end{aligned} \tag{29}$$

for all $p \in \mathbf{P}$.

Since the matrix $C(i\omega;p)$ is a scalar, condition (iii) is included in (ii) which is satisfied because

$$\begin{aligned} g(\omega;p) &= (1+0.2p_1+0.3p_2)(1.6+0.5p_1-0.7p_2) \\ &\quad +(0.9-0.1p_1-0.3p_2) \\ &\quad \times(0.8-0.5p_1+0.3p_2)\omega \\ &\geq 0.4473\omega+1.0670 \end{aligned} \tag{30}$$

is obviously $\mathbb{R}_+$–positive.

Our analysis is elementary when compared to the stability testing procedure of Xiao *et al.* (1999), which involves extensive computation required by the Edge Theorem.

Let us consider more complex examples which well require the use of Bernstein's algorithm.

Example 2. A two–variable polynomial is given as

$$\begin{aligned} h(s,z;p) = s^2z^2 &+h_{21}(p)s^2z+h_{12}(p)sz^2+h_{20}(p)s^2 \\ &+h_{02}(p)z^2+h_{11}(p)sz \\ &+h_{10}(p)s+h_{01}(p)z+h_{00}(p) \end{aligned} \tag{31}$$

where

$$\begin{aligned} h_{21}(p) &= 3-p_1 \\ h_{20}(p) &= p_1p_2 \\ h_{11}(p) &= 3p_1p_2-p_1^2p_2 \\ h_{01}(p) &= 6-5p_1+3p_2-p_1p_2+p_1^2 \\ h_{12}(p) &= p_1p_2 \\ h_{02}(p) &= 2-p_1+p_2 \\ h_{10}(p) &= p_1^2p_2^2 \\ h_{00}(p) &= 2p_1p_2-p_1^2p_2+p_1p_2^2 \end{aligned} \tag{32}$$

and

$$\mathbf{P} = \{p \in \mathbb{R}^2 : \quad p_1 \in [1,2], \quad p_2 \in [1,2]\}. \tag{33}$$

The polynomial $f(s;p)$ is computed as

$$\begin{aligned} f(s;p) &= (4-p_1+p_1p_2)s^2+(4p_1p_2-p_1^2p_2+p_1^2p_2^2)s \\ &\quad +8-6p_1+4p_2+p_1p_2+p_1^2-p_1^2p_2+p_1p_2^2 \end{aligned} \tag{34}$$

The corresponding minimizing polynomial

$$\underline{f}(s) = 4s^2+4s+4 \tag{35}$$

is obtained by minimizing each coefficient using Bernstein's algorithm. Obviously, $\mathcal{F}$ is $\mathbb{C}_-$–stable since $\underline{f}(s)$ has positive coefficients.

Next, we compute

$$c(z;p) = c_2(p)z^2+c_1(p)z+c_0(p) \ , \tag{36}$$

where

$$\begin{aligned} c_2(p) &= 2-p_1+p_2-\omega^2+ip_1p_2\omega \\ c_1(p) &= 6-5p_1+3p_2-p_1p_2+p_1^2-3\omega^2+p_1\omega^2 \\ &\quad +i(3p_1p_2-p_1^2p_2)\omega \\ c_0(p) &= 2p_1p_2-p_1^2p_2-p_1p_2\omega^2+ip_1^2p_2^2\omega \ . \end{aligned} \tag{37}$$

In this case, the 2×2 matrix $C(i\omega;p)$ turns out to be a diagonal matrix

$$C(i\omega;p) = \text{diag}\ \{c_{11}(i\omega;p), \quad c_{22}(i\omega;p)\} \tag{38}$$

and conditions (ii) and (iii) of Theorem 5 reduce to positivity of the coefficients

$$\begin{aligned} c_{11}(i\omega;p) &= \tilde{c}_{11}(\omega^2;p) \\ &= (3-p_1)\omega^4+(-12+10p_1-6p_2+2p_1p_2 \\ &\quad -2p_1^2+3p_1^2p_2^2-p_1^3p_2^2)\omega^2 \\ &\quad +12-16p_1+12p_2-10p_1p_2 \\ &\quad +7p_1^2+3p_2^2-p_1p_2^2 \\ &\quad +2p_1^2p_2-p_1^3 \\[1ex] c_{22}(i\omega;p) &= \tilde{c}_{22}(\omega^2;p) \\ &= (3p_1p_2-p_1^2p_2)\omega^4 \\ &\quad +(-12p_1p_2+10p_1^2p_2-6p_1p_2^2 \\ &\quad +2p_1^2p_2^2-2p_1^3p_2+3p_1^3p_2^3-p_1^4p_2^3)\omega^2 \\ &\quad +12p_1p_2-16p_1^2p_2+12p_1p_2^2 \\ &\quad -10p_1^2p_2^2+7p_1^3p_2+3p_1p_2^3-p_1^2p_2^3 \\ &\quad +2p_1^3p_2^2-p_1^4p_2 \ . \end{aligned} \tag{39}$$

By using Bernstein's minimization algorithm we compute the minorizing polynomials (Šiljak and Stipanović, 1999) and establish positivity of the two polynomials $c_{11}(i\omega;p)$ and $c_{22}(i\omega;p)$ by obtaining the minima

$$\begin{aligned} \min_{\substack{p\in\mathbf{P}\\ \omega\in\mathbb{R}_+}} \tilde{c}_{11}(\omega;p) &= 1.3724 \text{ at } \ p_1=2, \quad p_2=1, \\ &\qquad \omega = 0.1715 \\ \min_{\substack{p\in\mathbf{P}\\ \omega\in\mathbb{R}_+}} \tilde{c}_{22}(\omega;p) &= 3.1185 \text{ at } \ p_1=2, \quad p_2=1, \\ &\qquad \omega = 0.2487 \ . \end{aligned} \tag{40}$$

Positivity of the minima implies robust stability property 25 for the polynomial $h(s,z;p)$ of 31.

4. TIME–DELAY SYSTEMS

Our objective in this section is to show how the tools presented in this chapter can be applied to test robust stability of linear systems of the retarded type described by a differential–difference equation

$$x^{(n)}(t)+\sum_{j=0}^{n-1}\sum_{k=0}^{m}h_{jk}(p)x^{(j)}(t-k\tau) = 0 \ , \tag{41}$$

where $\tau \geq 0$. The coefficients $h_{jk}(p)$ are polynomials in the uncertain parameter vector $p \in \mathbb{R}^\ell$ which belongs to a box $\mathbf{P}$.

It is well known (Bellman and Cooke, 1963) that for a fixed parameter p, a system (41) is stable if and only if

$$h(s,e^{-\tau s};p) = s^n+\sum_{j=0}^{n-1}\sum_{k=0}^{m}h_{jk}(p)s^je^{-k\tau s} \neq 0 \ , \qquad \text{Re } s \geq 0 \ . \tag{42}$$

The system is robustly stable if (42) holds for all $p \in \mathbf{P}$.

The following theorem is a straightforward robustification of a theorem by Kamen (1983):

Theorem 6. System (41) is robustly stable independent of delay if

$$h(s,z;p) \neq 0, \quad \{s \in \mathbb{C}_-^c\} \cap \{z \in \mathbf{K}\} \cap \{p \in \mathbf{P}\} \ . \tag{43}$$

This condition is also necessary if

$$h(0,z;p) \neq 0, \quad \{z \in \mathbf{K}\} \cap \{p \in \mathbf{P}\} \ . \tag{44}$$

To test condition (43) we first use the bilinear transformation

$$z = \begin{cases} \dfrac{1+i\omega}{1-i\omega} \ , & \omega \in \mathbb{R} \text{ when } z \in \mathbf{K}\backslash\{-1\} \\ -1 \ , & z = -1 \end{cases} \tag{45}$$

to define the polynomials

$$\begin{aligned} \tilde{h}(s,i\omega;p) &= (1-i\omega)^m h(s,\frac{1+i\omega}{1-i\omega};p) \\ f(s;p) &= h(s,-1;p) \ . \end{aligned} \tag{46}$$

Then, by following Ansell's approach in, we consider the polynomial $c(s;p) = \tilde{h}(s,i\omega;p)$,

$$c(s;p) = \sum_{j=0}^{n} c_j(p) s^j \tag{47}$$

where

$$c_j(p) = \sum_{k=0}^{n} \tilde{h}_{jk}(p)(i\omega)^k \ . \tag{48}$$

With $c(s;p)$ we associate the symmetric $n \times n$ Hermite matrix $C = (c_{jk})$ having elements c_{jk} defined in (6), and obtain the polynomial

$$g(\omega^2;p) = \det\, C(i\omega;p) \ . \tag{49}$$

Finally, with polynomials $f(s;p)$ and $g(\omega^2;p)$ at hand, we can imitate Theorem 5 to state the following:

Theorem 7. System (41) is robustly stable independent of delay if

(i) $\hat{\mathcal{F}}$ is $\mathbb{R}_+$–positive and $f(s;p')$ is $\mathbb{C}_-$–stable for some $p' \in \mathbf{P}$.
(ii) $\mathcal{G}$ is $\mathbb{R}_+$–positive.
(iii) $C(0;p'')$ is positive definite for some $p'' \in \mathbf{P}$.

It is obvious that condition (44) of Theorem 6, which is included in (45), can be tested *via* positivity as well.

To illustrate the application of Theorem 7 let us use the following:

Example 3. A time–delay system (41) is given as

$$\begin{aligned} x^{(2)}(t) + p_2 x^{(1)}(t-\tau) + p_1 x(t-\tau) + x^{(1)}(t) \\ +(1+p_1p_2^2)x(t) = 0 \end{aligned} \tag{50}$$

with the uncertainty box

$$\mathbf{P} = \{p \in \mathbb{R}^2 : \ p_1 \in [-0.5, 0.5], \ p_2 \in [-0.5, 0.5]\} \ . \tag{51}$$

From (50), we compute the associated polynomial

$$h(s,z;p) = s^2 + (p_2 s + p_1)z + s + 1 + p_1 p_2^2 \ , \tag{52}$$

and test first the necessity of condition (43) by checking condition (44). Since

$$h(0,z;p) = p_1 z + 1 + p_1 p_2^2 \tag{53}$$

and $1 + p_1p_2^2 > |p_1|$, we conclude that (44) is satisfied. This implies that condition (43) is necessary and sufficient for robust stability of system (50), and we proceed to compute the polynomial

$$f(s;p) = s^2 + (1-p_2)s + 1 - p_1 + p_1 p_2^2 \ . \tag{54}$$

To test robust stability of this polynomial we do not need to construct the family $\hat{\mathcal{F}}$. It suffices to check positivity of each coefficient, which we do by using the Bernstein algorithm. The resulting minorizing polynomial

$$\underline{f}(s) = s^2 + 0.5s + 0.5 \tag{55}$$

implies robust stability of $f(s;p)$, that is, condition (i) of Theorem 7 is satisfied.

For testing condition (ii) we need the polynomial

$$\begin{aligned} \tilde{h}(s,i\omega;p) = & (1-i\omega)s^2 + [1 + p_2 + (-1+p_2)i\omega]s \\ & +1 + p_1 + p_1p_2^2 \\ & +(-1 + p_1 - p_1p_2^2)i\omega \ . \end{aligned} \tag{56}$$

Using equations (47)–(49), we compute

$$\begin{aligned} g(\omega;p) = & 4(1 - p_1 - 2p_2 + 2p_2^2 + 2p_1p_2 \\ & -2p_1p_2^3 + p_1p_2^4)\omega^2 \\ & +8(1 - 2p_1^2 - p_2^2 + p_1p_2^2 - p_1p_2^4)\omega \\ & +4(1 + p_1 + 2p_2 + 2p_1p_2 + p_2^2 + 2p_1p_2^2 \\ & +2p_1p_2^3 + p_1p_2^4. \end{aligned} \tag{57}$$

By applying the Bernstein algorithm to each coefficient of $g(\omega;p)$, we obtain the minorizing polynomial

$$\underline{g}(\omega) = 0.625\omega^2 + 1.25\omega + 0.375 \ , \tag{58}$$

which is clearly $\mathbb{R}_+$–positive, and (ii) of Theorem 7 is satisfied.

Finally, the matrix of condition (iii) is computed as $C(0,0) = 2I_2$, where I_2 is the identity matrix of dimension 2, and robust stability independent of delay of system (50) is established with respect to the uncertainty box $\mathbf{P}$ in (51).

5. CONCLUSIONS

We have shown how stability of 2D polynomials with interval parameters can be tested *via* polynomial positivity. To test stability of polynomials with multiaffine and polynomic uncertainty structures, positivity of only two interval polynomials is required. A remarkable efficiency of the proposed stability criteria is due to their suitability for applications of Bernstein's expansion algorithms.

ACKNOWLEDGMENT

The research reported herein has been supported by the National Science Foundation under the grant ECS-0099469.

REFERENCES

Ansell, H.G. (1964). On certain two-variable generalizations of circuit theory with applications to networks of transmission lines and lumped reactances. *IEEE Transactions on Circuit Theory, CT–11*, 214–223.

Basu, S. (1990). On boundary implications of stability and positivity properties of multidimensional systems. *Proceedings of IEEE, 78*, 614–626.

Bistritz, Y. (1999). Stability testing of two-dimensional discrete linear system polynomials by a two-dimensional tabular form. *IEEE Transactions on Circuits and Systems, Part 1, 46*, 666-667.

Bistritz, Y. (2000). Immitance-type tabular stability for 2-D LSI systems based on a zero location test for 1-D complex polynomials. *Circuits, Systems and Signal Processing, 19*, 245–265.

Bose, N. K., and E. Zeheb (1986). Kharitonov's theorem and stability test of multidimensional digital filters. *IEE Proceedings, 133*, 187–190.

Dudgeon, D., and R. Mersereau (1984). *Multidimensional Digital Signal Processing.* Englewood Cliffs, NJ: Prentice–Hall.

Garloff, J. (1993). Convergent bounds for the range of multivariable polynomials, *Lecture Notes in Computer Science* (Vol. 212, pp. 37–56). Berlin: Springer.

Hu, X. (1994). Techniques in the stability testing of discrete time systems. In *Control and Dynamic Systems* (Vol. 66, pp. 153–216). C. T. Leondes (Ed.). San Diego, CA: Academic Press.

Huang, T. S. (1981). *Two-Dimensional Digital Signal Processing I.* Berlin: Springer.

Jury, E. I. (1982). *Inners and Stability of Dynamic Systems.* (2nd ed.). Melbourne, FL:Kreiger.

Jury, E. I. (1986). Stability of multidimensional systems and related problems. In *Multidimensional systems, Techniques and Applications* (pp. 89–159). S. G. Tzafestas (ed.). New York: Marcel Dekker.

Kamen, E. W. (1982). Linear systems with commensurate time delays: Stability and stabilization independent of delay. *IEEE Transactions on Automatic Control, 4*, 367–375.

Lehnigh, S. H. (1966). *Stability Theorems for Linear Motions.* Englewood Cliffs, NJ: Prentice–Hall.

Lim, J. S. (1990). *Two-Dimensional Signal and Image Processing.* Englewood Cliffs, NJ: Prentice–Hall.

Malan, S., M. Milanese, M. Taragna and J. Garloff (1992). B^3 algorithm for robust performance analysis in presence of mixed parametric and dynamic perturbations, *Proceedings of the 31st IEEE Conference on Decision and Control* (pp. 128–133).

Marden, M. (1966). *Geometry of Polynomials.* Providence, RI: American Mathematical Society.

Mastorakis, N. E. (2000). New necessary stability conditions for 2-D systems. *IEEE Transactions on Circuits and Systems, Part 1, 47*, 1103–1105.

Premaratne, K. (1993). Stability determination of two-dimensional discrete-time systems. *Multidimensional Systems and Signal Processing, 4*, 331–354.

Rajan, P. K., and H. C. Reddy (1991). A study of two-variable very strict Hurwitz polynomials with interval coefficients. *Proceedings of the 1991 IEEE International Symposium on Circuits and Systems* (pp. 1093–1096).

Rogers, E., and D. H. Owens (1993). Stability tests and performance bounds for a class of 2D linear systems. *Multidimensional Systems and Signal Processing, 4*, 355–391.

Šiljak, D. D. (1975). Stability criteria for two-variable polynomials. *IEEE Transactions on Circuits and Systems, CAS–22*, 185–189.

Šiljak, D. D. (1989). Parameter space methods for robust control design: A guided tour. *IEEE Transactions on Automatic Control, 34*, 674–688.

Šiljak, D. D., and D. M. Stipanović (1999). Robust *D*-stability *via* positivity. *Automatica, 35*, 1477–1484.

Šiljak, D. D. (YEAR) Algebraic criteria for positive realness relative to the unit circle. *Journal of the Franklin Institute, 296*, 115–122.

Stipanović, D. M., and D. D. Šiljak (2001). SPR criteria for uncertain rational matrices *via* polynomial positivity and Bernstein's expansions. *IEEE Transactions of Circuits and Systems, Part I* (to appear).

Xiao, Y., R. Unbehauen and X. Du (1999). Robust Hurwitz stability conditions of polytopes of bivariate polynomials. *Proceedings of the 38th Conference on Decision and Control* (pp. 5030–5035).