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RECURSIVE IDENTIFICATION OF LINEAR MULTIVARIABLE SYSTEMS WITH BOUNDED DISTURBANCES

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Abstract: In this contribution a simple and useful recursive identification method for linear multivariable systems in the presence of bounded disturbances is presented. We show in particular how to design a selection rule for some weighting matrices so that the estimated parameters are consistent with the measurements and the noise constraints. Conditions for asymptotic convergence are given. Performances and simplicity of the design will be shown through a numerical example to estimate physical parameters of a mechanical system. *Copyright ©2002 IFAC*

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1. INTRODUCTION

Set membership identification of linear systems in the presence of bounded disturbances has received a huge attention in the last two decades. This approach was introduced to deal with parameters estimation when the standard assumptions on the noise (such as zero mean white Gaussian noise with a known variance) usually used in the well-known identification methods (Ljung and Soderstrom, 1983) are not satisfied, the only hypothesis required is that disturbances in the model are bounded. We refer the reader to the pioneering works of Fogel and Huang (Fogel and Huang, 1982), Milanese and Belforte (Milanese and Belforte, 1982). Since then several techniques have been performed to improve the existing results (even for a class of nonlinear models) and also to introduce new algorithms, for more details see (Belforte *et al.*, 1990; Dasgupta and Huang, 1987; Norton, 1987; Canudas-De-Wit and Carrillo, 1990; Boutayeb, 2000) and the references inside. Many researchers are still interested by this kind of problem (see (L. Sheng-ping and Tianyou, 1999; Sen, 1999; Milanese and Taragna, 2001; Sun and Fan, 2001)), but until now, only SISO case has been treated.

In this note we address the problem of recursive identification, of linear multivariable systems in the presence of bounded disturbances. One of the main features is to design a selection rule for a weighting matrix computation. Indeed, we notice that among on-line (or recursive) methods only single output models have been considered in the literature. Unfortunately, most of physical and complex processes are described by multi-input and multi-output differential equations. The main difficulty in treating multivariable systems, by a recursive algorithm, lies in determining some weighting factors to assure both consistency and output error constraints at each sampling time. The proposed approach may be seen as a generalization to the multivariable case of the results performed in (Dasgupta and Huang, 1987), (Canudas-De-Wit and Carrillo, 1990) and (Tan *et al.*, 1997; Sun and Fan, 2001).

On the other hand, properties as well as conditions for asymptotic stability of the algorithm are given. Finally, the obtained results will be applied to estimate physical parameters of a mechanical system.

2. THE IDENTIFICATION ALGORITHM

Consider a discrete-time multi-input multi-output system of the form :

$$y_k = F_k\theta^* + v_k \quad (1a)$$
$$v_k^T R_k^{-1} v_k \leq 1 \quad \text{for all} \quad k \in \mathbb{R}^+ \quad (1b)$$

where $y_k \in \mathbb{R}^p$ is a measurable system outputs vector of the system, $\theta^* \in \mathbb{R}^n$ is an unknown parameters vector to be identified, $F_k \in \mathbb{R}^{p\times n}$ is a measurable regressor, $v_k \in \mathbb{R}^p$ is an unobservable bounded noise vector including the measurement noise, the modelling inaccuracy, the discretization errors and eventually the computer round-off errors and $R_k \in \mathbb{R}^{p\times p}$ is a symmetric positive definite known matrix which reflects a known upper bound on noise covariance matrix.

Our aim in this paper is to design an identification algorithm for the system (1a)-(1b), that maintains the output error into the noise bounds, i.e. that ensures $(y_k - F\theta_k)^T R_k^{-1}(y_k - F\theta_k) < 1$ with $\tilde{y}_k = y_k - F_k\hat{\theta}_{k-1}$. Hereafter, we propose to estimate the parameters vector θ^* of the MIMO system (1a-b) by the recursive algorithm described bellow :

$$\hat{\theta}_k = \hat{\theta}_{k-1} + K_k(y_k - F_k\hat{\theta}_{k-1}) \quad (3)$$
$$K_k = P_{k-1}F_k^T\Lambda_k(F_kP_{k-1}F_k^T\Lambda_k + \lambda I)^{-1} \quad (4)$$
$$P_k = \frac{1}{\lambda}(I - K_kF_k)P_{k-1} \quad (5)$$

where $\Lambda_k \in \mathbb{R}^{p\times p}$ is a symmetric positive semi-definite weighting matrix given by :

$$\Lambda_k = \begin{cases} \lambda\left(F_kP_{k-1}F_k^T\right)^{-1}\left(\|\tilde{y}_k\|_{MR_k^{-1}M}M^{-1} - I\right) \\ \qquad \text{if } \|\tilde{y}_k\|_{MR_k^{-1}M}M^{-1} - I > 0 \\ \qquad \text{and } F_kP_{k-1}F_k^T > 0 \\ 0 \qquad \text{otherwise} \end{cases} \quad (6)$$

where $\|x\|_W = (x^TWx)^{\frac{1}{2}}$ is the weighted Euclidean norm of a vector x, W is a symmetric positive definite matrix of appropriate dimension and $M \in \mathcal{R}^{p\times p}$ is an arbitrary positive definite matrix.

The estimation algorithm's equations (3), (4) and (5) and the expression of the weighting matrix Λ_k (6) are obtained by minimizing the following cost function :

$$J(\hat{\theta}_k) = \sum_{t=1}^{k}\lambda^{k-t}(y_t - F_t\hat{\theta}_k)^T\Lambda_t(y_t - F_t\hat{\theta}_k) \quad (7)$$

subject to the constraint :

$$\mathcal{H}(\hat{\theta}_{k-1},\Lambda_k) = (y_k - F_k\hat{\theta}_k)^T \times R_k^{-1}(y_k - F_k\hat{\theta}_k) - 1 \leq 0 \quad (8)$$

where $\lambda \in]0,1]$ is a forgetting factor. To solve this problem, let us introduce the following Lagrangian function :

$$\mathcal{L}(\hat{\theta}_k,\mu) = J(\hat{\theta}_k) + \mu\mathcal{H}(\hat{\theta}_{k-1},\Lambda_k) \quad (9)$$

where $\mu \geq 0$ is the Lagrangian multiplier such that $\mu = 0$ if $\mathcal{H}(\hat{\theta}_{k-1},\Lambda_k) < 0$ and $\mu > 0$ otherwise. When $\mathcal{H}(\hat{\theta}_{k-1},\Lambda_k) \geq 0$ ($\mu > 0$), the minimization problem under the inequality constraint is solved by :

$$\frac{\partial\mathcal{L}(\hat{\theta}_k,\mu)}{\partial\hat{\theta}_k} = \frac{\partial J(\hat{\theta}_k)}{\partial\hat{\theta}_k} = 0 \quad (10.a)$$
$$\frac{\partial\mathcal{L}(\hat{\theta}_k,\mu)}{\partial\mu} = \mathcal{H}(\hat{\theta}_{k-1},\Lambda_k) = 0 \quad (10.b)$$

As $\frac{\partial^2 J(\hat{\theta}_k)}{\partial\hat{\theta}_k^2} = \sum_{t=1}^{k}\lambda^{k-t}F_t^T\Lambda_tF_t \geq 0$, $J(\hat{\theta}_k)$ is convex in $\hat{\theta}_k$ and this minimization problem admits an unique solution. Algorithm's equations (3), (4) and (5) are implicitly derived from (10.a), and now we use(10.b) to determine the expression of the weighting matrix Λ_k given by (6). Indeed, substituting (3) and (4) into (8), relation (10.b) becomes :

$$\Big[y_k - F_k\Big(\hat{\theta}_{k-1} + P_{k-1}F_k^T\big(F_kP_{k-1}F_k^T + \lambda\Lambda_k^{-1}\big)^{-1}(y_k - F_k\hat{\theta}_{k-1})\Big)\Big]^T(y_k - F_k\hat{\theta}_k) = 0$$

$$\Leftrightarrow \Big[(y_k - F_k\hat{\theta}_{k-1}) - F_kP_{k-1}F_k^T\big(F_kP_{k-1}F_k^T + \lambda\Lambda_k^{-1}\big)^{-1}(y_k - F_k\hat{\theta}_{k-1})\Big]^T R_k^{-1}(y_k - F_k\hat{\theta}_k) - 1 = 0$$

Using a simple factorization technique and the identity :

$$I = \left(F_kP_{k-1}F_k^T + \lambda\Lambda_k^{-1}\right)^{-1}\left(F_kP_{k-1}F_k^T + \lambda\Lambda_k^{-1}\right)$$

it follows that :

$$\lambda^2(y_k - F_k\hat{\theta}_{k-1})^T\left(F_kP_{k-1}F_k^T\Lambda_k + \lambda I\right)^{-T}R_k^{-1} \times \left(F_kP_{k-1}F_k^T\Lambda_k + \lambda I\right)^{-1}(y_k - F_k\hat{\theta}_{k-1}) - 1 = 0 \quad (11)$$

Using the a priori estimation error $\tilde{y}_k = y_k - F_k\hat{\theta}_{k-1}$ in equation (11) yields :

$$\lambda^2\tilde{y}_k^T\left(F_kP_{k-1}F_k^T\Lambda_k + \lambda I\right)^{-T}R_k^{-1} \times \left(F_kP_{k-1}F_k^T\Lambda_k + \lambda I\right)^{-1}\tilde{y}_k = 1 \quad (12)$$

It should be noticed that (12) can be viewed as one single equation with $p(p-1)$ variables (since Λ_k is a symmetric $p\times p$ matrix), this quadratic scalar relation then admits an infinity of solutions Λ_k. For any arbitrary positive definite matrix $M \in \mathbb{R}^{p\times p}$, it is obvious that $\frac{\tilde{y}_k^TM^TR_k^{-1}M\tilde{y}_k}{\left(\sqrt{\tilde{y}_k^TM^TR_k^{-1}M\tilde{y}_k}\right)^2} = 1$ and all the solutions of equation (12) may be parameterized by M in the following way :

$$\lambda\left(F_kP_{k-1}F_k^T\Lambda_k + \lambda I\right)^{-1} = \frac{M}{\sqrt{\tilde{y}_k^TM^TR_k^{-1}M\tilde{y}_k}} \quad (13)$$

From (13), the weighting matrix Λ_k is given by :

$$\Lambda_k = \lambda\left(F_kP_{k-1}F_k^T\right)^{-1}$$

$$\times\left(M^{-1}\sqrt{\tilde{y}_k^T M^T R_k^{-1} M \tilde{y}_k} - I\right) \quad (14)$$

As M is a free design parameter, we set $M = I$ for simplicity in the rest of the paper. In this particular case, Λ_k rewrites as :

$$\Lambda_k = \lambda\left(\sqrt{\tilde{y}_k^T R_k^{-1}\tilde{y}_k} - 1\right)\left(F_k P_{k-1} F_k^T\right)^{-1} \quad (15)$$

Since Λ_k must be a positive definite matrix, the equality (15) is true only if $\tilde{y}_k^T R_k^{-1}\tilde{y}_k > 1$ and if $F_k P_{k-1} F_k^T$ is positive definite, so invertible. Otherwise, i.e., either if $\tilde{y}_k^T R_k^{-1}\tilde{y}_k \leq 1$, that is the a priori estimation error is already inside the noise bounds defined in (1b) so the object is attained and we can't find better estimate for θ at time k, or if $F_k P_{k-1} F_k^T$ is not invertible, that is the measurement sequence $\{F_k\}$ is not persistently exciting, in those cases it is useless to update the estimated parameters and the only possible solution is to set $\Lambda_k = 0$ then the gain of (4) $K_k = 0$ and $\hat{\theta}_k = \hat{\theta}_{k-1}$. Using (15) and for the reasons explained above, the weighting matrix Λ_k is then given by :

$$\Lambda_k = \begin{cases} \lambda\left(\|\tilde{y}_k\|_{R_k^{-1}} - 1\right)\left(F_k P_{k-1} F_k^T\right)^{-1} & \text{if } \|\tilde{y}_k\|_{R_k^{-1}} > 1 \\ & \text{and } \quad F_k P_{k-1} F_k^T > 0 \\ 0 & \text{otherwise} \end{cases} \quad (16)$$

Theorem 1. Given (3) to (5), if the weighting matrix Λ_k introduced in (4) is given by (16), then the estimated parameters vector $\hat{\theta}_k$ has the following properties :

i. $\left\|\hat{\theta}_k - \theta^*\right\|$ is upper bounded i.e. :
$$\left\|\hat{\theta}_k - \theta^*\right\| \leq \kappa\left\|\hat{\theta}_0 - \theta^*\right\| ;$$

ii. there exists a scalar $\sigma_k > 0$ such that $|\hat{\theta}_{ik} - \theta_i^*| \leq \sigma_k\sqrt{P_{iik}}$ for all $k \geq 0$, $i = 1, 2, \ldots, n$;

where $\kappa = \dfrac{\lambda_{\max}(P_0)}{\lambda_{\min}(P_0)}$, $\sigma_k^2 = \dfrac{\lambda^k}{\lambda_{\min}(P_0)}\left\|\tilde{\theta}_0\right\|^2$, $\hat{\theta}_{ik}$ (resp. θ_i^*) is the i^{th} element of the vector $\hat{\theta}_k$ (resp. θ^*) and P_{iik} is the i^{th} diagonal element of the matrix P_k.
Furthermore, if the measurement matrix sequence $\{F_k\}$ is persistently exciting, i.e. for some constant integer $m \geq n$ and all k, there exist positive constants α and β such that

$$\alpha I \leq \sum_{i=k}^{k+m} F_i^T \Lambda_i F_i \quad \leq \quad \beta I \quad (17)$$

then the estimation algorithm (3)-(5) has the following additional properties :

iii. $\left\|\hat{\theta}_k - \theta^*\right\|^2 \leq \gamma\left\|\hat{\theta}_0 - \theta^*\right\|^2 \lambda^k$ for all $k \geq m+1$;

iv. $\lim\limits_{k\to\infty}\left\|y_k - F_k\hat{\theta}_k\right\|_{R_k^{-1}} \leq 1.$

where
$$\gamma = \begin{cases} \dfrac{1}{\alpha\lambda_{\min}(P_0)}\left(\dfrac{\lambda^{-(m+1)} - 1}{\lambda^{-1} - 1}\right) & \text{if} \quad \lambda < 1 \\ \dfrac{1}{\alpha\lambda_{\min}(P_0)} & \text{if} \quad \lambda = 1 \end{cases}$$

■

PROOF. : Each part of the theorem will be demonstrated

i. Let us define the estimation error vector $\tilde{\theta}_k = \theta^* - \hat{\theta}_k$ and consider the candidate Lyapunov function :
$$V_k = \tilde{\theta}_k^T P_k^{-1}\tilde{\theta}_k \quad (18)$$
Using (4) and (5) and after some linear manipulations, the following relations are obvious :
$$K_k = P_k F_k^T \Lambda_k \quad (19)$$
$$P_k^{-1} = \lambda P_{k-1}^{-1} + F_k^T \Lambda_k F_k \quad (20)$$
Substituting (19) in (3) and (3) in (18) yields :
$$V_k = \left(\tilde{\theta}_{k-1} - P_k F_k^T \Lambda_k (y_k - F_k\hat{\theta}_{k-1})\right)^T P_k^{-1} \times\left(\tilde{\theta}_{k-1} - P_k F_k^T \Lambda_k (y_k - F_k\hat{\theta}_{k-1})\right) \quad (21)$$
Using (20) and a priori estimation error vector definition, (21) becomes :
$$\begin{aligned} V_k &= \tilde{\theta}_{k-1}^T(\lambda P_{k-1}^{-1} + F_k^T\Lambda_k F_k)\tilde{\theta}_{k-1} \\ &\quad -\tilde{\theta}_{k-1}^T F_k^T \Lambda_k \tilde{y}_k - \tilde{y}_k^T \Lambda_k F_k \tilde{\theta}_{k-1} \\ &\quad +\tilde{y}_k^T \Lambda_k F_k P_k F_k^T \Lambda_k \tilde{y}_k \\ &= \lambda V_{k-1} + \tilde{y}_k^T \Lambda_k\left(F_k P_k F_k^T - \Lambda_k^{-1}\right)\Lambda_k \tilde{y}_k \\ &\quad + \left(\tilde{y}_k - F_k\tilde{\theta}_{k-1}\right)^T \Lambda_k \left(\tilde{y}_k - F_k\tilde{\theta}_{k-1}\right) \end{aligned} \quad (22)$$
where $V_{k-1} = \tilde{\theta}_{k-1}^T P_{k-1}^{-1}\tilde{\theta}_{k-1}$. Using a priori and parameters estimation error vectors definitions, it comes that :
$$\tilde{y}_k - F_k\tilde{\theta}_{k-1} = y_k - F_k\theta \quad = \quad v_k \quad (23)$$
and, using (4), (5) and the matrix inversion lemma we obtain :
$$\begin{aligned} F_k P_k F_k^T - \Lambda_k^{-1} &= \frac{1}{\lambda}\left[F_k P_{k-1} F_k^T - F_k P_{k-1} F_k^T \right. \\ &\quad \left. \times\left(F_k P_{k-1} F_k^T + \lambda\Lambda_k^{-1}\right)^{-1} F_k P_{k-1} F_k^T - \lambda\Lambda_k^{-1}\right] \\ &= \frac{1}{\lambda}\left\{\left[\left(F_k P_{k-1} F_k^T\right)^{-1} + \lambda^{-1}\Lambda_k\right]^{-1} - \lambda\Lambda_k^{-1}\right\} \\ &= -\lambda\Lambda_k^{-1}\left(F_k P_{k-1} F_k^T + \lambda\Lambda_k^{-1}\right)^{-1}\Lambda_k^{-1} \end{aligned} \quad (24)$$
Introducing (23) and (24) into (22) gives :
$$V_k = -\lambda\tilde{y}_k^T \Lambda_k\left(F_k P_{k-1} F_k^T \Lambda_k + \lambda I\right)^{-1}\tilde{y}_k + \lambda V_{k-1} + v_k^T \Lambda_k v_k \quad (25)$$
If $\tilde{y}_k^T R_k^{-1}\tilde{y}_k \leq 1$ or if $F_k P_{k-1}^{-1} F_k^T$ is not invertible, the algorithm is not updated i.e. $\Lambda_k = 0$ and (25) reduces to

$$\begin{aligned} V_k &= \lambda V_{k-1} \\ V_k - V_{k-1} &= (\lambda-1)V_{k-1} \qquad (26) \end{aligned}$$

Otherwise (i.e. if $\tilde{y}_k^T R_k^{-1}\tilde{y}_k > 1$ and $F_k P_{k-1}^{-1} F_k^T$ is invertible), from (15) we have :

$$\left(F_k P_{k-1} F_k^T + \lambda \Lambda_k^{-1}\right)^{-1} = \frac{\Lambda_k}{\lambda\sqrt{\tilde{y}_k^T R_k^{-1}\tilde{y}_k}} \qquad (27)$$

Using (27), (25) may be rewritten as :

$$V_k - V_{k-1} = (\lambda-1)V_{k-1} + q \qquad (28)$$

where $q \in \mathbb{R}$ is defined by :

$$\begin{aligned} q &= v_k^T\Lambda_k v_k - \lambda \tilde{y}_k^T\left[F_k P_{k-1}F_k^T + \lambda\Lambda_k^{-1}\right]^{-1}\tilde{y}_k \\ &= v_k^T\Lambda_k v_k - \tilde{y}_k^T\frac{\Lambda_k}{\sqrt{\tilde{y}_k^T R_k^{-1}\tilde{y}_k}}\tilde{y}_k \qquad (29) \end{aligned}$$

Now let us study the sign of q. As $\tilde{y}_k^T R_k^{-1}\tilde{y}_k \geq 0$ and if $\tilde{y}_k^T R_k^{-1}\tilde{y}_k \neq 0$, multiplying $\tilde{y}_k^T R_k^{-1}\tilde{y}_k$ by q doesn't change its sign :

$$\begin{aligned} \tilde{y}_k^T R_k^{-1}\tilde{y}_k q &= (\tilde{y}_k^T R_k^{-1}\tilde{y}_k)(v_k^T\Lambda_k v_k) \\ &\quad -\sqrt{\tilde{y}_k^T R_k^{-1}\tilde{y}_k}(\tilde{y}_k^T\Lambda_k\tilde{y}_k) \\ &= \tilde{y}_k^T\left((v_k^T\Lambda_k v_k)R_k^{-1} - (\sqrt{\tilde{y}_k^T R_k^{-1}\tilde{y}_k})\Lambda_k\right)\tilde{y}_k \\ &= -\tilde{y}_k^T Q\tilde{y}_k \qquad (30) \end{aligned}$$

where

$$Q = (\sqrt{\tilde{y}_k^T R_k^{-1}\tilde{y}_k})\Lambda_k - (v_k^T\Lambda_k v_k)R_k^{-1} \qquad (31)$$

The scalar q defined in (29) is negative if the matrix Q is semi-positive definite. $\forall v_k \in \mathbb{R}^n$ we have :

$$\begin{aligned} v_k^T Q v_k &= (\sqrt{\tilde{y}_k^T R_k^{-1}\tilde{y}_k})(v_k^T\Lambda_k v_k) \\ &\quad -(v_k^T\Lambda_k v_k)(v_k^T R_k^{-1}v_k) \\ &= v_k^T\Lambda_k v_k\left(\sqrt{\tilde{y}_k^T R_k^{-1}\tilde{y}_k} - v_k^T R_k^{-1}v_k\right) \qquad (32) \end{aligned}$$

Recall that we are in the case where

$\tilde{y}_k^T R_k^{-1}\tilde{y}_k > 1$ with $v^T R_k^{-1} v < 1$, thus, it comes from (32) that :

$$v_k^T Q v_k \geq v_k^T\Lambda_k v_k\left(\sqrt{\tilde{y}_k^T R_k^{-1}\tilde{y}_k} - 1\right) \geq 0 \qquad (33)$$

Consequently, we can write :
$\forall v_k \in \mathbb{R}^n$: $\|v_k\|_{R_k^{-1}} < 1$, $v_k^T Q v_k \geq 0$
Hence it is clear that the matrix Q given by (31) is semi-positive definite which implies that $q \leq 0$.
Therefore we have :

$$\begin{aligned} V_k - V_{k-1} &= (\lambda-1)V_{k-1} + q \\ &\leq (\lambda-1)V_{k-1} \leq 0 \qquad \lambda \in]0,1[\end{aligned}$$

Furthermore, by the aid of (26), we have :

$$\forall\ \lambda \in\ ]0,1], \qquad V_k \leq \lambda V_{k-1}.$$

Next, from (18) we obtain :

$$\begin{aligned} \tilde{\theta}_k^T P_k^{-1}\tilde{\theta}_k &\leq \lambda\tilde{\theta}_{k-1}^T P_{k-1}^{-1}\tilde{\theta}_{k-1} \\ &\leq \lambda^2\tilde{\theta}_{k-2}^T P_{k-2}^{-1}\tilde{\theta}_{k-2} \\ &\leq \ldots \\ &\leq \lambda^k\tilde{\theta}_0^T P_0^{-1}\tilde{\theta}_0 \qquad (34) \end{aligned}$$

and using (20) :

$$\begin{aligned} P_k^{-1} &\geq \lambda P_{k-1}^{-1} \geq \lambda^2 P_{k-2}^{-1} \\ &\geq \ \ldots \ \geq \ \lambda^k P_0^{-1} \qquad (35) \end{aligned}$$

By the aid of Rayleigh principle (Golub and Loan, 1993), a result of the Weyl inequality [1], (34) and (35), we show that :

$$\begin{aligned} \lambda^k\lambda_{\min}\left(P_0^{-1}\right)\left\|\tilde{\theta}_k\right\|^2 &\leq \lambda_{\min}\left(P_k^{-1}\right)\left\|\tilde{\theta}_k\right\|^2 \\ &\leq \tilde{\theta}_k^T P_k^{-1}\tilde{\theta}_k \\ &\leq \lambda^k\tilde{\theta}_0^T P_0^{-1}\tilde{\theta}_0 \\ &\leq \lambda^k\lambda_{\max}\left(P_0^{-1}\right)\left\|\tilde{\theta}_0\right\|^2 \qquad (36) \end{aligned}$$

and hence : $\left\|\tilde{\theta}_k\right\|^2 \leq \dfrac{\lambda_{\max}\left(P_0^{-1}\right)}{\lambda_{\min}\left(P_0^{-1}\right)}\left\|\tilde{\theta}_0\right\|^2$
that proves i.

ii. From inequality (36), we have

$$\tilde{\theta}_k^T P_k^{-1}\tilde{\theta}_k \leq \sigma_k^2$$

where

$$\sigma_k^2 = \lambda^k\lambda_{\max}\left(P_0^{-1}\right)\left\|\tilde{\theta}_0\right\|^2$$

Let us write the Sherman-Morrison formula (Golub and Loan, 1993) :

$$\begin{aligned} \left(\sigma_k^2 P_k - \tilde{\theta}_k\tilde{\theta}_k^T\right)^{-1} &= (\sigma_k^2 P_k)^{-1} \\ &\quad + \frac{(\sigma_k^2 P_k)^{-1}\tilde{\theta}_k\tilde{\theta}_k^T(\sigma_k^2 P_k)^{-1}}{1 - \tilde{\theta}_k^T(\sigma_k^2 P_k)^{-1}\tilde{\theta}_k} \end{aligned}$$

$$\sigma_k^2 P_k - \tilde{\theta}_k\tilde{\theta}_k^T = (\sigma_k^2 P_k)\left(I + \frac{P_k^{-1}\tilde{\theta}_k\tilde{\theta}_k^T}{\sigma_k^2 - \tilde{\theta}_k^T P_k^{-1}\tilde{\theta}_k}\right)^{-1}$$

If $\tilde{\theta}_k^T P_k^{-1}\tilde{\theta}_k < \sigma_k^2$ then the matrix $\left(I + \dfrac{P_k^{-1}\tilde{\theta}_k\tilde{\theta}_k^T}{\sigma_k^2 - \tilde{\theta}_k^T P_k^{-1}\tilde{\theta}_k}\right)$ is positive definite and so it is for $\left(\sigma_k^2 P_k - \tilde{\theta}_k\tilde{\theta}_k^T\right)$. Moreover, as the matrix $\left(\sigma_k^2 P_k - \tilde{\theta}_k\tilde{\theta}_k^T\right)$ is symmetric, its diagonal elements are positive and it follows that :

$$\sigma_k^2 P_{iik} - \tilde{\theta}_{ik}^2 \geq 0 \qquad \text{for } i = 1, 2, \ldots, n$$

where $\tilde{\theta}_{ik}^2$ are the diagonal elements of $\tilde{\theta}_k\tilde{\theta}_k^T$. Thus, property ii is proven.

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[1] if A, B are two square positive definite matrices then : $\lambda_{min}(A) - \lambda_{min}(B) \geq \lambda_{min}(A-B)$. Furthermore if $A - B \geq 0$ then $\lambda_{min}(A) \geq \lambda_{min}(B)$.

iii. Let $k \geq m+1$. (20) yields :

$$\begin{aligned} P_k^{-1} &= \lambda P_{k-1}^{-1} + F_k^T \Lambda_k F_k \\ &= \lambda^2 P_{k-2}^{-1} + \lambda F_{k-1}^T \Lambda_{k-1} F_{k-1} + F_k^T \Lambda_k F_k \\ &\vdots \\ &= \lambda^{m+1} P_{k-m-1}^{-1} + \lambda^m F_{k-m}^T \Lambda_{k-m} F_{k-m} \\ &\quad + \ldots + \lambda F_{k-1}^T \Lambda_{k-1} F_{k-1} + F_k^T \Lambda_k F_k \\ &= \lambda^{m+1} P_{k-m-1}^{-1} + \sum_{i=0}^{m} \lambda^i F_{k-i}^T \Lambda_{k-i} F_{k-i} \end{aligned} \tag{37}$$

Let us define $G_i = F_{k-i}^T \Lambda_{k-i} F_{k-i}$.
As G_i is a positive semi-definite matrix and as $0 < \lambda \leq 1$, it comes that :

$$\begin{aligned} &\sum_{i=0}^{m} \lambda^i G_i - \frac{1}{\sum_{i=0}^{m} \lambda^{-i}} \sum_{i=0}^{m} G_i \\ &= \sum_{i=0}^{m} \left\{ \left(\frac{1}{\lambda^{-i}} - \frac{1}{\sum_{i=0}^{m} \lambda^{-i}} \right) G_i \right\} \ \geq \ 0 \end{aligned} \tag{38}$$

If $\lambda \neq 1$ and as $\lambda^{m+1} P_{k-m-1}^{-1} > 0$, it comes from (37) and (38) that :

$$\begin{aligned} P_k^{-1} &= \lambda^{m+1} P_{k-m-1}^{-1} + \sum_{i=0}^{m} \lambda^i G_i \\ &\geq \sum_{i=0}^{m} \lambda^i G_i \\ &\geq \frac{1}{\sum_{i=0}^{m} \lambda^{-i}} \sum_{i=0}^{m} G_i = \frac{\lambda^{-1} - 1}{\lambda^{-(m+1)} - 1} \sum_{i=0}^{m} G_i \\ &\geq \frac{\lambda^{-1} - 1}{\lambda^{-(m+1)} - 1} \alpha I \end{aligned} \tag{39}$$

If $\lambda = 1$, $P_k^{-1} \geq \sum_{i=0}^{m} G_i \geq \alpha I$. Using (34), (39) and Rayleigh principle again and setting

$$\rho = \begin{cases} \dfrac{\lambda^{-1} - 1}{\lambda^{-(m+1)} - 1} & \text{if } \ \lambda < 1 \\ 1 & \text{if } \ \ \lambda = 1 \end{cases}$$

we have :

$$\begin{aligned} \rho\alpha \left\| \tilde{\theta}_k \right\|^2 &\leq \lambda_{\min}\left(P_k^{-1}\right) \left\| \tilde{\theta}_k \right\|^2 \\ &\leq \tilde{\theta}_k^T P_k^{-1} \tilde{\theta}_k \\ &\leq \lambda^k \tilde{\theta}_0^T P_0^{-1} \tilde{\theta}_0 \\ &\leq \lambda^k \lambda_{\max}\left(P_0^{-1}\right) \left\| \tilde{\theta}_0 \right\|^2 \end{aligned}$$

this yields to the result :

$$\left\| \tilde{\theta}_k \right\|^2 \leq \frac{1}{\alpha \lambda_{\min}(P_0)\, \rho} \ \lambda^k \left\| \tilde{\theta}_0 \right\|^2 \tag{40}$$

iv. The inequality (40) garanties that the parameter error norm $\left\| \tilde{\theta}_k \right\|$ is upper bounded by an exponentially decreasing term as long as the persistence condition (17) is fulfilled and while $\|\tilde{y}_k\|_{R_k^{-1}} > 1$, in other words, as long as the estimation algorithm is updated (when $\Lambda_k \neq 0$). This yields to the asymptotic convergence of $\hat{\theta}_k$ to some value contained in a certain neighborhood of the true parameter θ^* where the measure error satisfies $\|\tilde{y}_k\|_{R_k^{-1}} \leq 1$.

This completes the proof. ❏

Remark : We can choose any symmetric positive definite matrix M instead of $M = I$ in the theorem. The weighting matrix Λ_k in (16) is then replaced by (6). The matrix M introduces more weight to some composants of estimation error vector rather than others, and a judicious choice of this matrix can improve the algorithm convergence.

3. A SIMULATION EXAMPLE

Consider the following car suspension continuous two inputs/two outputs model (for $t \in [0, +\infty[$) :

$$\begin{aligned} m_1 \ddot{x}_1(t) + f(\dot{x}_1(t) - \dot{x}_2(t)) + k_1 x_1(t) = u_1(t) \\ m_2 \ddot{x}_2(t) + f(\dot{x}_2(t) - \dot{x}_1(t)) + k_2 x_2(t) = u_2(t) \end{aligned}$$

By using first order Euler discretization, we obtain (for $k = 0, 1, 2, \ldots$) :

$$\begin{aligned} m_1 x_{d1}''(k) + f(x_{d1}'(k) - x_{d2}'(k)) + k_1 x_{d1}(k) + \varepsilon_{d1}(k) \\ = u_{1d}(k) \\ m_2 x_{d2}''(k) + f(x_{d2}'(k) - x_{d1}'(k)) + k_2 x_{d2}(k) + \varepsilon_{d2}(k) \\ = u_{2d}(k) \end{aligned}$$

where (for $i = 1, 2$)

$$\begin{aligned} u_{id}(k) &= u_i(t = k) \\ x_{di}(k) &= x_i(t = k) \\ x_{di}'(k) &= \frac{x_{di}(k) - x_{di}(k-1)}{T} \\ x_{di}''(k) &= \frac{x_{di}'(k) - x_{di}'(k-1)}{T} \\ &= \frac{x_{di}(k) - 2x_{di}(k-1) + x_{di}(k-2)}{T^2} \end{aligned}$$

and ε_{d1} and ε_{d2} are the discretization errors and T is the sampling period.
We can now rewrite the above discretized model in the form :

$$u_d(k) = F(k)\theta + \varepsilon_d(k) + v(k)$$

where

$$u_d(k) = \begin{pmatrix} u_{1d}(k) \\ u_{2d}(k) \end{pmatrix}, \quad \theta = \begin{pmatrix} m_1 \\ m_2 \\ f \\ k_1 \\ k_2 \end{pmatrix},$$

$$\varepsilon_d(k) = \begin{pmatrix} \varepsilon_{d1}(k) \\ \varepsilon_{d2}(k) \end{pmatrix}, \quad v(k) = \begin{pmatrix} v_1(k) \\ v_2(k) \end{pmatrix},$$

$$F(k) = \begin{pmatrix} x''_{d1} & 0 & x'_{d1} - x'_{d2} & x_{d1} & 0 \\ 0 & x''_{d2} & x'_{d2} - x'_{d1} & 0 & x_{d2} \end{pmatrix}.$$

The inputs $u_1(t)$ and $u_2(t)$ are rectangular signals with linearly varying frequencies and sinusoidal magnitudes. In addition of unmodeled discretization errors, the measurements are subject to unknown but bounded noises $v_1(k)$ and $v_2(k)$ (uniformly distributed random sequences) of known bounds which are chosen so that the signal-to noise ratio is 20 dB.

Table 1 shows the good performances of the proposed algorithm. The final estimated parameters given in Table 1 represent the mean values of 100 simulation results.

	True parameters	Initial values	Final estimated values
m_1 (kg)	30	0	30.5
m_2 (kg)	285	0	285.8
f (Nm^{-1}s)	2000	0	1987.7
k_1 (Nm^{-1})	20210	0	19714.9
k_2 (Nm^{-1})	850000	0	852329.2

Table 1. Numerical values of the true parameters, initial and final values of estimated parameters

4. CONCLUSION

In this paper, we have addressed the problem of set membership identification of linear multivariable systems with unknown bounded disturbances. From minimization of a prescribed Lagrangian function, a recursive identification algorithm is deduced so that consistency of the estimated parameters with the measurements and noise constraints are guaranteed. Properties and convergence of the proposed approach were established. Finally, the proposed technique was successfully applied to the identification of physical parameters of a car suspension model.

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