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A MIN-MAX MPC ALGORITHM FOR LPV SYSTEMS SUBJECT TO ACTUATOR SATURATION [1]

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Abstract: In this paper, a new MPC algorithm is developed for the polytopic LPV systems subject to actuator saturation. First, set invariance of discrete-time LPV systems subject to actuator saturation is analyzed and the invariant set is determined by solving an LMI optimization problem. Then, based on set invariance, a min-max MPC algorithm is proposed for LPV systems subject to actuator saturation. A gain-scheduling MPC algorithm is also proposed for the design of a parameter-dependent controller.

Keywords: model predictive control, linear parameter-varying systems, gain-scheduling, actuator saturation.

1. INTRODUCTION

Model predictive control (MPC) refers to a class of control algorithms in which a dynamic process model is used to predict and optimize process performance. The first MPC techniques were developed in the 1970s because conventional single-loop controllers were unable to satisfy increasingly stringent performance requirements (Clarke, 1994). It is well-known that the receding horizon implementation of the open-loop optimal control profile gives rise to a stationary feedback control law (Bitmead *et al.*, 1990). Although there are many reported successful industrial applications, it is very difficult to analyze the finite horizon MPC algorithms theoretically since closed-loop asymptotic stability depends on many tuning parameters in an unnecessarily complicated way and no guarantee is provided (Bitmead *et al.*, 1990). Fortunately, several researchers proved that the constrained infinite horizon MPC algorithms can have very good stability property. For example, (Rawlings and Muske, 1993) showed that feasibility of the infinite horizon MPC optimization guarantees closed-loop stability by using only the first N control moves and setting the remaining (infinitely many) moves to zero. Instead of setting the control inputs to zero after a certain horizon, (Scokaert and Rawlings, 1998) and (Chmielewski and Manousiouthakis, 1996) used a fixed feedback control law to obtain a finite parameterization of the inputs over an infinite prediction horizon. (Michalska and Mayne, 1993) proposed a similar infinite horizon MPC algorithm for a class of non-linear plants. (Kothare *et al.*, 1996) studied the robust MPC control problem for uncertain systems with time-varying parameter uncertainty by the LMI technique. A gain-scheduling Quasi-Min-Max MPC algorithm was presented for polytopic LPV systems by (Lu and Arkun, 2000).

In the recent years there has been significant interest in linear parameter-varying (LPV) systems, which is motivated by the gain scheduling control design methodology (Shamma and Athans, 1990; Rugh, 1991). LPV systems are systems that depend on unknown but measurable time-varying parameters. The measurement of these parameters provides real-time information on the variations of the plant's characteristics. Hence, it is desirable to design controllers that are scheduled based on this information. The approach to gain-scheduling involves the design of several LTI controllers for a parameterized family of linearized system models and the interpolation of the controller gains. The class of

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systems is different from its standard linear time-varying (LTV) counterpart due to the causal dependence of its controller gains on the variation of the plant dynamics. LPV control theory has been proven useful in simplifying the interpolation and realization problems associated with conventional gain-scheduling. Specially, it allows one to treat the gain-scheduled controller as a single entity, with the gain-scheduling achieved entirely by the parameter-dependent controller (Apkarian *et al.*, 1995; Wu and Packard, 1995).

Actuator saturation can severely degrade the closed-loop system performance and sometimes even make the otherwise stable closed-loop system unstable by some large perturbation. The analysis and synthesis of controllers for dynamic systems with actuator nonlinearities have received increasing attention recently, see, for example, (Hu and Lin, 2001; Lin, 1998) and the references therein. Often, actuator saturation is dealt with by either designing low gain control laws that, for a given bound on the initial conditions, avoid the saturation limits, or estimating the domain of attraction in the presence of actuator saturation, especially in the design of MPC controller. It is known that attempts to penalize the control output variables so that the actuator limits are never violated for any expected reference commands often leads to conservative designs in which the control system for the most part operates far from its full capacity. In general, most MPC control algorithms are based on this scheme (Chmielewski and Manousiouthakis, 1996; Kothare *et al.*, 1996; Rawlings and Muske, 1993).

The approach in the present paper consists of looking at the infinite horizon LQ regulation as a receding horizon regulation strategy for LPV systems with actuator saturation. More specifically, we minimize on-line a quadratic cost with actuator saturation considered. The minimization is over linear state feedback gain matrices.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider a discrete-time polytopic LPV system subject to actuator saturation,

$$x_{k+1} = A(p_k)x_k + B(p_k)\sigma(u_k), \tag{1}$$
$$z_k = C(p_k)x_k + D(p_k)\sigma(u_k), \tag{2}$$

where $x \in \mathbb{R}^n$ denotes the state, $u \in \mathbb{R}^m$ the control input, $z \in \mathbb{R}^p$ the control output and $[A(p_k), B(p_k), C(p_k), D(p_k)] = \sum_{j=1}^{r} p_{k,j}[A_j, B_j, C_j, D_j]$. The time-varying parameter $p_k \in \mathbb{R}^r$ varies inside the simplex $\mathcal{P}$, *i.e.*, $\sum_{j=1}^{r} p_{k,j} = 1,\ \ 0 \le p_{k,j} \le 1$. The function $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is the standard saturation function defined as follows

$$\sigma(u_k) = \left[\, \sigma(u_{k,1})\ \sigma(u_{k,2})\ \cdots\ \sigma(u_{k,m}) \,\right]^T,$$

where $\sigma(u_{k,i}) = \mathrm{sign}(u_{k,i})\min\{1, |u_{k,i}|\}$. The aim of this paper is to find an LTV state feedback

$$u_k = F_k x_k, \tag{3}$$

and an LTV parameter-dependent state feedback

$$u_k = \sum_{i=1}^{r} p_i F_{k,i} x_k, \tag{4}$$

which asymptotically stabilizes the LPV systems (1) at the origin, by the MPC approach. Control law (4) is the so-called gain-scheduled controller.

In this paper, we will consider the infinite horizon MPC algorithm for the systems subject to actuator saturation. That is, if we assume that in the kth-step the state $x_{k|k} = x_k$ is known, then we will design the control $u_{k+l|k}, l \ge 0$, which minimizes the following worst-case performance:

$$\min_{u_{k+l|k}, l\ge 0}\ \max_{p_{k+l|k}\in\mathcal{P}} \left\{ J_k = \sum_{l=0}^{\infty} z_{k+l|k}^T z_{k+l|k} \right\}. \tag{5}$$

In the receding horizon framework, only the first computed control move $u_{k|k}$ is implemented. At time $k+1$, the optimization is solved again with new measurements from the plant. The purpose of taking measurements at each time step is to compensate for unmeasured disturbances and model uncertainty. This is the main feature of the receding horizon control.

Let f_i be the i-th row of the matrix F. We define the symmetric polyhedron

$$\mathcal{L}(F) = \{x \in \mathbb{R}^n : |f_i x| \le 1,\ i = 1, 2, \cdots, m\}.$$

For a matrix $P > 0$ and a real $\rho > 0$, the ellipsoid $\Omega(P,\rho) = \left\{x \in \mathbb{R}^n : x^T P x \le \rho\right\}$, is inside $\mathcal{L}(F)$ if and only if

$$f_i (P/\rho)^{-1} f_i^T \le 1, \quad i = 1, 2, \cdots m.$$

Let $\mathcal{V}$ be the set of $m \times m$ diagonal matrices whose elements are either 1 or 0. There are 2^m elements in $\mathcal{V}$. Suppose that each element of $\mathcal{V}$ is labeled as $E_i,\ i = 1, 2, \cdots, 2^m$, and denote $E_i^- = I - E_i$. Clearly, E_i^- is also an element of $\mathcal{V}$ if $E_i \in \mathcal{V}$.

Lemma 1. (Hu and Lin, 2001) Let $F, H \in \mathbb{R}^{m\times n}$ be given. For $x \in \mathbb{R}^n$, if $||Hx||_\infty \le 1$, then

$$\sigma(Fx) \in \mathrm{co}\{E_i F x + E_i^- H x :\ i \in [1, 2^m]\},$$

where $\mathrm{co}\{\cdot\}$ denotes the convex hull of a set. This means that we can rewrite $\sigma(Fx)$ as

$$\sigma(Fx) = \sum_{i=1}^{2^m} \eta_i (E_i F + E_i^- H)x,$$

where η_i is a state-dependent parameter satisfying $0 \le \eta_i \le 1,\ \sum_{i=1}^{2^m} \eta_i = 1$.

Lemma 2. Given two matrices $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{m\times n}$ and two positive semi-definite matrices $P \in \mathbb{R}^{m\times m}, Q \in \mathbb{R}^{n\times n}$, such that $A^TPA - Q < 0$, and $B^TPB - Q < 0$, then $A^TPB + B^TPA - 2Q < 0$.

Lemma 3. Suppose that matrices $A_i \in \mathbb{R}^{m\times n}, i = 1, 2, \cdots, r$, and two positive semi-definite matrices $P \in \mathbb{R}^{m\times m}, Q \in \mathbb{R}^{n\times n}$ are given. The following matrix inequality holds

$$\left(\sum_{i=1}^{r} p_i A_i^T\right) P \left(\sum_{i=1}^{r} p_i A_i\right) - Q < 0, \qquad (6)$$

where $\sum_{i=1}^{r} p_i = 1,\ 0 \leq p_i \leq 1$, if

$$A_i^T P A_i - Q < 0,\ \forall i = 1, 2, \cdots, r. \qquad (7)$$

3. A SATURATED MPC ALGORITHM

For LPV system (1), we first establish a set invariance condition for the control law $u_{k+l|k} = F_k x_{k+l|k}$. We then propose an MPC algorithm with actuator saturation considered. In what follows, we denote $\hat{A}_{i,j} = A_i + B_i(E_jF_k + E_j^- H_k)$, and $\hat{C}_{i,j} = C_i + D_i(E_jF_k + E_j^- H_k)$.

3.1 Set Invariance Condition

Theorem 4. Let $x_k = x_{k|k}$ be the state of system (1) measured at time k and state feedback matrix F_k be given. The ellipsoid $\Omega(P_k, \gamma)$ is invariant if there exists a matrix $H_k \in \mathbb{R}^{m\times n}$ satisfying the following matrix inequalities

$$\hat{A}_{i,j}^T P_k \hat{A}_{i,j} - P_k + \hat{C}_{i,j}^T \hat{C}_{i,j} < 0, \quad i \in [1, r], j \in [1, 2^m], \qquad (8)$$

and $\Omega(P_k, \gamma_k) \subset \mathcal{L}(H_k)$. Moreover, for any $x_k \in \Omega(P_k, \gamma_k)$, the performance objective function (5) satisfies $J_k < \gamma$.

Proof. Choose a Lyapunov function as

$$V(x_{k+l|k}) = x_{k+l|k}^T P_k x_{k+l|k},\ l \geq 0.$$

By Lemma 1, we have

$$\Delta V(x_{k+l|k}) = x_{k+l|k}^T (\hat{A}_{k+l|k}^T P_k \hat{A}_{k+l|k} - P_k) x_{k+l|k}$$

where $\hat{A}_{k+l|k} = \sum_{i=1}^{r} \sum_{j=1}^{2^m} p_{k+l|k,i} \eta_{l|k,j} \hat{A}_{i,j}$ and $\eta_{l|k,j}$ is dependent on $x_{k+l|k}$.

On the other hand, (8) implies that the following matrix inequalities hold

$$\hat{A}_{i,j}^T P_k \hat{A}_{i,j} - P_k < 0,\ \forall i \in [1, r],\ j \in [1, 2^m].$$

By Lemma 3, we have

$$\Delta V(x_{k+l|k}) < 0,\ \ \forall x_{k+l|k} \in \Omega(P_k, \gamma) \backslash \{0\}.$$

Thus if $x_{k|k}^T P_k x_{k|k} \leq \gamma$ then $x_{k+l|k}^T P_k x_{k+l|k} \leq \gamma$ for $l \geq 1$, *i.e.* $\Omega(P_k, \gamma)$ is an invariant set. This also implies that system (1) is asymptotically stable at the origin with all $\Omega(P_k, \gamma)$ contained in the domain of attraction.

Note that

$$J_k = \sum_{l=0}^{\infty} J_{k,l} + x_{k|k}^T P_k x_{k|k},$$

where $J_{k,l} = z_{k+l|k}^T z_{k+l|k} + \Delta V(x_{k+l|k})$. By Lemma 1, we can rewrite (2) as

$$z_{k+l|k} = \sum_{i=1}^{r} \sum_{j=1}^{2^m} p_{k+l|k,i} \eta_{l|k,j} \hat{C}_{i,j} x_{k+l|k}.$$

Hence

$$J_{k,l} = x_{k+l|k}^T \left(\tilde{A}_{k+l|k}^T \tilde{P}_k \tilde{A}_{k+l|k} - P_k\right) x_{k+l|k},$$
$$\tilde{A}_{k+l|k} = \sum_{i=1}^{r} \sum_{j=1}^{2^m} p_{k+l|k,i} \eta_{l|k,j} \begin{bmatrix} \hat{A}_{i,j} \\ \hat{C}_{i,j} \end{bmatrix},$$
$$\tilde{P}_k = \begin{bmatrix} P_k & 0 \\ 0 & I \end{bmatrix}.$$

By Lemma 3, it is easy to see that if (8) hold, then $J_{k,l} < 0$, which implies $J_k \leq x_k^T P_k x_k < \gamma$. ■

If we don't consider the optimal performance index (5), Theorem 4 is the set invariance condition for discrete-time LPV systems subject to actuator saturation. For the special case $r = 1$, Theorem 4 recovers the set invariance condition of a LTI discrete-time system subject to actuator saturation (Hu and Lin, 2001).

3.2 MPC Algorithm

Based on Theorem 4, we can use the following optimization problem minimizing the upper bound of the performance function (5) for a given x_k :

$$\max_{P_k>0, F_k, H_k} \gamma, \ \text{s.t.} \qquad (9)$$

$a)$ $x_k^T P_k x_k < \gamma$,

$b)$ matrix inequalities (8),

$c)$ $|h_{k,i} x| \leq 1,\ \ \forall x \in \Omega(P_k, \gamma),\ i = [1, m]$,

where $h_{k,i}$ denotes the i-th row of H_k.

The feasibility of optimization problem (9) ensures the existence of a stabilizing state feedback control law $u_{k+l|k} = F_k x_{k+l|k}$ which is able to steer the state from x_k to zero, *i.e.*, x_k is contained in the domain of attraction of the origin, and minimizes the performance index. On the other hand, for a given constant control matrix F_k designed without considering actuator saturation, optimization problem (9) can also be used to determine if x_k is contained in the domain of attraction of the origin in the presence of actuator saturation. Note that optimization problem (9) can also be used to determine the invariant ellipsoid. In (Hu and Lin, 2001), the authors presented an approach such that the invariant ellipsoid $\Omega(P_k, \gamma_k)$ is as large as possible, *i.e.*, leading to a large domain of attraction for the closed-loop system. Here, in order to minimize the performance function (5), we are interested in the minimum value of γ_k. In what follows, we will show that optimization problem (9) can be solved by an LMI optimization problem.

Let $Q_k = (P_k/\gamma)^{-1}$, $Y_k = F_k Q_k$ and $Z_k = H_k Q_k$. Also, denote the i-th row of Z_k be $z_{k,i}$. Then, Condition $a)$ is equivalent to

$$x_k^T P_k x_k < \gamma \Longleftrightarrow \begin{bmatrix} 1 & x_k^T \\ x_k & Q_k \end{bmatrix} > 0. \qquad (10)$$

Condition $b)$ is equivalent to

$$\begin{bmatrix} -Q_k & * & * \\ A_i Q_k + B_i(E_j Y_k + E_j^- Z_k) & -Q_k & 0 \\ C_i Q_k + D_i(E_j Y_k + E_j^- Z_k) & 0 & -\gamma I \end{bmatrix} < 0, \quad \forall i \in [1, r],\ j \in [1, 2^m]. \qquad (11)$$

Condition $c)$ is equivalent to

$$h_{k,i}(P_k/\gamma)^{-1} h_{k,i}^T \leq 1 \Longleftrightarrow \begin{bmatrix} 1 & z_{k,i} \\ z_{k,i}^T & Q_k \end{bmatrix} \geq 0,\ \forall i \in [1, m]. \qquad (12)$$

So, optimization problem (9) can be transformed into the following one with LMI constraints,

$$\min_{Q_k > 0, Y_k, Z_k} \gamma, \quad \text{s.t.} \quad \text{LMIs (10), (11) and (12).} \qquad (13)$$

Theorem 5. Let $x_k = x_{k|k}$ be the state of the system (1) measured at time k. Then the state feedback control matrix F_k at time k that minimizes the upper bound of performance function (5) can be solved by

$$F_k = Y_k Q_k^{-1},$$

where $(Q_k > 0, Y_k)$ is a solution of optimization problem (13).

The MPC algorithm solves on-line the optimization problem (13) at each time instant and implements only the first element of the optimal control profile. The optimization is repeated at the next sampling time by updating the initial condition with the new state.

In (13), if we require $Y_k = Z_k$, then we recover the MPC algorithm presented in (Kothare *et al.*, 1996), which can be described as:

$$\min_{Q_k > 0, Y_k} \gamma, \quad \text{s.t.} \qquad (14)$$

$a)$ LMI (10),

$b)$ $\begin{bmatrix} -Q_k & * & * \\ A_i Q_k + B_i Y_k & -Q_k & 0 \\ C_i Q_k + D_i Y_k & 0 & -\gamma I \end{bmatrix} < 0,\ \forall i \in [1, r],$

$c)$ $\begin{bmatrix} 1 & y_{k,i} \\ y_{k,i}^T & Q_k \end{bmatrix} \geq 0, \ \ \forall i \in [1, m].$

Note that the constraints in (14) are only sufficient for $x_k \in (Q_k^{-1}, 1)$ and hence the control $u_{k|k} = F_k x_{k|k}$ will never reach saturation limits. In (13), we permit the control to saturate and hence our algorithm would result in a less conservative closed-loop performance. For this reason, we refer to our algorithm (13) as *saturated MPC algorithm*. On the other hand, it is known that low-gain controllers that avoid saturation will often result in low levels of performance, especially for the cases where the disturbance is intermediate or small amplitude.

3.3 Feasibility and Stability

In the receding horizon framework, only the first computed control move $u_{k|k}$ is implemented. At time $k+1$, the optimization is re-solved with new measurements $x_{k+1|k+1}$ from the plant. The purpose of taking measurements at each time step is to compensate for unmeasured disturbances and model uncertainty. This is the main feature of the receding horizon control. The following lemma ensures that the solvability of the MPC algorithm at time $k > 0$, provided that the optimization problem (9) is solvable at $k = 0$.

Lemma 6. The existence of solution $(P_k, F_k, H_k, \gamma_k)$ to optimization problem (9) with a given x_k at time k implies the existence of solution $(P_{k+1}, F_{k+1}, H_{k+1}, \gamma_{k+1})$ to optimization problem (9) at time $k+1$.

Proof. Assume that $(P_k, F_k, H_k, \gamma_k)$ is a solution of the optimization problem (9) with the given x_k at time k, then, by the proof of Theorem 4, we have

$$x_{k+1|k}^T P_k x_{k+1|k} < x_{k|k}^T P_k x_{k|k} < \gamma_k$$

for any permissible p_k and the resulting state $x_{k+1|k}$ under the control $u_{k|k} = F_k x_{k|k}$. This implies that if $p_k = p_k^*$ at time k and the resulting state is $x_{k+1|k+1}^*$ under the control $u_k = F_k x_k$, consequently, we have

$$x_{k+1|k+1}^{*T} P_k x_{k+1|k+1}^* < \gamma_k.$$

Note that the unique constraint relating to the initial state $x_{k+1} = x_{k+1|k+1}^*$ in optimization problem (9) at $k+1$ is $x_{k+1}^T P_k x_{k+1} < \gamma_{k+1}$.This implies that $(P_k, F_k, H_k, \gamma_k)$ is also a solution of (9) at $k+1$, and thus (9) is feasible at $k+1$. ■

Theorem 7. For a given x_0, if it is feasible for optimization problem (13), then the receding horizon state feedback control obtained by (13) asymptotically stabilizes the system (1) at the origin.

Proof. Choose a Lyapunov function as $V(x_k) = x_k^T P_k x_k$, where $P_k > 0$ is obtained by solving optimization problem (9). The stability can then be easily proved with the routine way in (Kothare *et al.*, 1996) and hence the proof is omitted. ■

4. GAIN-SCHEDULING CONTROL LAW DESIGN

The approach to gain-scheduling involves the design of several LTI controllers for a parameterized family of linearized system models and the interpolation of the controller gains.

In general, the time-varying parameter vector p_k can be measured or estimated on-line. In this case, we can design a scheduling control law

$$u_{k+l|k} := \tilde{F}_{l|k} x_{k+l|k} \quad l \geq 0, \tag{15}$$

where $\tilde{F}_{l|k} := \tilde{F}_k(p_{k+l|k}) = \sum_{j=1}^r p_{k+l|k,j} F_{k,j}$, and $F_{k,j}$ is the "local" state feedback matrix of the local model $[A_j, B_j]$ at step k. It is shown that this kind of control laws can stabilize a larger class of LPV plants than the single control law (3). Note that F_k in (3) is a constant matrix for all $l > 0$, while $\tilde{F}_{l|k}$ in (15) is a time-varying matrix function of $p_{k+l|k}$ although $F_{k,j}$'s are constant for all $j = 1, 2, \cdots, r$.

With control law (15), the closed-loop system can be rewritten as

$$x_{k+l+1|k} = \sum_{i=1}^r p_{k+l|k,i}(A_i x_{k+l|k} + B_i \sigma(\tilde{F}_{l|k} x_{k+l|k})),$$
$$z_{k+l|k} = \sum_{i=1}^r p_{k+l|k,i}(C_i x_{k+l|k} + D_i \sigma(\tilde{F}_{l|k} x_{k+l|k})).$$

Let $\tilde{H}_{l|k} = \sum_{j=1}^r p_{k+l|k,j} H_{k,j}$. By Lemma 1, we have

$$x_{k+l+1|k} = \sum_{s=1}^{2^m} \sum_{i,j=1}^r \eta_{l|k,s} p_{k+l|k,i} p_{k+l|k,j} \tilde{A}_{s,i,j} x_{k+l|k},$$
$$z_{k+l|k} = \sum_{s=1}^{2^m} \sum_{i,j=1}^r \eta_{l|k,s} p_{k+l|k,i} p_{k+l|k,j} \tilde{C}_{s,i,j} x_{k+l|k},$$

where $\tilde{A}_{s,i,j} = A_i + B_i(E_s F_{k,j} + E_s^- H_{k,j})$, $\tilde{C}_{s,i,j} = C_i + D_i(E_s F_{k,j} + E_s^- H_{k,j})$, $s \in [1, 2^m]$, $i, j \in [1, r]$.

Remark 1. It can be seen that the above system can be further simplified if the subsystem (A_i, B_i, C_i, D_i) possesses a common input matrices B and D, namely $B_i = B, D_i = D$ for all i. In this case, the closed-loop system can be simplified to

$$x_{k+l+1|k} = \sum_{s=1}^{2^m} \eta_{l|k,s} \sum_{i=1}^r p_{k+l|k,i} \tilde{A}_{s,i,i} x_{k+l|k},$$
$$z_{k+l|k} = \sum_{s=1}^{2^m} \eta_{l|k,s} \sum_{i=1}^r p_{k+l|k,i} \tilde{C}_{s,i,i} x_{k+l|k}.$$

Theorem 8. Let $x_k = x_{k|k}$ be the state of system (1) measured at sampling time k and local state feedback matrices $F_{k,j}$, $j = 1, 2, \cdots, r$, be given. The ellipsoid $\Omega(P_k, \gamma)$ is invariant if there exist matrices $H_{k,j} \in \mathbb{R}^{m \times n}$ satisfying

$$\tilde{A}_{s,i,j}^T P_k \tilde{A}_{s,i,j} - P_k + \tilde{C}_{s,i,j}^T \tilde{C}_{s,i,j} < 0,$$
$$i, j \in [1, r], \ s \in [1, 2^m], \tag{16}$$

and $\Omega(P_k, \gamma_k) \subset \bigcap_{j=1}^r \mathcal{L}(H_{k,j})$. Moreover, for any $x_k \in \Omega(P_k, \gamma_k)$, the performance objective function (5) satisfies $J_k < \gamma$.

The proof is similar to that of Theorem 4.

Corollary 9. For the special case with $B_i = B$ and $D_i = D$ for all i, the ellipsoid $\Omega(P_k, \gamma)$ is invariant if there exist r matrices $H_{k,i} \in \mathbb{R}^{m \times n}$ satisfying

$$\tilde{A}_{s,i,i}^T P_k \tilde{A}_{s,i,i} - P_k + \tilde{C}_{s,i,i}^T \tilde{C}_{s,i,i} < 0,$$
$$i \in [1, r], s \in [1, 2^m], \tag{17}$$

and $\Omega(P_k, \gamma_k) \subset \bigcap_{j=1}^r \mathcal{L}(H_{k,j})$. Moreover, for any $x_k \in \Omega(P_k, \gamma_k)$, the performance objective function (5) satisfies $J_k < \gamma$.

In what follows, we will present a less conservative condition. Let

$$\bar{p}_{k+l|k,t} = \begin{cases} p_{k+l|k,i}^2, & t = i^2, \\ 2p_{k+l|k,i} p_{k+l|k,j}, & t = ij, j < i \in [1, r], \end{cases}$$
$$\bar{A}_{s,t} = \begin{cases} \tilde{A}_{s,i,i}, & t = i^2, \\ (\tilde{A}_{s,i,j} + \tilde{A}_{s,j,i})/2, & t = ij, j < i \in [1, r], \end{cases}$$
$$\bar{C}_{s,t} = \begin{cases} \tilde{C}_{s,i,i}, & t = i^2, \\ (\tilde{C}_{s,i,j} + \tilde{C}_{s,j,i})/2, & t = ij, j < i \in [1, r], \end{cases}$$

Then we have

$$0 \leq \bar{p}_{k+l|k,t} \leq 1, \quad \sum_{t=1}^{r(r+1)/2} \bar{p}_{k+l|k,t} = 1.$$

Hence

$$\Delta V(x_{k+l|k}) = x_{k+l|k}^T \left\{ \sum_{s=1}^{2^m} \eta_{l|k,s} \sum_{t=1}^{r(r+1)/2} \bar{p}_{k+l|k,t} \bar{A}_{s,t}^T P_k \times \sum_{s=1}^{2^m} \eta_{l|k,s} \sum_{t=1}^{r(r+1)/2} \bar{p}_{k+l|k,t} \bar{A}_{s,t} - P_k \right\} x_{k+l|k},$$

$$J_{k,l} = x_{k+l|k}^T \left\{ \sum_{s=1}^{2^m} \eta_{l|k,s} \sum_{t=1}^{r(r+1)/2} \bar{p}_{k+l|k,t} \begin{bmatrix} \bar{A}_{s,t} \\ \bar{C}_{s,t} \end{bmatrix}^T \tilde{P}_k \times \sum_{s=1}^{2^m} \eta_{l|k,s} \sum_{t=1}^{r(r+1)/2} \bar{p}_{k+l|k,t} \begin{bmatrix} \bar{A}_{s,t} \\ \bar{C}_{s,t} \end{bmatrix} - P_k \right\} x_{k+l|k}.$$

Theorem 10. Let $x_k = x_{k|k}$ be the state of system (1) measured at sampling time k and local state feedback control matrices $F_{k,j},\ j = 1, 2, \cdots, r$, be given. The ellipsoid $\Omega(P_k, \gamma)$ is invariant if there exist matrices $H_{k,j} \in \mathbb{R}^{m\times n}$ satisfying

$$\tilde{A}_{s,i,i}^T P_k \tilde{A}_{s,i,i} - P_k + \tilde{C}_{s,i,i}^T \tilde{C}_{s,i,i} < 0, \quad i \in [1,r], s \in [1,2^m], \quad (18)$$

$$(\tilde{A}_{s,i,j}^T + \tilde{A}_{s,j,i}^T) P_k (\tilde{A}_{s,i,j} + \tilde{A}_{s,j,i}) - 4P_k + (\tilde{C}_{s,i,j}^T + \tilde{C}_{s,j,i}^T)(\tilde{C}_{s,i,j} + \tilde{C}_{s,j,i}) < 0, \quad i < j \in [1,r], s \in [1,2^m], \quad (19)$$

and $\Omega(P_k, \gamma_k) \subset \bigcap_{j=1}^r \mathcal{L}(H_{k,j})$. Moreover, for any $x_k \in \Omega(P_k, \gamma_k)$, the performance objective function (5) satisfies $J_k < \gamma$.

Remark 2. In Comparison with Theorem 8, the number of matrix inequalities in Theorem 10 is reduced by $r(r-1)2^{m-1}$. In the special case of $B_i = B,\ \forall i$, another $r(r-1)2^{m-1}$ matrix inequalities in (19) can be removed as in Corollary 9.

Let $Y_{k,j} = F_{k,j} Q_k$ and $Z_{k,j} = H_{k,j} Q_k,\ j \in [1,r]$. Then (18) and (19) are equivalent to the following LMIs

$$\begin{bmatrix} -Q_k & * & * \\ A_i Q_k + B_i(E_s Y_{k,i} + E_s^- Z_{k,i}) & -Q_k & 0 \\ C_i Q_k + D_i(E_s Y_{k,i} + E_s^- Z_{k,i}) & 0 & -\gamma I \end{bmatrix} < 0, \quad i \in [1,r],\ s \in [1,2^m], \quad (20)$$

$$\begin{bmatrix} -4Q_k & * & * \\ A_i Q_k + B_i(E_s Y_{k,j} + E_s^- Z_{k,j}) + A_j Q_k + B_j(E_s Y_{k,i} + E_s^- Z_{k,i}) & -Q_k & 0 \\ C_i Q_k + D_i(E_s Y_{k,j} + E_s^- Z_{k,j}) + C_j Q_k + D_j(E_s Y_{k,i} + E_s^- Z_{k,i}) & 0 & -\gamma I \end{bmatrix} < 0, \quad i < j \in [1,r],\ s \in [1,2^m]. \quad (21)$$

Theorem 11. Let $x_k = x_{k|k}$ be the state of system (1) measured at sampling time k. Then gain-scheduled state feedback control law (15) at step k that minimizes the upper bound of performance function (5) can be solved by

$$F_{k,j} = Y_{k,j} Q_k^{-1},\ \forall j \in [1,r],$$

where $(Q_k > 0, Y_{k,j})$ is the solution of the following LMI optimization problem

$$\min_{Q_k > 0, Y_{k,j}, Z_{k,j}} \gamma, \quad \text{s.t.} \quad (22)$$

a) LMI (10),

b) LMI (20), (21),

c) $\begin{bmatrix} 1 & z_{k,i}^j \\ (z_{k,i}^j)^T & Q_k \end{bmatrix} \geq 0,\ \forall i \in [1,m],\ \forall j \in [1,r].$

5. CONCLUSIONS

In this paper, we have addressed the model predictive control algorithm for the linear parameter-varying systems subject to actuator saturation. The set invariance and the optimal control problem for the LPV systems subject to actuator saturation have been solved by solving an LMI optimization problem. An MPC algorithm based on the set invariance is proposed. The gain-scheduling MPC algorithm is also studied by using the linear matrix inequality techniques.

REFERENCES

Apkarian, P., P. Gahinet and G. Becker (1995). Self-scheduled H_∞ control of linear parameter-varying systems. *Automatica* **31**, 1251–1261.

Bitmead, R.R., M. Gevers and V. Wertz (1990). *Adaptive optimal control: The thinking man's GPC.* Englwood Cliffs, NJ: Prentice-Hall.

Chmielewski, D. and V. Manousiouthakis (1996). On constrained infinite-time linear quadratic optimal control. *Sys. Con. Let.* **29**, 121–129.

Clarke, D. W. (1994). *Advances in model-based predictive control.* Oxford University Press.

Hu, T. and Z. Lin (2001). *Control Systems with Actuator Saturation: Analysis and Design.* Birkhäuser. Boston.

Kothare, M. V., V. Balakrishnan and M. Morari (1996). Robust constrained model predictive control using linear matrix inequalities. *Automatica* **32**, 1361–1379.

Lin, Z. (1998). *Low Gain Feedback.* Springer. London.

Lu, Y. and Y. Arkun (2000). Quasi-min-max MPC algorithms for LPV systems. *Automatica* **36**, 527–540.

Michalska, H. and D.Q. Mayne (1993). Robust receding horizon control of constrained nonlinear systems. *IEEE T AC* **38**(11), 1623–1633.

Rawlings, J. and K. Muske (1993). The stability of constrained receding-horizon control. *IEEE T AC* **38**, 1512–1516.

Rugh, W.J. (1991). Analytical framework for gain scheduling. *IEEE Con. Sys. Mag.* **11**, 74–84.

Scokaert, P.O.M. and J. Rawlings (1998). Constrained linear quadratic regulation. *IEEE T AC* **43**(8), 1163–1169.

Shamma, J.S. and M. Athans (1990). Analysis of nonlinear gain-scheduled control systems. *IEEE T AC* **35**, 898–907.

Wu, F. and A. Packard (1995). LQG control design for LPV systems. *Proc. of American Control Conf.* pp. 4440–4444.