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SHAPE OPTIMIZATION PROBLEM FOR 3-DIMENSIONAL BODIES WITH MINIMAL SURFACE HEAT

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Abstract: Spacecraft shape has a great influence on its aerodynamic and heat characteristics. This paper presents the statements and analytical solutions of new optimization problems of finding optimal 3-dimensional body shapes from viewpoint of minimum of radiation heat transfer (including the problems of optimal longitudinal and transversal contours of bodies). *Copyright ©2002 IFAC*

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1. INTRODUCTION

Optimal aerodynamic design of space apparatus is one of the most interesting and promising fields of the applications of the optimal control methods. It is known that shuttle-like spacecrafts moving in planetary atmosphere with hypersonic speeds are exposed to intensive radiation heat. One of the efficient method of solving problem of minimizing heat fluxes to apparatus surface is a choice of an optimal body shape.

In addition to improving traditional configurations (axis-symmetric bodies and flat wings) the new shapes of bodies with optimum thermophysical and aerodynamic characteristics are now under investigation. It is known that bodies with circular transversal sections are not optimal with viewpoint of drag minimization. In particular, in (Miele and Saaris, 1965) it was shown that the drag of star-like bodies is less then that of equivalent bodies of revolution.

Note that identical results take place in the optimal design of three-dimensional bodies with minimal surface heat. In paper (Arguchintseva, 1999) the problems of minimization of total (convective and radiation) heat fluxes along flight trajectory in atmosphere for 3-dimensional bodies with the elliptic base section were investigated.

However, the review of scientific references on this problem (Arguchintseva, 1992, 1999) has shown that in spite of wide research of optimization problems of super- and hypersonic aerodynamics many questions connected with influence of heat fluxes on the shape of the 3-dimensional bodies with noncircular transversal sections are not well studied.

This paper presents investigation of a new problem of minimization of radiation heat flux to the surfaces of three-dimensional bodies. In a class of slender bodies possessing homotetic property the initial optimization problem may be reduced to two separated problems of finding optimal longitudinal and transverse contours. From mathematical point of view these problems are variational ones with a glance to various isoperimetric and boundary conditions.

The investigation of the problem of determining the optimal transversal contour has shown that there exists a class of variational solutions composed of n identical cycles. A distinguishing feature of the suggested approach is that the minimization procedure includes not only a search for each extremal segment but also the number of these segments. This leads to the additional optimizing condition on the number of cycles. Joint integration of the Euler-Lagrange equation and the condition mentioned above has permitted the author to obtain three classes of analytical solutions.

2. PROBLEM STATEMENT

Consider a hypersonic motion of a three-dimensional body in a cylindrical coordinate system (r, θ, z). The origin of coordinates is located at the critical point of the body; the z-axis is opposite to the direction of the body motion. Restrict our consideration to shapes possessing a homotetic property; that is, any transversal section of the body, perpendicular to axis z, should be geometric similar to its base section. Consequently, the geometry of the body is described by the equation:

$$f(r, \theta, z) = r - \varphi(z)\rho(\theta) = 0,$$

where $\varphi(z)$, $\rho(\theta)$ are the longitudinal and transversal contours of the body.

It was shown (Pilyugin and Tirskii, 1989) that the surface distribution of the local radiation heat flux is determined as:

$$q_R = q_{R0}(- < N, e_z >)^m,$$

$$< N, e_z > = \frac{-\dot{\varphi}(z)\rho(\theta)}{\left[1 + (\dot{\rho}(\theta)/\rho(\theta))^2 + (\dot{\varphi}(z)\rho(\theta))^2\right]^{1/2}},$$

where N is the unit vector normal to element of wetted area ds, positively oriented outward; (e_r, e_θ, e_z) are unit vectors of the cylindrical coordinate system; q_{R0} is the radiation heat flux in the critical point of the body; m is a parameter determined by radiation properties of the medium and the velocity of the body motion ($m \in [3; 10]$).

By integrating the local radiation heat flux q_R along the wetted surface S, we obtain the formula for the coefficient of the radiation heat:

$$C_R = \int\limits_S \int \frac{q_R}{q_{R0}}\, ds; \quad ds = \frac{r}{< N, e_r >} d\theta dz;$$

$$< N, e_r > = \left[1 + (\dot{\rho}(\theta)/\rho(\theta))^2 + (\dot{\varphi}(z)\rho(\theta))^2\right]^{-1/2},$$

or

$$C_R = \int\limits_0^L \varphi(z)\dot{\varphi}^m(z) dz \times$$

$$\times \int\limits_0^{2\pi} \frac{\rho^{m+1}(\theta) d\theta}{\left[1 + (\dot{\rho}(\theta)/\rho(\theta))^2 + (\dot{\varphi}(z)\rho(\theta))^2\right]^{(m-1)/2}},$$

where L is a body length. In the class of slender bodies ($(\dot{\varphi}(z)\rho(\theta))^2 << 1$) the expression for C_R may be presented as a product of two functionals $J_1(\varphi)$ $J_2(\rho)$, depending correspondingly only from longitudinal $\varphi(z)$ or transversal $\rho(\theta)$ contours of the body:

$$C_R = J_1(\varphi) \cdot J_2(\rho),$$

$$J_1(\varphi) = \int\limits_0^l \varphi(z)\dot{\varphi}^m(z) dz, \tag{1}$$

$$J_2(\rho) = \int\limits_0^{2\pi} \frac{\rho^{m+1}(\theta) d\theta}{\left[1 + (\dot{\rho}(\theta)/\rho(\theta))^2\right]^{(m-1)/2}}. \tag{2}$$

Consequently, the general problem of finding shapes of a three-dimensional bodies from the viewpoint of the minimum of surface radiation heat may be reduced to following ones:

1. The problem on optimal longitudinal contour $\varphi(z)$ minimizing the functional $J_1(\varphi)$ (1).

2. The problem on optimal transversal contour $\rho(\theta)$ minimizing the functional $J_2(\rho)$ (2).

It is investigated the following restrictions on the body shape:

1) boundary conditions:
   - on the longitudinal contour

$$\varphi(0) = 0, \quad \varphi(L) = 1; \tag{3}$$

   - on the transversal contour

$$\rho(0) = \rho(2\pi); \tag{4}$$

2) isoperimetric conditions:
   - on the given base area of the body S_B

$$\int\limits_0^{2\pi} \rho^2(\theta) d\theta = 2S_B; \tag{5}$$

   - on the given area of the wetted surface

$$\int\limits_0^{2\pi} \int\limits_0^L \varphi(z)\rho(\theta) \left\{1 + (\dot{\rho}(\theta)/\rho(\theta))^2 + \right.$$

$$\left. + (\dot{\varphi}(z)\rho(\theta))^2\right\}^{1/2} d\theta dz = S,$$

which, because of the slender body approximation ($(\dot{\varphi}(z)\rho(\theta))^2 << 1$), reduces to:

$$\int\limits_0^{2\pi} \int\limits_0^L \varphi(z) \left[\rho^2(\theta) + \dot{\rho}^2(\theta)\right]^{1/2} d\theta dz = S. \tag{6}$$

3. OPTIMAL LONGITUDIONAL CONTOUR

Consider the variational problem 1: "In the class of smooth functions $\varphi(z)$, which are consistent with the end conditions (3), find the function, which minimize the functional $J_1(\varphi)$ (1)".

The first integral of the Euler-Lagrange equation for this problem is given by:

$$(1-m)\varphi(z)\dot{\varphi}^m(z) = C, \quad C = const.$$

The solution of this differential equation consistent with the end conditions leads to:

$$\varphi(z) = (z/L)^{\frac{m}{m+1}}. \tag{7}$$

Obtained longitudinal contour satisfies Legendre necessary condition. The associated minimum value of the goal functional is given by:

$$J_1^{min}(\varphi) = L^{1-m}(\frac{m}{m+1})^m.$$

Because of the result (7), the isoperimetric condition on the wetted surface area (6) becomes:

$$\int_0^{2\pi} \sqrt{\rho^2(\theta) + \dot{\rho}^2(\theta)}d\theta = S_*, \tag{8}$$

$$S_* = \frac{S}{L} \cdot \frac{(2m+1)}{m+1} = const.$$

4. OPTIMAL TRANSVERSAL CONTOUR

The variational problem 2 on the optimal transversal contour is formulated as: "In the class of the piecewise-smooth functions $\rho(\theta)$, which are consistent with boundary condition (4) and isoperimetric conditions on the given areas of the base S_B (5) and the wetted surface S_* (8), find the transversal contour of the body minimized the radiation heat functional $J_2(\rho)$ (2)".

The Lagrange functional for variational problem under the consideration is written as:

$$J(\rho) = \int_0^{2\pi} F(\lambda_1, \lambda_2, \rho, \dot{\rho})d\theta;$$

$$F(\lambda_1, \lambda_2, \rho, \dot{\rho}) = \frac{\rho^{m+1}}{[1+\dot{\rho}^2/\rho^2]^{(m-1)/2}} + \lambda_1\rho^2 + \\ + \lambda_2\sqrt{\rho^2 + \dot{\rho}^2}, \tag{9}$$

where λ_1, λ_2 are undetermined constant Lagrange multipliers.

A solution of the optimization problem 2 must satisfy:

1) the Euler-Lagrange equation for the functional (9) that admits the first integral:

$$F - \dot{\rho}F_{\dot{\rho}} = C, \quad C = const; \tag{10}$$

2) the transversality condition:

$$F_{\dot{\rho}}|_0 = F_{\dot{\rho}}|_{2\pi} = 0, \tag{11}$$

3) the Legendre condition:

$$F_{\dot{\rho}\dot{\rho}} \geq 0; \tag{12}$$

4) the Erdmann-Weierstrass condition in the corner points:

$$\Delta[F - \dot{\rho}F_{\dot{\rho}}] = \Delta[F_{\dot{\rho}}]\delta\rho = 0,$$

where the symbol $\Delta[...]$ denotes the difference between quantities evaluated after ("+") and before ("-") the corner θ_C.

The Erdmann-Weierstrass condition leads to relationship: $\dot{\rho}_+(\theta_C) + \dot{\rho}_-(\theta_C) = 0$. It follows, that there exists a class of the transversal sections composed n identical cycles, each of which covers the angle interval $2\pi/n$ and is symmetric with respect to the radial line joining the origin of the polar coordinate system with the peak point of the cycle. Each cycle involves a pair of symmetric segments, and each segment does not contain corners. Such contour satisfies the boundary condition (4).

We shall now restrict analysis to one segment of composed transversal section considered above. Consequently, the goal integral and isoperimetric conditions can be rewritten in the form:

$$J_2(\rho, n) = 2n \int_0^{\pi/n} \frac{\rho^{2m}(\theta)d\theta}{[\rho^2(\theta) + \dot{\rho}^2(\theta)]^{(m-1)/2}}; \tag{13}$$

$$2n \int_0^{\pi/n} \rho^2(\theta)d\theta = 2S_B; \tag{14}$$

$$2n \int_0^{\pi/n} \sqrt{\rho^2(\theta) + \dot{\rho}^2(\theta)}d\theta = S_*; \tag{15}$$

So the Lagrange functional of the variational problem is given as:

$$J(\rho) = 2n \int_0^{\pi/n} F(\lambda_1, \lambda_2, \rho, \dot{\rho})d\theta;$$

where the function F is defined by formula (9). It must be noticed that the object of the minimization procedure is to find not only the equation of each segment of the extremal but also the number n of segments.

Concerning the shape optimization of each segment, the Euler-Lagrange equation (10) is still valid with consideration of transversality condition (11) in the end points $\theta_0 = 0,\ \ \theta_f = \pi/n$.

Optimizing condition on the number of extremal segments can be written in the form:

$$\int_0^{\pi/n} F d\theta - \frac{\pi}{n}[F]_{\pi/n} = 0. \qquad (16)$$

Note that only the integer values of the cycle parameter n are physically realistic.

Equation (10) and transversality condition (11) in the end points $\theta_0 = 0,\ \ \theta_f = \pi/n$ lead to:

$$(F)_0 = (F)_{\pi/n} = C.$$

The first integral of the Euler-Lagrange equation (10) is integrated over the interval $[0, \pi/n]$:

$$\int_0^{\pi/n} F d\theta - \int_0^{\pi/n} \Phi d\theta = \int_0^{\pi/n} C d\theta \equiv \frac{\pi}{n}[F]_{\pi/n}; \quad (17)$$

$$\Phi = \dot{\rho} F_{\dot{\rho}}.$$

With regard (16), (17), the following conditions must be satisfied along the optimal contour:

$$\int_0^{\pi/n} \Phi d\theta = 0, \qquad \Phi|_0 = \Phi|_{\pi/n} = 0. \qquad (18)$$

There is need to examine a case when the function $\Phi = 0$ at every point of the interval $[0, \pi/n]$. Other case (when the function Φ changes a sign along the mentioned interval) is impossible because of violating the optimal necessary conditions for the functional (13).

Condition (18) leads to the expression:

$$\dot{\rho}^2(\dot{\rho}^2 + \rho^2 - a^2\rho^4) = 0; \quad a = \left(\frac{m-1}{\lambda_2}\right)^{1/m}.$$

It implies that:

$$\dot{\rho} = 0, \qquad (19)$$

$$or \quad \Psi = \dot{\rho}^2 + \rho^2 - a^2\rho^4 = 0. \qquad (20)$$

The Legendre condition (12) shows that equations (19), (20) are valid correspondingly for $\rho \leq \rho_{kr} = 1/a$ and $\rho \geq \rho_{kr}$.

Assuming that the initial and final points of the extremal are arbitrary located on the lines (19),(20), three classes of bodies must be analyzed.

4.1. Bodies of Class I

Consider the case when both end points of the extremal are located on the line (19):

$$\dot{\rho}|_0 = 0, \qquad \dot{\rho}|_{\pi/n} = 0.$$

So the optimal transversal contour is described by differential equation (19), which admits the general integral: $\rho = C_1$, $C_1 = const$.

Thus, the optimal body of the class I is a body of revolution. The constant C_2 is defined from the isoperimetric conditions:
1) $C_1 = \sqrt{S_B/\pi}$ - in the case of the given base area (5);
2) $C_1 = S_*/(2\pi)$ - in the case of the given area of the wetted surface of body (8). The parameters of the problem have to be connected ($S_B = S_*^2/(4\pi)$), if the isoperimetric conditions (5), (8) are satisfied together.

The following minimum value of the radiation heat coefficient (2) is:

$$J_2^{min} = 2\pi C_1^{m+1}.$$

4.2. Bodies of Class II

Consider the extremal with both end points on the line (20) $\Psi = 0$:

$$\Psi|_0 = 0, \qquad \Psi|_{\pi/n} = 0.$$

In this case a solution of the differential equation (20) is given by:

$$\rho(\theta) = \frac{1}{a\cos(\theta + C_2)}, \quad C_2 = const.$$

Consequently, optimal bodies of the class II have star-like transversal sections.

The unknown parameters a and C_2 are defined from isoperimetric conditions (5), (8):

$$a = \frac{S_*}{2S_B}; \quad \frac{S_*^2}{S_B} = 4n(tg\,(\frac{\pi}{n} + C_2) - tg\,C_2).$$

Consider the optimizing condition (16) on the cycle parameter n of the extremal. It leads to:

$$\left[m\left(\frac{\lambda_2}{m-1}\right)^{\frac{m-1}{m}} + \lambda_1\right] \times$$

$$\times \left\{\int_0^{\pi/n} \rho^2(\theta) d\theta - \frac{\pi}{n}\rho^2(\frac{\pi}{n})\right\} = 0.$$

The expression in the braces is not zero, because in this case the Euler-Lagrange equation is violated.

Consequently, we have a connection between the Lagrange multipliers:

$$\lambda_1 = -m \left(\frac{\lambda_2}{m-1} \right)^{\frac{m-1}{m}},$$

$$\lambda_2 = (m-1)(2S_B/S_*)^m.$$

Then the optimizing condition (16) is valid for any value of the cycle parameter n. Thus, the integral of radiation heat is independent from the number of cycles n:

$$J_2^{min} = S_*{}^{1-m}(2S_B)^m. \quad (21)$$

4.3. Bodies of Class III.

In this case the extremal is subject to the conditions:

$$\dot{\rho}|_0 = 0, \quad \Psi|_{\pi/n} = 0.$$

Consequently, the transversal contour of these bodies is described by the differential equations (19), (20) and involves two subarcs: one is circular and other is straight line tangent to circular. Analytically, the extremal arc is represented by:

$$\rho = \begin{cases} 1/, \quad 0 \leq \theta \leq \varepsilon, \\ 1/[a\cos(\theta - \varepsilon)], \quad \varepsilon \leq \theta \leq \pi/n, \end{cases}$$

where ε is the angular interval corresponding to circular portion, that determined from isoperimetric condition (5), (8):

$$\frac{S_*^2}{S_B} = 4n\left[\varepsilon + tg(\frac{\pi}{n} - \varepsilon)\right]; \; a = S_*/(2S_B).$$

The expression for radiation heat flux to body surface of the class III does not depend on the number of cycles n and is defined by formula (21).

5. ANALYSIS OF RESULTS

Thus, we have three types of bodies with optimal transversal contours:

1) bodies of class I with circular contours ($\rho \leq \rho_{kr}$);
2) bodies of class II with star-like contours ($\rho \geq \rho_{kr}$);
3) bodies of class III having star-like contours with circular arcs ($\rho \geq \rho_{kr}$).

Fig.1 presents the optimal shape of three-dimensional body of the class II for $m = 3$, $n = 6$. On Fig.2 there are optimal transversal sections of class III for spectrum of parameters (S_*^2/S_B) and $n = 3$.

Analysis of results shows that there are two limit cases of optimal transversal contour:

a) the circular contour, when ($S_*^2/S_B = 4\pi$);
b) the regular polygon contour ("mn"), when $S_*^2/S_B = 4n\, tg(\pi/n)$.

It must be noticed, that solutions of the class III take place when $S_*^2/S_B \in [4\pi; 4n\, tg(\pi/n)]$, and solutions of the class II are valid when $S_*^2/S_B \geq 4n\, tg(\pi/n)$.

Bodies of optimal shapes significantly reduce radiation surface heat by comparison with bodies of revolution having the same given geometric characteristics (until 90% -for the class II ; until 70% - for the class III) (Table 1).

Table 1. Comparison of heat characteristics of optimal bodies and equivalent bodies of revolution

	class III		class II	
S_*^2/S_B	$3.8n\, tg(\pi/n)$		$6n\, tg(\pi/n)$	
$\rho(\pi/n)/\rho(0)$				
$n = 4$	1.342		1.966	
$n = 6$	1.091		1.497	
m	5	10	5	10
J_2^{min}/J_2^{rev}				
$n = 4$	0.684	0.425	0.274	0.054
$n = 6$	0.911	0.811	0.366	0.104

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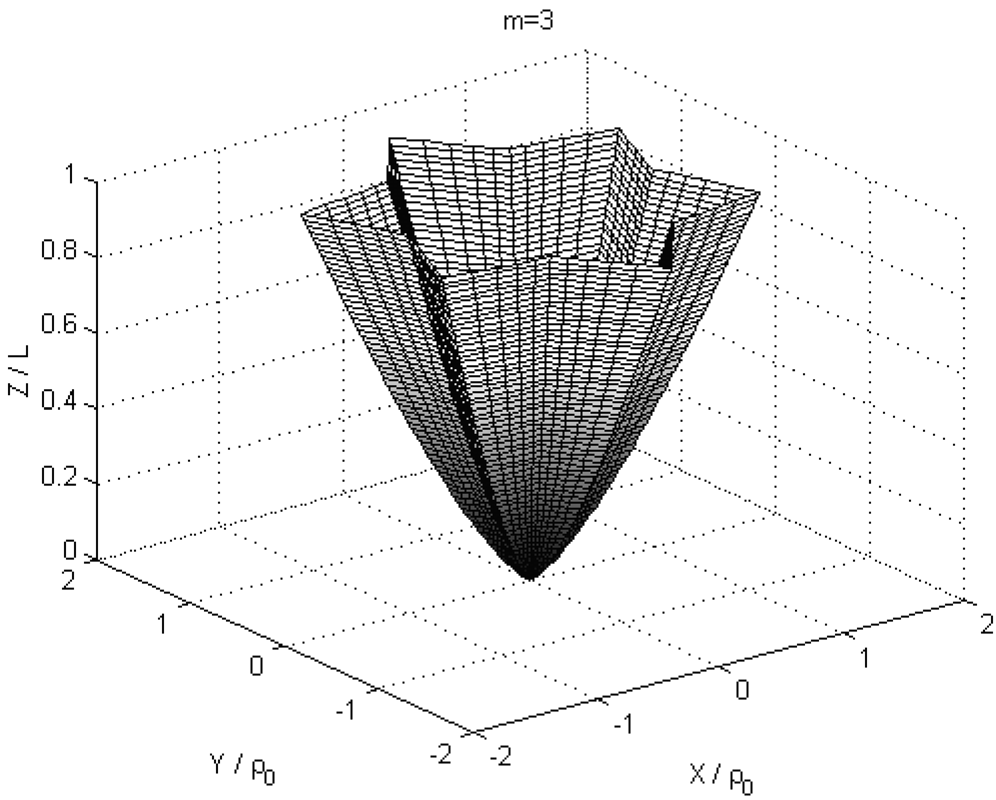

Fig.1. Optimal body of the class II with given (S_*, S_B) for n=6, $S_*^2/S_B = 6n\, tg\,(\pi/n)$.

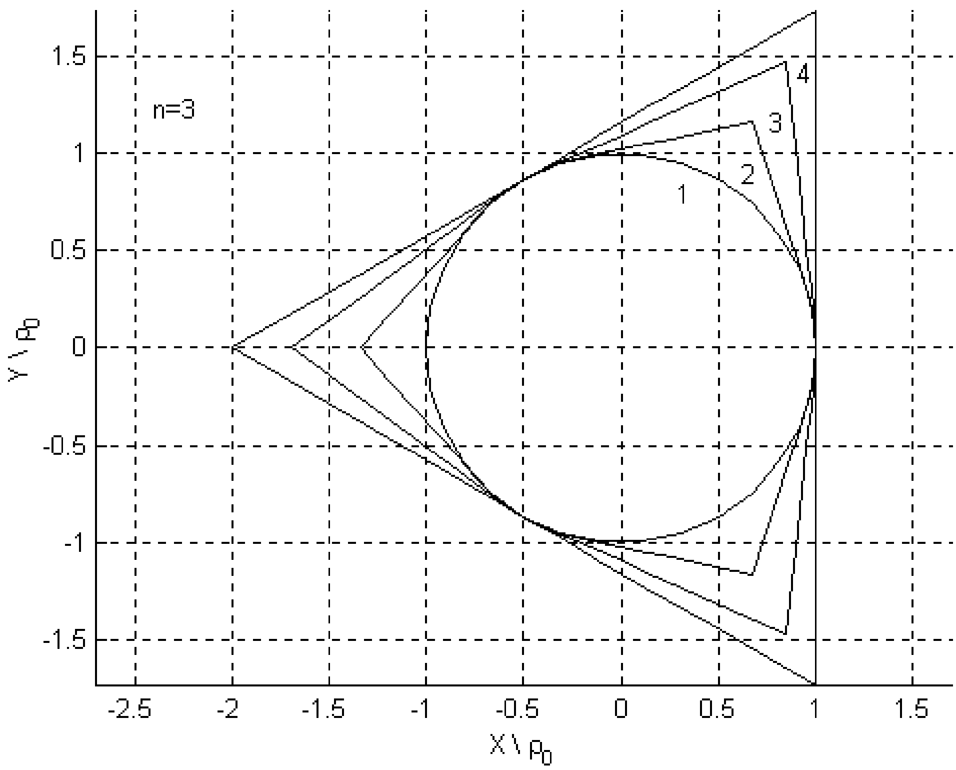

Fig. 2. The transversal sections of bodies of the class III with given (S_*, S_B) for $n=3$, $m=5$, S_*^2/S_B ={1. - 4π; 2. – $2.8\, n\, tg\,(\pi/n)$; 3. – $3.4n\, tg\,(\pi/n)$; 4. - $4n\, tg\,(\pi/n)$}.