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KALMAN-BUCY FILTERING FOR SINGULAR STOCHASTIC DIFFERENTIAL SYSTEMS

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Abstract: This work investigates the problem of state estimation for singular stochastic differential systems. A Kalman-Bucy-like filter is proposed, based on a suitable decomposition of the descriptor vector into two components. The first one is expressed as a function of the observation, and therefore does not need to be estimated, while the second component is described by a regular linear stochastic system and can be estimated by a Kalman-Bucy filter. Numerical simulations are presented on the case of a stochastic system with an unknown input, modeled as a singular system. *Copyright © 2002 IFAC*

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1. INTRODUCTION

The filtering problem for discrete-time *singular* systems (also named *descriptor* systems) has been widely considered in literature in recent years. In (Dai, 1987; Dai 1989) the case of square singular systems has been investigated, while rectangular systems were considered in (Darouach, *et al.*, 1993). Gaussian descriptor systems have been treated in (Nikoukhah, *et al.*, 1992; Nikoukhah, *et al.*, 1999), where an optimal filter, according to the Maximum Likelihood Criterion, has been presented. The case of non-Gaussian singular systems has been studied in (Germani, *et al.*, 2001), where a minimum error variance polynomial filter is constructed following the approach in (Carravetta, *et al.*, 1996). All filtering algorithms developed for the discrete-time case are based on a clever use of the time-shift of the output sequence, that allows to transform a *singular* problem into a *regular* one. Unfortunately, such algorithms can not be easily extended to continuous-time systems. The main reason is that for continuous-time systems the time-shift on the output should be replaced with a time-derivative on the noisy output, that is not available nor computable.

This work investigates and solves the filtering problem for continuous-time stochastic descriptor systems, described by the Ito differential formulation. The proposed filter is based on a suitable decomposition of the descriptor vector into two components, one of which is a function of the measured output, and therefore does not need to be estimated, while the other component is described by a regular linear stochastic system and can be estimated by a Kalman-Bucy filter. As an example, the filter is developed and tested on the case of a descriptor system that models a regular stochastic system in the presence of an unknown input. Numerical

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simulations show the effectiveness of the proposed filter.

The paper is organized as follows: in Section 2 the filtering problem is formulated, and some important properties concerning solvability and estimability of discrete-time singular systems are suitably extended to the continuous-time framework. These properties help in developing the filtering algorithm, that is described in Section 3. The example and numerical simulations are reported in Section 4.

2. SINGULAR STOCHASTIC DIFFERENTIAL SYSTEMS

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{\mathcal{F}_t,\ t \geq 0\}$ be a family of nondecreasing sub-σ-algebras of $\mathcal{F}$. A singular time-invariant stochastic differential system in the Ito formulation is described by the equations:

$$J dx_t = A x_t dt + F dW_t, \quad x_0 = \chi, \tag{1a}$$

$$dy_t = C x_t dt + G dW_t, \quad y_0 = 0, \tag{1b}$$

where $x_t \in I\!R^n$ is the descriptor vector, $y_t \in I\!R^q$ is the measured output, χ a Gaussian random variable with mean $\bar{\chi}$ and covariance Ψ_χ. The pair $(W_t, \mathcal{F}_t)$ is a standard Wiener process taking values in $I\!R^b$. J and A are $m \times n$, matrices, C is $q \times n$, F is $m \times b$ and G is $q \times b$. Obviously, if J is square and nonsingular, then system (1) can be put in a regular form, so that the filtering problem is solved by the well-known Kalman-Bucy algorithm.

This paper considers the filtering problem for systems of the type (1) in the more general setting of J not square and/or not full-rank. Since in this case the existence and the uniqueness of the solution process x_t of (1a) is not guaranteed for all triples (J, A, F), the solvability of the filtering problem requires that the singular system under investigation satisfies some structural properties. The problem of existence of a solution and of its uniqueness, and the property of the causality have been widely investigated for discrete time singular-systems (Luenberger, 1977; Luenberger, 1978; Darouach, *et al.*, 1995, Germani, *et al.*, 2001). Here follows an essential extension of this analysis to continuous-time singular systems modeled by the Ito stochastic equations (1).

First, let us state some definitions and results on deterministic singular systems. In the following the space of locally essentially bounded measurable functions from $[0, \infty)$ to $I\!R^p$ is denoted by $\mathcal{M}(I\!R_+; I\!R^b)$, while the space of absolutely continuous functions from $[0, \infty)$ to $I\!R^n$ is denoted by $W(I\!R_+; I\!R^n)$. The symbol $O_{m\times n}$ denotes the $m \times n$ zero matrix, while I_n denotes the identity $n \times n$ matrix.

Definition 1. *A continuous-time singular system of the type*

$$J\dot{x}_t = A x_t + F u_t, \qquad x_0 = \chi, \tag{2}$$

is said to be "causally solvable" if $\forall(\chi, u) \in I\!R^n \times \mathcal{M}(I\!R_+; I\!R^b)$*, there exists at least one solution* $x \in W(I\!R_+; I\!R^n)$ *of equation* (2).

Theorem 2. *A necessary and sufficient condition for causal solvability of system* (2) *is that* $\mathcal{R}([A\ F]) \subseteq \mathcal{R}(J)$.

Proof. The result derives from Rouché-Capelli theorem: at each time $t \geq 0$ condition $\mathcal{R}([A\ F]) \subseteq \mathcal{R}(J)$ is necessary and sufficient to guarantee existence of $\dot{x}_t$ that satisfies (2). ∎

Definition 3. *A causally-solvable singular system with observations* y_t *described by equations:*

$$J\dot{x}_t = A x_t + F u_t, \qquad x_0 = \chi, \tag{3a}$$

$$y_t = C x_t, \tag{3b}$$

is said to be "estimable from the measurements" if $\forall(\chi, u, y) \in I\!R^n \times \mathcal{M}(I\!R_+; I\!R^b) \times W(I\!R_+; I\!R^q)$ *is such that if a solution* x *of* (3) *exists in* $W(I\!R_+; I\!R^n)$*, this is unique.*

Stated in other words, singular systems that are estimable from the measurements are such that the evolution of x_t is univocally determined by the output y_t. An important role for *estimability from the measurements* of singular systems is played by the matrix

$$\overline{H} = \begin{bmatrix} J \\ C \end{bmatrix} \in I\!R^{(m+q)\times n}. \tag{4}$$

In particular, it will be required that $\overline{H}$ is a full column rank matrix, that means that has n independent columns. Obviously, a necessary condition for this is that $n \leq m + q$.

Theorem 4. *A solvable singular system of the type* (3) *is estimable from the measurements if and only if matrix* $\overline{H}$ *is full column rank.*

Proof. By differentiating the output equation (3b) system (3) can be put in the form

$$\begin{bmatrix} J \\ C \end{bmatrix} \dot{x}_t = \begin{bmatrix} A \\ O_{q\times n} \end{bmatrix} x_t + \begin{bmatrix} O_{m\times q} \\ I_q \end{bmatrix} \dot{y}_t + \begin{bmatrix} F \\ O_{q\times b} \end{bmatrix} u_t, \tag{5}$$

According to the solvability hypothesis of system (3) and to the full column rank assumption for matrix $\overline{H}$, the right hand side belongs to the range of $\overline{H}$ for any $t \geq 0$, and $\dot{x}_t$ is obtained premultiplying equation (5) by any left-inverse of matrix $\overline{H}$, denoted in the following with $\overline{H}^+$ (this means that $\overline{H}^+$ is such that $\overline{H}^+\overline{H} = I_n$). It follows that the evolution of (5) is univocally determined by the equation below:

$$\dot{x}_t = \overline{\mathcal{A}} x_t + \overline{\mathcal{D}} \dot{y}_t + \overline{\mathcal{F}} u_t, \qquad x_0 = \chi, \tag{6}$$

where the triple $(\overline{\mathcal{A}}, \overline{\mathcal{D}}, \overline{\mathcal{F}})$ is defined as

$$\overline{\mathcal{A}} = \overline{H}^+ \begin{bmatrix} A \\ O_{q\times n} \end{bmatrix}, \quad \overline{\mathcal{D}} = \overline{H}^+ \begin{bmatrix} O_{m\times q} \\ I_q \end{bmatrix}, \qquad (7)$$
$$\overline{\mathcal{F}} = \overline{H}^+ \begin{bmatrix} F \\ O_{q\times b} \end{bmatrix}.$$

■

Remark 5. (Özçaldiran *et al.*, 1992) gave the concept of *strong observability* of autonomous descriptor systems, and shown that the full rank condition of matrix $\overline{H}$ is a necessary condition for strong observability. •

Definition 6. *Any regular system of the type* (6) *that gives the same solution of the partially observed singular system* (3) *is called a "Complete Regular System" (CRS) for* (3).

Remark 7. Note that causal solvability and estimability from the output are necessary and sufficient conditions for the existence of a CRS for a singular systems. •

Now, before considering stochastic singular differential systems of the type (1), the following remark explains some facts about the measured variables.

Remark 8. Note that the observation model (1b) provides y_t as an "integrated measurement", with a covariance error linearly increasing with time ($I\!E\{(GW_tW_t^TG^T\} = GG^Tt$). On the other hand, physical sensors are affected by a noise with bounded covariance (constant, in stationary models). Hence, from a practical point of view, we can assume that a physical sensor provides the measurement ζ_t formally defined by the measure equation

$$\zeta_t = Cx_t + Gn_t, \qquad (8)$$

where n_t is the formal derivative of the Wiener process (white noise model). ζ_t such that $y_t = \int_0^t \zeta_\tau d\tau$. Although the observation model (8) is characterized by a measure error with constant covariance, it is not mathematically rigorous in the Ito formulation, and therefore the "integrated measurement" model (1b) must be used. However, the knowledge of ζ_t can be assumed, if required, for the filter implementation. (Note that in the Kalman-Bucy filter for regular stochastic systems the forcing term in the filter equation is the differential dy_t, that is assumed known.) •

Here follows some definitions and results for singular stochastic systems (1a).

Definition 9. *A stochastic singular system described by* (1a) *is said to be "causally solvable" if for any Gaussian random vector* χ *there exists at least one Gaussian solution process* x_t *that is* $\mathcal{F}_t$*-adapted in* $[0, \infty)$.

It could be shown that also for stochastic singular systems a necessary and sufficient condition for causal solvability is that $\mathcal{R}([A\ F]) \subseteq \mathcal{R}(J)$. In some cases it is possible to define a stochastic regular system that gives the same solutions of a partially observed singular system, if noise-free observations are available.

Definition 10. *Consider a stochastic singular system* (1) *with* $G = 0$ *(noise-free measurement):*

$$Jdx_t = Ax_tdt + FdW_t, \qquad x_0 = \chi, \qquad (9a)$$
$$\zeta_t = Cx_t. \qquad (9b)$$

Assume that system (9) *is causally solvable. A regular system described by the following explicit form:*

$$\begin{aligned} d\xi_t &= \overline{\mathcal{A}}\xi_tdt + \overline{\mathcal{D}}d\zeta_t + \overline{\mathcal{F}}dW_t, \qquad \xi_0 = \chi \\ \zeta_t &= C\xi_t \end{aligned} \qquad (10)$$

defined by a triple of matrices $(\overline{\mathcal{A}}, \overline{\mathcal{D}}, \overline{\mathcal{F}})$ *of suitable dimensions, is called a "Stochastic Complete Regular System" (SCRS) for* (9) *if and only if* ξ_t *is also a solution of* (9).

The following theorem can be given:

Theorem 11. *A stochastic singular system* (9) *admits a SCRS if and only if it is causally solvable* $\big(\mathcal{R}([A\ F]) \subseteq \mathcal{R}(J)\big)$ *and estimable from the output* ($\overline{H}$ *full column-rank). All SCRS have the form* (10) *with matrices* $(\overline{\mathcal{A}}, \overline{\mathcal{D}}, \overline{\mathcal{F}})$ *given by* (7).

Proof. A simple proof can be obtained following the same steps made to derive the CRS (6) for deterministic singular systems. Take the differential of ζ_t given by (9b), obtaining $d\zeta_t = Cdx_t$, and write the stochastic system

$$\overline{H}dx_t = \begin{bmatrix} A \\ O_{q\times n} \end{bmatrix} x_tdt + \begin{bmatrix} O_{m\times q} \\ I_q \end{bmatrix} d\zeta_t + \begin{bmatrix} F \\ O_{q\times b} \end{bmatrix} dW_t. \qquad (11)$$

From this, thanks to causal-solvability and estimability conditions, the equation can be solved for the differential dx_t using any left-inverse of matrix $\overline{H}$, obtaining the SCRS (10). ■

3. THE FILTER CONSTRUCTION

Consider a stochastic singular causally solvable system, described by the Ito equations (1). Let $\rho = \text{rank}(G)$. The main assumption needed in this paper for the derivation of a filter for (1) is the following:

$$\rho < q. \qquad (12)$$

Without loss of generality we will assume that the first ρ rows of G are independent. Then, a selection matrix of the form $T_1 = \big[I_\rho\ \ O_{\rho\times(q-\rho)}\big]$ can be used to define a new output $y_{1,t} = T_1y_t$ ($y_{1,t} \in I\!R^\rho$), that satisfies the equation:

$$dy_{1,t} = T_1Cx_tdt + T_1GdW_t, \quad y_{1,0} = 0. \qquad (13)$$

with T_1G a full rank matrix. Now let $T_2 \in I\!R^{(q-\rho)\times q}$ be a full rank matrix whose rows generate the left-null-space of G: $T_2G = O_{(q-\rho)\times b}$. Another output $y_{2,t} = T_2 y_t$ can be defined ($y_{2,t} \in I\!R^{q-\rho}$), that satisfies the equation

$$dy_{2,t} = T_2 C x_t dt, \quad y_{2,0} = 0. \tag{14}$$

This allows to define a noise-free measurements vector $z_t = T_2 C x_t$ that allows, under suitable assumptions, the construction of a SCRS for the singular system (1).

Lemma 12. *For the singular system* (1) *assume that* $\mathcal{R}([A\ F]) \subseteq \mathcal{R}(J)$ *and that* $\rho = \mathrm{rank}(G) < q$, *so that a noise-free measurement* $z_t = T_2 C x_t$ *can be defined. Assume that matrix*

$$H = \begin{bmatrix} J \\ T_2 C \end{bmatrix}, \tag{15}$$

is full column rank. Then the singular system

$$\begin{aligned} J dx_t &= A x_t dt + F dW_t, \qquad x_0 = \chi, \\ z_t &= T_2 C x_t, \end{aligned} \tag{16}$$

admits a SCRS given by:

$$dx_t = \mathcal{A} x_t dt + \mathcal{D} dz_t + \mathcal{F} dW_t, \qquad x_0 = \chi \tag{17}$$

with matrices

$$\begin{aligned} \mathcal{A} &= H^+ \begin{bmatrix} A \\ O_{(q-\rho)\times n} \end{bmatrix}, \quad \mathcal{D} = H^+ \begin{bmatrix} O_{m\times(q-\rho)} \\ I_{(q-\rho)} \end{bmatrix}, \\ \mathcal{F} &= H^+ \begin{bmatrix} F \\ O_{(q-\rho)\times b} \end{bmatrix}, \end{aligned} \tag{18}$$

in which H^+ denotes any left-inverse of H.

Proof. The proof easily comes by applying Theorem 11. ■

Remark 13. The condition for H to have n independent columns (full column rank) implies that m (the number of rows of J) plus $q-\rho$ (the dimension of the noise-free measurement vector z_t) must be greater than n, i.e. $m + q - \rho \geq n$. This means that the dimension of z_t must be at least $n - m$. •

In Lemma 12 the noise-free component of the observation vector has been used to remove the singular formulation of the state equation. Here follows how to exploit the noisy measures $y_{1,t}$ (13) for the construction of a filter for the SCRS associated to the singular system.

By using a suitable change of state and output coordinates, system (17) can be rewritten as stated by the following lemma.

Lemma 14. *Consider system* (1) *under the same assumptions of Lemma* 12. *Consider the triple* $(\mathcal{A}, \mathcal{D}, \mathcal{F})$ (18) *defined in Lemma* 12. *Define two processes $\mathcal{X}_t$ and $\mathcal{Y}_t$ as*

$$\mathcal{X}_t = x_t - \mathcal{D} z_t, \quad \mathcal{Y}_t = T_1(I_q - C\mathcal{D}T_2) y_t. \tag{19}$$

Then the processes $\mathcal{X}_t$ and $\mathcal{Y}_t$ satisfy:

$$\begin{aligned} d\mathcal{X}_t &= \mathcal{A}\mathcal{X}_t dt + \mathcal{B} dy_t + \mathcal{F} dW_t, \quad \mathcal{X}_0 = (I_n - \mathcal{D}T_2 C)\chi \\ d\mathcal{Y}_t &= \mathcal{C}\mathcal{X}_t dt + \mathcal{G} dW_t, \qquad \mathcal{Y}_0 = 0. \end{aligned} \tag{20}$$

with:

$$\mathcal{B} = \mathcal{A}\mathcal{D}T_2, \quad \mathcal{C} = T_1 C, \quad \mathcal{G} = T_1 G. \tag{21}$$

Proof. Direct computation of the differentials of $\mathcal{X}_t$ and $\mathcal{Y}_t$ as defined by (19), taking into account the expression of the SCRS (17) for the singular system (16), provides equations (20). ■

In order to properly take into account the presence of the output y_t as a forcing term in the state equation (20), a suitable decomposition of the system is required, as given by the following proposition.

Proposition 15. The processes $\mathcal{X}_t$ and $\mathcal{Y}_t$ defined in (20) can be split as

$$\mathcal{X}_t = \mathcal{X}_t^d + \mathcal{X}_t^s, \tag{22a}$$
$$\mathcal{Y}_t = \mathcal{Y}_t^d + \mathcal{Y}_t^s, \tag{22b}$$

with

$$\begin{aligned} d\mathcal{X}_t^d &= \mathcal{A}\mathcal{X}_t^d dt + \mathcal{B} dy_t, \quad \mathcal{X}_0^d = I\!E[\mathcal{X}_0], \\ d\mathcal{Y}_t^d &= \mathcal{C}\mathcal{X}_t^d dt, \quad \mathcal{Y}_0^d = 0. \end{aligned} \tag{23}$$

$$\begin{aligned} d\mathcal{X}_t^s &= \mathcal{A}\mathcal{X}_t^s dt + \mathcal{F} dW_t, \quad \mathcal{X}_0^s = \mathcal{X}_0 - I\!E[\mathcal{X}_0], \\ d\mathcal{Y}_t^s &= \mathcal{C}\mathcal{X}_t^s dt + \mathcal{G} dW_t, \quad \mathcal{Y}_0^s = 0. \end{aligned} \tag{24}$$

Proof. The proof is readily obtained by direct computation, summing up the differentials $d\mathcal{X}_t^d$ and $d\mathcal{X}_t^s$ of systems (23) and (24), respectively, to obtain the differential $d\mathcal{X}_t$ of system (20), and summing up the differentials $d\mathcal{Y}_t^d$ and $d\mathcal{Y}_t^s$ to obtain the differential $d\mathcal{Y}_t$ of equation (20). ■

Remark 16. Proposition 15 shows the decomposition of the new state $\mathcal{X}_t$ into two terms: $\mathcal{X}_t^d$ is the totally-observed component and $\mathcal{X}_t^s$ is the partially-observed, zero-mean component $\mathcal{F}_t^{\mathcal{Y}^s}$-adapted, where $\mathcal{F}_t^{\mathcal{Y}^s}$ is the σ-algebra generated by the measurement process $\mathcal{Y}^s$ up to time t. •

From the definitions of Lemma 14 and Proposition 15 it follows that $x_t = \mathcal{X}_t^d + \mathcal{X}_t^s + \mathcal{D} z_t$. On the other hand it must be stressed that $\mathcal{X}_t^d$ is completely determined by the measurements y_t, through equation (23a), and therefore only $\mathcal{X}_t^s$, the state of system (24), has to be filtered. This is the reason why we give the following:

Definition 17. *A $\mathcal{P}$-estimate for the descriptor vector of the singular system* (1) *is an estimate with the following structure:*

$$\tilde{x}_t = \mathcal{X}_t^d + \widetilde{\mathcal{X}}_t^s + \mathcal{D}z_t \tag{25}$$

where $\widetilde{\mathcal{X}}_t^s$ is any $\mathcal{F}_t^{\mathcal{Y}^s}$-measurable function.

Remark 18. Note that the estimation error of a $\mathcal{P}$-estimate is given by $x_t - \tilde{x}_t = \mathcal{X}_t^s + \widetilde{\mathcal{X}}_t^s$, and therefore the error covariance matrix coincides with the covariance of the estimation error of the partially-observed component of the state:

$$\mathrm{Cov}\big(x_t - \tilde{x}_t\big) = \mathrm{Cov}\big(\mathcal{X}_t^s - \widetilde{\mathcal{X}}_t^s\big). \tag{26}$$

•

Let us denote with $L_t(\mathcal{Y}^s)$ the space of all linear functions of the random process $\{\mathcal{Y}_\tau^s,\ \tau \in [0,t]\}$ with values in $I\!R^n$.

Definition 19. *A linear $\mathcal{P}$-estimate for the descriptor vector of the singular system* (1) *is any $\mathcal{P}$-estimate given by equation* (25)*, where $\widetilde{\mathcal{X}}_t^s \in L_t(\mathcal{Y}^s)$.*

It is known that the minimum error variance estimate for $\mathcal{X}_t^s$, given the observation process $\{\mathcal{Y}_\tau^s,\ \tau \in [0,t]\}$, is the linear function given by the projection of $\mathcal{X}_t^s$ onto the space $L_t(\mathcal{Y}^s)$, denoted $\widehat{\mathcal{X}}_t^s = \Pi\big[\mathcal{X}_t^s | L_t(\mathcal{Y}^s)\big]$.

Thanks to expression (26) for the covariance error of a $\mathcal{P}$-estimate, it follows that the optimal linear $\mathcal{P}$-estimate of x_t is

$$\begin{aligned} \hat{x}_t &= \mathcal{X}_t^d + \widehat{\mathcal{X}}_t^s + \mathcal{D}z_t, \\ \text{with} \quad & \widehat{\mathcal{X}}_t^s = \Pi\big[\mathcal{X}_t^s | L_t(\mathcal{Y}^s)\big]. \end{aligned} \tag{27}$$

The following theorem gives an algorithm that computes the optimal linear $\mathcal{P}$-estimate of the descriptor vector x_t of system (1).

Theorem 20. *The linear optimal $\mathcal{P}$-estimate for system* (1) *under the same assumptions of Lemma* 12 *is given by*

$$\tilde{x}_t = \widehat{\mathcal{X}}_t + \mathcal{D}z_t \tag{28}$$

where $\widetilde{\mathcal{X}}_t$ is given by the filter equation

$$\begin{aligned} d\widehat{\mathcal{X}}_t &= \mathcal{A}\widehat{\mathcal{X}}_t dt + \mathcal{B}dy_t \\ &\quad + \big(\mathcal{F}\mathcal{G}^T + P_t\mathcal{C}^T\big)\big(\mathcal{G}\mathcal{G}^T\big)^{-1}\Big(d\mathcal{Y}_t - \mathcal{C}\widehat{\mathcal{X}}_t dt\Big), \\ \widehat{\mathcal{X}}_0 &= (I_n - \mathcal{D}T_2C)\bar{\chi}, \end{aligned} \tag{29}$$

in which matrices $\mathcal{A}, \mathcal{DF}$ are defined by equations (18) *and $\mathcal{B}, \mathcal{C}, \mathcal{G}$ are defined by* (21)*, and matrix P_t is the estimation error covariance matrix computed solving*

$$\begin{aligned} \dot{P}_t &= \mathcal{A}P_t + P_t\mathcal{A}^T + \mathcal{G}\mathcal{G}^T \\ &\quad - \big(\mathcal{F}\mathcal{G}^T + P_t\mathcal{C}^T\big)\big(\mathcal{G}\mathcal{G}^T\big)^{-1}\big(\mathcal{F}\mathcal{G}^T + P_t\mathcal{C}^T\big)^T, \\ P_0 &= \Psi_\chi, \end{aligned} \tag{30}$$

Proof. Equations (29) and (30) are obtained defining $\widehat{X}_t$, the optimal linear $\mathcal{P}$-estimate $\mathcal{X}_t$, as

$$\widehat{\mathcal{X}}_t = \mathcal{X}_t^d + \widehat{\mathcal{X}}_t^s, \tag{31}$$

where $\widehat{\mathcal{X}}_t^s$ is obtained by the Kalman-Bucy filter applied to system (24), that is

$$\begin{aligned} d\widehat{\mathcal{X}}_t^s =& \mathcal{A}\widehat{\mathcal{X}}_t^s dt + \big(\mathcal{F}\mathcal{G}^T + P_t\mathcal{C}^T\big) \\ & \cdot \big(\mathcal{G}\mathcal{G}^T\big)^{-1}\Big(d\mathcal{Y}_t^s - \mathcal{C}\widetilde{\mathcal{X}}_t^s dt\Big), \end{aligned} \tag{32}$$

with the error covariance matrix P_t given by (30). The sum of the differential $d\mathcal{X}_t^d$ given from (23) with $d\widehat{\mathcal{X}}_t^s$ from (32), after simple computations, gives the filter (29). Note that the choice of T_1 operated at the beginning of the section guarantees that matrix $\mathcal{G}\mathcal{G}^T = T_1GG^TT_1^T$ is nonsingular, so that the filter equations are well-posed. ■

4. SIMULATION RESULTS

This section reports some simulation results obtained by the application of the proposed filter (29) on an unknown-input system modeled as a singular system. The unknown-input system here considered has the following structure

$$\begin{aligned} d\zeta_t &= \widetilde{A}\zeta_t dt + Bdu_t + FdW_t^1, \qquad \zeta_{t_0} = \zeta_0 \\ dy_t &= \widetilde{C}\zeta_t dt + GdW_t^2 \end{aligned} \tag{33}$$

where $\zeta(t) \in I\!R^3$, $y(t) \in I\!R^2$ are the state and the output, respectively, $u(t) \in I\!R$ is the unknown-input and the W_t^1, W_t^2 are scalar standard Wiener processes. The system matrices used in the simulations are:

$$\widetilde{A} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0.4 & 0 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.5 \\ -1.5 \end{bmatrix}, \quad F = \begin{bmatrix} -0.5 \\ 0.6 \\ -0.2 \end{bmatrix}, \tag{34}$$

$$\widetilde{C} = \begin{bmatrix} 1 & 0.5 & -2 \\ 1 & 1 & -0.2 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}. \tag{35}$$

According to a procedure borrowed from the discrete-time case (Darouach, *et al.*, 1995), the unknown-input system (33) can be modeled as a singular system of the type (1) by the definition of the extended state

$$x_t = \begin{pmatrix} \zeta_t \\ u_t \end{pmatrix} \in I\!R^4. \tag{36}$$

Since input u_t is unknown, an equation for the differential of the state variable $x_4 = u_t$ can not be written, and this leads to a singular system of the type (1) with

$$J = \big[I_3 \ \ -B\big], \quad A = \big[\widetilde{A}\ O_{3\times 1}\big], \quad C = \big[\widetilde{C}\ O_{2\times 1}\big]. \tag{37}$$

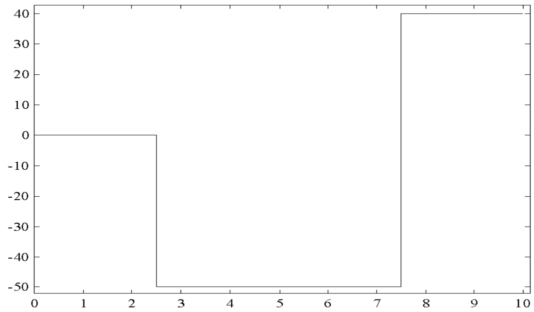

Fig. 1. The unknown input u_t.

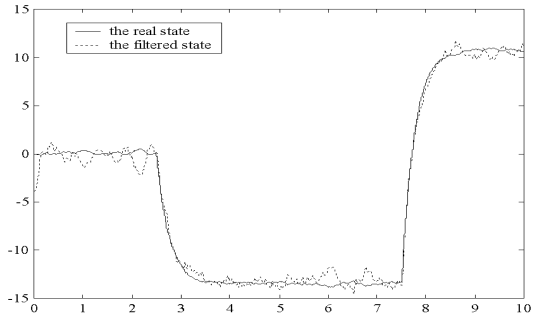

Fig. 2. True and estimated $x_{1,t}$.

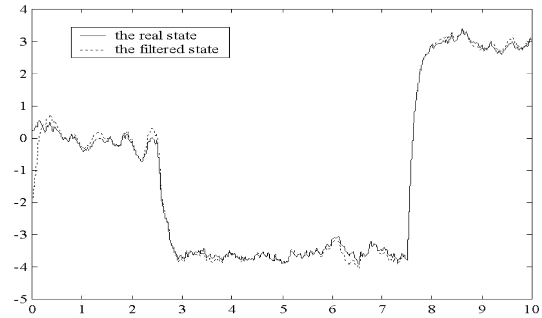

Fig. 3. True and estimated $x_{2,t}$.

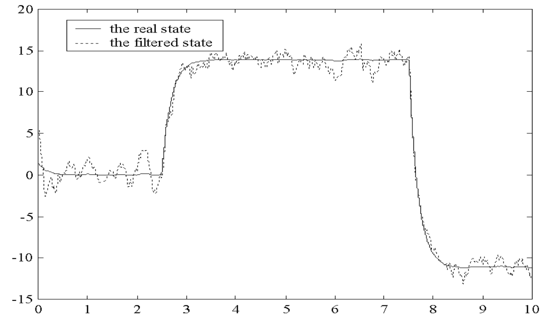

Fig. 4. True and estimated $x_{3,t}$.

The piece-wise constant input u_t shown in fig. 1 has been used in the simulations. Fig.'s 2–4 report the true and the filtered state variables. Numerical simulations are obtained trough the construction of an exact stochastic realization of system (33) at discrete times $t_k = k\Delta$, with $\Delta = 0.001$. The Kalman-Bucy filter is integrated using the Euler-Maruyama method (Higham, 2001) with fixed step $dt = 0.001$.

5. CONCLUSIONS

This paper presents a minimum variance approach to solve the filtering problem for stochastic continuous-time descriptor systems described using the Ito formulation. The solution is a linear, Kalman-Bucy-like algorithm, which estimates the descriptor vector of a singular system onto the Hilbert space spanned by the family of a suitable class of transformations of the measured outputs, denoted as linear $\mathcal{P}$-estimates. Numerical simulations show the effectiveness of the proposed filter.

REFERENCES

Carravetta, F., A. Germani and M. Raimondi (1996). Polynomial filtering for linear discrete-time non-Gaussian systems, *SIAM J. Control and Optim.*, **Vol. 34**, No. 5, pp. 1666–1690.

Dai, L., (1987). State estimation schemes in singular systems, *Proc. 10th IFAC World Congress*, Munich, **Vol. 9**, pp. 211–215.

Dai, L. (1989). Filtering and LQG problems for discrete-time stochastic singular systems, *IEEE Trans. Autom. Contr.*, No. 34, pp. 1105–1108.

Darouach, M., M. Zasadzinski, A. Basson Onana and S. Nowakowski (1995). Kalman filtering with unknown inputs via optimal state estimation of singular systems, *Int. J. Syst. Sci.*, **Vol. 26**, No. 10, pp. 2015–2028.

Darouach, M., M. Zasadzinski and D. Mehdi (1993). State estimation of stochastic singular linear systems, *Int. J. Syst. Sci.*, **Vol. 2**, No. 2, pp. 345–354.

Germani, A., C. Manes and P. Palumbo (2000). Polynomial filtering for stochastic non-Gaussian descriptor systems, *IASI-CNR Research Report* **No. 526**.

Germani, A., C. Manes, P. Palumbo (2001). Optimal Linear Filtering for Stochastic Non-Gaussian Descriptor Systems, *Proc.* 40^{th} *Conf. on Decision and Control*, Orlando, Fl, USA.

Higham, D.J. (2001). An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Review*, Vol. 43, No. 3, pp. 525–546.

Liptser, R.S., A. N. Shiryaev (1974). *Statistics of Random Processes*, Springer-Verlag. Berlin.

Luenberger, D.G. (1977). Dynamic Equations in Descriptor Form, *IEEE Trans. Autom. Contr.*, No. 3, pp. 312–321.

Luenberger, D.G. (1978). Time-Invariant Descriptor Systems, *Automatica*, No. 14, pp. 473–480.

Nikoukhah, R., S. L. Campbell and F. Delebecque (1999). Kalman filtering for general discrete-time linear systems, *IEEE Trans. Autom. Contr.*, **Vol. 44**, No. 10, pp. 1829–1839.

Nikoukhah, R., B. C. Levy and A. S. Willsky (1992). Kalman filtering and Riccati equations for descriptor systems, *IEEE Trans. Autom. Contr.*, No. 9, pp. 1325–1342.

Özçaldiran, K., D. W. Fountain, and F. L. Lewis (1992). Some generalized notions of observability, *IEEE Trans. Autom. Contr.*, Vol. 37, No. 6, pp. 856–860.