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A ROLLING HORIZON STATE ESTIMATOR WITH CONSTRAINT HORIZON ONE

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Abstract: This paper describes a method for constrained state estimation based on receding horizon optimization. The case here studied corresponds to an optimization horizon of size two and a constraint horizon of size one. It is shown that, in this case, a simple *closed-form* solution can be obtained. The resulting estimator is called a Rolling Horizon Estimator with Constraint Horizon One. It is shown that this estimator is analogous to a class of anti-windup control algorithms. Simulation results confirm the merits of using this scheme for state estimation in the presence of state constraints.

Keywords: constraints, estimators, Kalman filter, state estimation, windup

1. INTRODUCTION

It is well known that *unconstrained* state estimation for linear systems is elegantly solved by the Kalman filter and related schemes (Anderson *et al.* [1979]). Recent research has focused on the problem of adding *constraints* to the estimation problem (Muske *et al.* [1993], Muske *et al.* [1995], Robertson *et al.* [1996], Rao *et al.* [2001]). A particularly attractive idea (Muske *et al.* [1993]) for *constrained* state estimation is to formulate it as a receding horizon optimization problem with quadratic cost. This keeps the size of the problem constant and allows standard Quadratic Programming (QP) methods to be utilized for its solution.

An important ingredient in constrained state estimation via a receding horizon formulation is the "entry estimate" (Başar *et al.* [1995], Verdu *et al.* [1987]). It has been shown that this estimate plays an important role in the accuracy and stability of the estimator. Several ways of defining the entry estimate have been explored in Rao *et al.* [2001]. One of the more appealing of these strategies is to simply utilize the estimate $\hat{x}_{k-L}$ obtained as the final estimate from an earlier block of data as the entry estimate when dealing with the block of data from $(k - L + 1)$ to k. This idea seems to be a natural choice, in the sense that, in the absence of constraints, the scheme simply reduces to the Kalman Filter. To implement this idea requires that L prior estimates be stored. The idea is illustrated graphically in Figure 1 for the case $L = 3$.

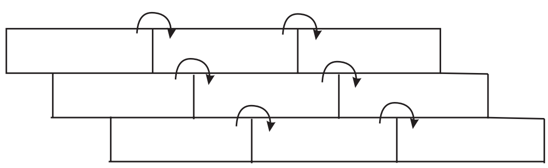

Fig. 1. Entry estimates given by early block estimates

This idea is captured in the terminology "Rolling Horizon Estimation". Note that the key idea is to store past estimates to be used as initial estimates in future optimization blocks. This formulation

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leads to the natural question: "Is it possible, or sensible, to use estimator blocks with constraint horizon of length one?" Clearly this would not make sense if past estimates were discarded but it seems to have some merit when past estimates are stored as initial estimates for the next block. Here, we call this class of estimators, "Rolling Horizon Estimators with Constraint Horizon One" or CH1 for short.

We show that the CH1 estimator is closely related to standard schemes used for anti-windup control. This creates and interesting connection between constrained estimation and widely used methods for constrained control.

The remainder of the paper proceeds as follows. In the next section we briefly describe the general case (arbitrary horizon) of the *rolling horizon* optimal state estimator. In Section 3 we develop the theory for the rolling horizon estimator with constraint horizon one (CH1) and derive the *analytical* solution for this simple case. In Section 4, the performance of the CH1 estimator is illustrated with a simulation example. Some connections of the CH1 estimator with anti-windup algorithms are discussed in Section 5, and in Section 6 some conclusions are drawn.

2. ROLLING HORIZON ESTIMATION

Consider a linear time invariant state space model of the form:

$$x_{k+1} = Ax_k + Bu_k + Dw_k \tag{1}$$

$$y_k = Cx_k + v_k \tag{2}$$

where $x_k \in \mathbf{R}^n$, $u_k \in \mathbf{R}^m$, $w_k \in \mathbf{R}^q$, $y_k \in \mathbf{R}^\ell, v_k \in \mathbf{R}^\ell$ are, respectively, the state, known deterministic input, process noise of covariance Q, output, and measurement noise of covariance R. The process and measurement noises are assumed to be uncorrelated.

2.1 *Unconstrained case*

The unconstrained best linear unbiased steady state estimate for x_k is well known to satisfy:

$$\hat{x}^*_{k+1|k} = A\hat{x}^*_{k|k} + Bu_k \tag{3}$$

$$\hat{x}^*_{k|k} = \hat{x}^*_{k|k-1} + K\left(y_k - C\hat{x}^*_{k|k-1}\right) \tag{4}$$

where

$$K = PC^T\left(CPC^T + R\right)^{-1} \tag{5}$$

and P satisfies the following Algebraic Riccati Equation

$$P = APA^T + Q - APC^T\left(CPC^T + R\right)^{-1} CPA^T \tag{6}$$

It is also well known that the estimate $\hat{x}^*_{k|k}$ can be determined, equivalently, by optimizing the following cost function w.r.t. $\{\hat{x}_{k-L+1}, \hat{w}_{k-L+1}, ..., \hat{w}_{k-1}\}$

$$\begin{aligned} J = & \left(\hat{x}_{k-L+1} - \hat{x}^*_{k-L+1|k-L}\right)^T \\ & P^{-1}\left(\hat{x}_{k-L+1} - \hat{x}^*_{k-L+1|k-L}\right) \\ & + \sum_{j=k-L+1}^{k} \left(y_k - C\hat{x}_k\right)^T R^{-1}\left(y_k - C\hat{x}_k\right) \\ & + \sum_{j=k-L+1}^{k-1} \hat{w}_k^T Q^{-1} \hat{w}_k \end{aligned} \tag{7}$$

where $\{\hat{x}_j\}$ and $\{\hat{w}_j\}$ satisfy the constraint

$$\hat{x}_{j+1} = A\hat{x}_j + Bu_j + D\hat{w}_j;\ j = k-L+1, ..., k-1 \tag{8}$$

We then set $\hat{x}^*_{k|k}$ as the *current* estimate $\hat{x}_k$ obtained from the above optimization problem.

Notice that the cost function (7) uses the prior estimate $\hat{x}^*_{k-L+1|k-L}$ as part of the data. The cost function can be motivated by maximum likelihood arguments in the case of Gaussian noise or simply as a convenient way to express the trade-off inherent in finding an estimate which matches the given observations $\{y_j;\ j = k-L+1,\ ...,\ k\}$ whilst not causing the associated estimates $\{\hat{w}_j;\ j = k-L+1,\ ..., k-1\}$ to be unreasonably large.

2.2 *Constrained case*

The cost function (7) together with the linear constraint (8) has also been suggested as a way of estimating the state of (1), (2) when the estimates are required to satisfy various *hard constraints*. Indeed, we may define $\hat{x}^L_{k-L+j|k}$ for $j = 1,\ ...\ L$ as the sequence $\left\{\hat{x}_{k-L+j};\ j = 1,\ ...L\right\}$ that minimizes (7) subject to (8) plus the additional constraints

$$\hat{w}_{k-L+j} \in \mathbb{W};\ \hat{x}_{k-L+j} \in \mathbb{X};\ j = 1,\ ...,\ L \tag{9}$$

where $\mathbb{W}$ and $\mathbb{X}$ are allowable constraint sets.

A remark is in order regarding the notation $\hat{x}^L_{k-L+j|k}$. The superscript refers to the length of the block of data being processed (that is, $y_{k-L+1}, \ldots, y_k$), the first component of the subscript denotes the sample time for the state and the second index denotes the end of the current block, i.e. the last data point available. Note that we use $\hat{x}^L_{k|k}$ as the *current* state estimate.

The minimization problem defined by (7), (8), (9) can be solved via Quadratic Programming techniques. An important ingredient in the cost function (7) is the, so called, *entry cost* which is the first term on the right hand side of (7). In the case of *unconstrained* estimation, this *entry cost* summarizes the data up to time $k-L$ via the pair $(\hat{x}^*_{k-L+1|k-L}, P)$. Indeed, in the case of Gaussian noise, $\hat{x}^*_{k-L+1|k-L}$ is a sufficient statistic for the past data. It is this fact which leads to the well known recursive form of the optimal estimator in the *unconstrained linear* case. Unfortunately, in the *constrained* case, it is not generally possible to have a finitely parameterized sufficient statistic for past data on the semi-infinite interval $(-\infty,\ k-L)$. In this case, we use the first term in (7) as an approximate way of capturing the *entry cost* via a quadratic function. However, this begs the question of how one should define $\hat{x}^*_{k-L+1|k-L}$ and P in the *constrained* case. Several options are discussed in Rao *et al.* [2001]. Arguably the most important ingredient is the *entry estimate* $\hat{x}^*_{k-L+1|k-L}$. Thus we take P to be the solution of (6) without further comment. Similarly, we could take $\hat{x}^*_{k-L+1|k-L}$ as the estimate provided by an *unconstrained* linear Kalman Filter. This would seem to be reasonable when the block length L is large since, under these conditions, the initial estimate will be *swamped* by the data from $k-L+1$ to k. However, this seems less reasonable in the case of small block lengths as we are effectively ignoring the constraints prior to time $k-L+1$.

Here, we will be interested in making the block length as short as possible. In part this is motivated by a desire to simplify the computations. However, we also want to make a connection with anti-windup control. For the short block case, it seems highly desirable that $\hat{x}^*_{k-L+1|k-L}$ is, in fact, a *constrained* estimate obtained from an earlier block of data. Indeed, this is the key recommendation made in Rao *et al.* [2001].

In the sequel, we take $u_k = 0$ since this deterministic input plays no part in the problem in view of the linearity of (1), (2).

3. THE CH1 ESTIMATOR

The Rolling Horizon Estimator with Constraint Horizon One is essentially as described in Section 2, save that the block size is taken equal to $L = 2$ and the constraint horizon is equal to one.

Since the system is time invariant, we can simplify the notation by setting $k = 1$ without loss of generality. The cost function and constraints then become:

$$\begin{aligned} J = &\left(\hat{x}_0 - \hat{x}^*_{0|-1}\right)^T P^{-1} \left(\hat{x}_0 - \hat{x}^*_{0|-1}\right) \qquad (10)\\ &+ (y_0 - C\hat{x}_0)^T R^{-1} (y_0 - C\hat{x}_0)\\ &+ (y_1 - C\hat{x}_1)^T R^{-1} (y_1 - C\hat{x}_1)\\ &+ \hat{w}_0^T Q^{-1} \hat{w}_0 \end{aligned}$$

where

$$\hat{x}_1 = A\hat{x}_0 + D\hat{w}_0 \qquad (11)$$

Combining (10) and (11), (and, assuming that A is invertible), gives a function that is quadratic in $\hat{x}_1, \hat{w}_0$ which can be written (modulo a constant) as:

$$J = \left[\hat{x}_1 - \hat{x}_1^{\mathrm{uc}} \quad \hat{w}_0 - \hat{w}_0^{\mathrm{uc}}\right] \begin{bmatrix} \Gamma & S \\ S^T & \Omega \end{bmatrix} \begin{bmatrix} \hat{x}_1 - \hat{x}_1^{\mathrm{uc}} \\ \hat{w}_0 - \hat{w}_0^{\mathrm{uc}} \end{bmatrix} \qquad (12)$$

where $\hat{x}_0^{\mathrm{uc}}$ and $\hat{w}_0^{\mathrm{uc}}$ are the *unconstrained* minimizing estimates. The latter quantities can be obtained from a standard linear Kalman Filter on the interval $[0,\ 1]$ with initial condition $\hat{x}^*_{0|-1}$. (Remember, however, that the latter estimate accounts for earlier constraints.) Thus, we have

$$\hat{x}_{0|0} = \hat{x}^*_{0|-1} + K\left[y_0 - C\hat{x}^*_{0|-1}\right] \qquad (13)$$

$$\hat{x}_{1|0} = A\hat{x}_{0|0} \qquad (14)$$

$$\hat{x}_1^{\mathrm{uc}} \triangleq \hat{x}_{1|1} = \hat{x}_{1|0} + K\left[y_1 - C\hat{x}_{1|0}\right] \qquad (15)$$

$$\begin{aligned} \hat{w}_0^{\mathrm{uc}} = &\left[D^T A^{-T} P_{1|0}^{-1} A^{-1} D + Q^{-1}\right]^{-1}\\ &D^T A^{-T} P_{1|0}^{-1}\left(\hat{x}_{0|0} - A^{-1}\hat{x}_{1|1}\right) \qquad (16) \end{aligned}$$

where $K \triangleq PC^T\left[CPC^T + R\right]^{-1}$ and $P_{1|0} \triangleq APA^T + Q$. Then,

$$\hat{x}_0^{\mathrm{uc}} \triangleq \hat{x}_{0|1} = A^{-1}\hat{x}_1^{\mathrm{uc}} - A^{-1}D\hat{w}_0^{\mathrm{uc}} \qquad (17)$$

In Seron *et al.* [2000], the geometry of the constrained quadratic optimization problem is exploited to obtain a finitely parameterized solution for the case of model predictive control of arbitrary *prediction* and *constraint horizons*. The *unconstrained* solution is first obtained analytically, then the variables are transformed such that the *constrained* optimum is given by the orthogonal projection into the allowed set. Finally, the space is partitioned into regions such that the orthogonal projection into the allowed variables attains the same closed-form expression for the constrained minimizer. The result is a controller which is finitely parameterized as a piece-wise affine function of the state. (See Seron *et al.* [2000] for the details.)

The same methodology can be used to add constraints to the block size two optimization problem described above. As in Seron *et al.* [2000], the first step is to carry out a transformation so as to turn the ellipsoidal cost contours in (12) into spheres. Thus, let

$$L^T L = \begin{bmatrix} \Gamma & S \\ S^T & \Omega \end{bmatrix} \qquad (18)$$

and perform the following change of variables

$$\hat{\eta}_0^{\mathrm{uc}} = L \begin{bmatrix} \hat{x}_1^{\mathrm{uc}} \\ \hat{w}_0^{\mathrm{uc}} \end{bmatrix} \qquad (19)$$

Then $\hat{\eta}_0^*$ (the *constrained* optimum in the transformed variables) is simply the closest point (in terms of the Euclidean norm) to $\hat{\eta}_0^{\mathrm{uc}}$ in the allowed region. The *constrained* optimum in the original variables is, thus

$$\begin{pmatrix} \hat{x}_1^* \\ \hat{w}_0^* \end{pmatrix} = L^{-1}\hat{\eta}_0^* \tag{20}$$

Another alternative in the case of the rolling horizon estimator of block size two, is to consider the simpler case of *constraint* horizon one. In fact, we have found that, for simple systems, the estimator with constraint horizon one performs remarkably better than the unconstrained estimator and very close to estimators with longer constraint horizons (we illustrate this with an example in Section 4).

Therefore, as a simple special case, we consider here the situation where the state is scalar and the constraint set is the interval $\mathbb{X} = [-\Delta, \Delta]$ and the disturbances are unbounded (e.g., they have a Gaussian distribution). Furthermore, we take the constraint horizon equal one, i.e., we only constrain $\hat{x}_1$ to be such that $|\hat{x}_1| \leq \Delta$. Notice that, with this choice, the actual estimate $\hat{x}_{k|k}^* \triangleq \hat{x}_k^*$ satisfies the constraints. Moreover, as shown below, this case has a very simple closed-form solution and, as discussed in Section 5, has close connections with anti-windup control schemes. For this simple case, it can be readily seen that the optimal *constrained* estimate $\hat{x}_1^*$ is

$$\hat{x}_1^* = \mathrm{sat}_\Delta\left\{\hat{x}_1^{\mathrm{uc}}\right\} \tag{21}$$

where $\mathrm{sat}_\Delta(\cdot)$ saturates $\hat{x}_1^{\mathrm{uc}}$ at $\pm\Delta$. Substituting back into (12) gives a quadratic function in $\hat{w}_0$. The latter function reaches its global minimum at $\hat{w}_0^*$ where

$$\hat{w}_0^* = \hat{w}_0^{\mathrm{uc}} + \Omega^{-1}S\left(\hat{x}_1^{\mathrm{uc}} - \hat{x}_1^*\right) \tag{22}$$

We can then propagate $\hat{x}_1^*$ to the next time instant to generate

$$\hat{x}_{2|1}^* = A\hat{x}_1^* + D\hat{w}_{1|1}^*;\ \hat{w}_{1|1}^* \equiv 0 \tag{23}$$

Here we have used the fact that the best estimate of w_1 given data up to time 1 is zero. The estimate $\hat{x}_{2|1}^*$ is then stored ready for the next block processing. Of course, there are two interlaced estimators required in this case, so the next update begins with $\hat{x}_{1|0}^*$ which would have been previously stored. (This form of propagating past information, which is illustrated in Figure 1 for the case $L = 3$, is called here *Rolling Horizon technique.*)

4. SIMULATION EXAMPLE

To investigate the performance of the CH1 estimator, discussed in the previous section, as compared with the Kalman filter and with a Long Horizon estimator, we consider the following simple model

$$x_{k+1} = \begin{cases} 0.8x_k + w_k\,; & \text{if } |0.8x_k + w_k| \leq \Delta \\ \Delta \times \mathrm{sgn}(0.8x_k + w_k)\,; & \text{otherwise} \end{cases}$$
$$y_k = x_k + v_k$$

where w_k, v_k are stationary, independent Gaussian white noises of variance Q and R, respectively. The noise variance is taken as $Q = 1$ and the value of R is varied from 0 to 200.

Note that in our model, the state x_k evolves linearly until it hits a barrier ($\pm\Delta$), in which case it remains saturated until future noise values take it inside the linear range. (This is a simplified model of a linear system with nonlinear "overflow" effects.) In Figure 2 we show the Mean Square Error of three different estimators:

(i) Kalman Filter, which satisfies (13)–(17);

(ii) CH1, i.e., with block size $L = 2$ and constraint horizon 1. In this case the estimator is given by the *analytical* expressions (13)–(17) and (21)–(23);

(iii) Long horizon estimator, with block size $L = 8$ and constraint horizon 8. In this last case, the optimization (7) subject to (8) and $|x_{k-L+j}| \leq \Delta$, $j = 1, \ldots, L$ is performed via a *numerical* quadratic programme (QP) routine. (No further performance improvements were obtained for horizons longer that 8.)

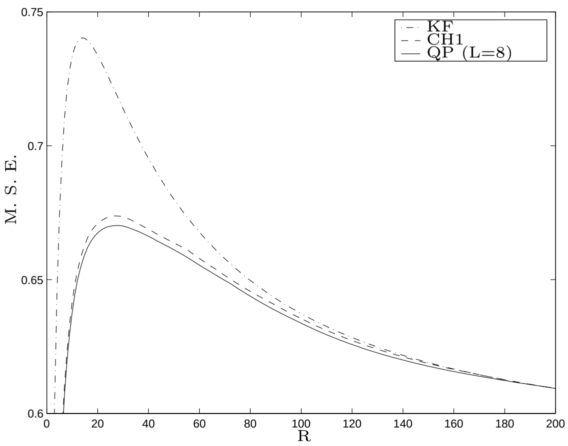

Fig. 2. Mean Square Error vs. Noise Variance R for: Kalman Filter, Rolling Horizon with Constraint Horizon One, and Long Horizon QP estimator with $L = 8$.

The three curves shown in Figure 2 pass through the origin. Notice, from Figure 2, that the CH1 estimator performs remarkably close to the Long Horizon estimator (specially when both are compared to the unconstrained Kalman Filter estimator). Notice also that, as R increases, the three estimators tend to perform similarly. This is due to the fact that, as $R \to \infty$, the Kalman gain in (5)

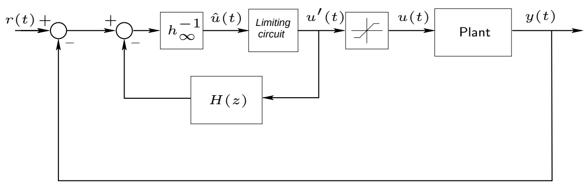

Fig. 3. A typical anti-windup control scheme

tends to $K \to 0$ and, hence, the state estimate $\hat{x}_k^{\text{uc}}$ becomes small and never reaches the constraints.

5. RELATION TO ANTI-WINDUP CONTROL

The scheme described in the last section reduces to the Kalman Filter when the saturation function in (21) is removed. Thus, the only *nonlinear element* in the constrained state estimator is a simple saturation function. A crucial feature of equations (21) and (23) is that the effect of the saturation is *remembered* via the state $\hat{x}^*_{2|1}$ which is passed onto the next block. We see that this is analogous to the standard anti-windup circuit (Goodwin *et al.* [2001]) shown in Figure 3 where current control saturations are *remembered* by the control law for future use.

In this context, we recall recent work reported in De Doná *et al.* [2000], which shows that anti-windup control is closely related to receding horizon control with constraint horizon one. Of course, in control, the effect of past constraints are automatically *remembered* by the plant. However, in constrained state estimation we have to make special provisions in the estimator so that it *remembers* the effect of past constraints. This has been achieved here by using $\hat{x}^*_{k+1|k}$ to pass data from one block to the next.

Moreover, it has been shown in De Doná *et al.* [2000] that a receding horizon control with constraint horizon one is actually also optimal for longer horizons in a non-trivial region of the state space. By similar arguments, one can show that the CH1 scheme is also optimal in a nontrivial sense.

Finally, a major advantage associated with having one single nonlinearity in the CH1 estimator, is that stability can be analyzed using standard ideas from nonlinear control, e.g., the Popov criterion (Vidyasagar [1993]). Thus, techniques analogous to those for anti-windup control (Teel [1999]) can be employed to study stability. This is a considerable simplification relative to the usual methods needed for receding horizon estimation (Rao *et al.* [2001]).

6. CONCLUSIONS

This paper has described a method for constrained state estimation which is analogous to anti-windup techniques in control. In particular, the constraint horizon has been restricted to one as is effectively done in anti-windup control. A major advantage of the proposed method is that it does not require on-line numerical computations as opposed to standard longer horizon estimators. Simulation results have confirmed that there are cases in which the constraint horizon one estimator performs very close to longer horizon estimators.

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