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A FRAMEWORK FOR DISTURBANCE ATTENUATION BY DISCONTINUOUS CONTROL

H. W. Knobloch [*] C. Ebenbauer [**] F. Allgöwer [**]

[*] *Institute of Mathematics, University of Würzburg, Germany*
[**] *Institute for Systems Theory in Engineering, University of Stuttgart, Germany*
{ce,allgower}@ist.uni-stuttgart.de

Abstract: We propose a framework for disturbance attenuation in case of nonlinear systems affine in a multivariable input. Roughly speaking, the underlying idea of our approach exploits the nature of nonlinear systems with a multivariable input. In particular, we extend the system under investigation by additional control possibilities, which were formed out by the so-called mixed Lie brackets. It is shown that this extension is reasonable and may allow to establish special dissipation inequalities by using a certain type of discontinuous feedback law. We illustrate this analysis technique by examples and suggest a synthesis framework for disturbance attenuation which fits very well into this concept.

Keywords: Discontinuous feedback, nonlinear multivariable control, stabilization, disturbance attenuation, dissipation inequalities.

1. INTRODUCTION

Since the publication of Brockett's paper (Brockett, 1983), it is a well known fact that discontinuous feedback laws are more powerful than their continuous counterpart. Indeed, as it was shown in (Clarke *et al.*, 1997), the class of asymptotically controllable system of the form $\dot{x} = f(x,u)$ can only be stabilized if the admissible feedback laws are extended to a certain class of discontinuous feedback laws. In the meantime, it has turned out that many important classes of nonlinear systems with interest in practice, for example nonholonomic mechanical systems, cannot be stabilized by using smooth time invariant feedback laws since they do not satisfy Brockett's necessary condition for smooth stabilization (Brockett, 1983). Additionally, discontinuity plays a dominant role in the classical calculus of variations and therefore also in the theory of optimal control. Beside these facts, discontinuous feedback laws may naturally appear in the context of nonlinear systems with a multivariable input as it is shown here.

In this paper an approach is presented that exploits the nature of nonlinear systems with a multivariable input. In particular, the system under investigation will be extended by additional control possibilities, which were formed out by the so-called mixed Lie brackets. It is shown that this extension is reasonable and may allow to establish special dissipation inequalities by using a certain type of discontinuous feedback law.

The paper is organized as follows. In Section 2 we outline the basic idea of our approach with a first application to stabilization problems. A formal statement of the underlying analysis technique is given in Section 3 as well as an application to a special type of dissipation inequality which form the basis for the disturbance attenuation strategy presented in Section 4. Additionally, some preliminary synthesis results with a special discontinuous feedback law are presented in Section 4. Finally, the results are summarized and critically discussed in Section 5.

2. THE BASIC IDEA AND A FIRST APPLICATION TO STABILIZATION PROBLEMS

Let us consider the global stabilization problem for a nonlinear system of the form

$$\dot{x} = p(x) + B(x)u \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state. To simplify matters, let us assume for the moment that the control input $u = [u_1\ u_2]^T$ is two-dimensional [1], $B(x)$ is the matrix given by the two column vectors $b_1(x)$, $b_2(x)$, and p, b_1, b_2 are smooth vector fields with $p(0) = b_1(0) = b_2(0) = 0$. One possible way to achieve global stability is to use control Lyapunov functions (CLF). The main idea behind CLF is, to pick a proper [2], positive definite, scalar-valued function $V(x)$ and then try to find a feedback law $u = u(x)$, which renders $\dot{V}(x, u(x))$ negative definite. In the case of system (1), this means that

$$\dot{V}(x, u(x)) = V_x(x)\,(p(x) + B(x)u(x)) \overset{!}{<} 0, \tag{2}$$

must be satisfied for all $x \neq 0$, where $V_x(x) = (\partial V/\partial x)(x)$. If $V_x(x)b_1(x) \neq 0$ or $V_x(x)b_2(x) \neq 0$ or both, then we have won because we can always find a smooth [3] $u = u(x)$ on $\mathbb{R}^n \setminus 0$ such that $\dot{V}(x, u(x))$ becomes negative definite. If $V_x(x)b_1(x) = 0$ *and* $V_x(x)b_2(x) = 0$, then $V(x)$ is a CLF if and only if $V_x(x)p(x) < 0$. To sum up, a proper, positive definite function $V(x)$ is a (global) CLF for system (1) if, for all $x \neq 0$

$$V_x(x)b_i(x) = 0\ \forall i \Rightarrow V_x(x)p(x) < 0. \tag{3}$$

As we can see from (3), the only real stumbling block of finding a CLF is thus the set of states where $V_x(x)b_i(x) = 0\ \forall i$, because on this set the uncontrolled system has to satisfy the property $V_x(x)p(x) < 0$. Now, if $V(x)$ is not a CLF, then we can try to find a 'better' $V(x)$ which is in general not an easy task. Here, we suggest the following approach. We extend our system with a certain additional 'virtual' control possibility given by the mixed Lie bracket, that is $[b_1, b_2](x) = \left(\frac{\partial b_2}{\partial x} b_1 - \frac{\partial b_1}{\partial x} b_2\right)(x)$, so that a new extended system can be defined as follows:

$$\dot{z} = p(z) + \hat{B}(z)\hat{u}, \tag{4}$$

where the matrix $\hat{B}(z)$ is given by the three column vectors $\hat{b}_1(z) = b_1(z), \hat{b}_2(z) = b_2(z), \hat{b}_3(z) = [b_1, b_2](z)$ and the extended control input is given by $\hat{u} = [u_1\ u_2\ v_{1,2}]^T$, where $v_{1,2}$ represents an additional 'virtual' control variable. To verify that $V(z)$ is a CLF of the extended system (4), we have to check for all $z \neq 0$

$$V_z(z)\hat{b}_i(z) = 0\ \forall i \Rightarrow V_z(z)p(z) < 0. \tag{5}$$

Now, one may ask the following question: Can we use condition (5) instead of the well-known condition (3) to verify that system (1) is stabilizable? If the answer to this question is yes, then this would simplify the process of finding an appropriate function V, because the additional control possibility implies that the significant set $V_x(x)b_i(x) = 0$ may be reduced, namely to the set $V_z(z)\hat{b}_i(z) = 0$. This would then constitute a new and less conservative sufficient condition to verify stabilizability that contains the well-known condition (3) as a subcase. And indeed, as a consequence of the results presented in (Knobloch and Wagner, 1984), the answer to the question can be shown to be a conditional "yes, we can!". Roughly speaking, the results presented in (Knobloch and Wagner, 1984) give the following relationship between system (1) and (4): Any trajectory of the extended (non-physical) system (4) can be 'tracked' by a trajectory of the original (physical) system (1) up to an arbitrary small error ε on a given finite time interval. What is essential, is the fact that this can be achieved by a certain type of discontinuous feedback law. A formal statement of this relationship is given in Section 3. Note that from the procedure above, it should be clear that this technique makes only sense, if we have at least a two-dimensional control input u and $B(x)$ is not constant.
To sum up, we can say that if a CLF for (4) is known, then, for any $\varepsilon > 0$, one can drive the state of (1) from a given initial position into a ε-neighborhood of the origin and keep it there for all times. This can be achieved by bounded discontinuous feedback, the bound depending upon ε. Finally, let us illustrate the procedure on a simple example.

Example 1. Consider the system

$$\dot{x} = \begin{bmatrix} x_1x_2 - 2x_1 \\ -x_1^2 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ -x_1 \end{bmatrix} u_1 + \begin{bmatrix} -x_2 \\ 0 \end{bmatrix} u_2, \tag{6}$$

and let us choose

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right) \tag{7}$$

as a CLF candidate. Then we get for the derivative

$$\dot{V}(x) = -2x_1^2 + x_1x_2(u_1 - u_2) \tag{8}$$

and we can see that for the set $\{x | x_1 = 0\}$ condition (3) is not satisfied and therefore $V(x)$ can not be used as CLF for system (6). However, for the extended system we get

$$\dot{z} = \begin{bmatrix} z_1z_2 - 2z_1 \\ -z_1^2 \end{bmatrix} + \begin{bmatrix} 2z_2 \\ -z_1 \end{bmatrix} u_1 + \begin{bmatrix} -z_2 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} z_1 \\ -z_2 \end{bmatrix} v_{1,2} \tag{9}$$

[1] A generalization for the higher dimensional case is straightforward (see also Section 3).
[2] $V(x) \to \infty$ as $\|x\| \to \infty$.
[3] See (Sontag, 1998*b*), Proposition 5.9.10.

and we have

$$\dot{V}(z) = -2z_1^2 + z_1 z_2 (u_1 - u_2) + (z_1^2 - z_2^2) v_{1,2}. \quad (10)$$

Now, $\hat{u} = [u_1\ u_2\ v_{1,2}]^T$ can be always chosen such that $\dot{V}(z) < 0$ for $z \neq 0$. Therefore, $V(z)$ is a CLF for system (9) which can thus be stabilized with a suitable feedback law on the basis of $V(z) = \frac{1}{2}\left(z_1^2 + z_2^2\right)$. As a consequence of the results presented in (Knobloch and Wagner, 1984) (see Section 3), this implies that the trajectories of the original system (6) can be driven arbitrarily close to the origin by applying a suitable (discontinuous) feedback law. Furthermore, if a local CLF around the origin of system (1) is known, then the original system (1) can be stabilized too.

In the same fashion as outlined above for the stabilization problem, a formal statement of the underlying technique in the more general case of dissipation inequalities is given in the next section.

3. ESTABLISHING DISSIPATION INEQUALITIES WITH THE HELP OF MIXED LIE BRACKETS

3.1 *Problem formulation*

We consider an affine control system

$$\dot{x} = p(x) + B(x)u + G(x)w, \quad (11)$$

where $u \in \mathbb{R}^{m_1}$ is the control input, $w \in \mathbb{R}^{m_2}$ is the disturbance, and $x \in \mathbb{R}^n$ is the state. $B(x)$ is the matrix given by m_1 column vectors $b_i(x)$, $G(x)$ is the matrix given by m_2 column vectors $g_j(x)$ and p, b_i, g_j are smooth vector fields.

We also assume that the disturbance w is a bounded continuous function of time t satisfying

$$||w(t)|| \leq \bar{\omega}. \quad (12)$$

Based on this information, we deal with the problem of designing control strategies in the general form $u = u(t,x)$ such that a special dissipation inequality

$$V(x(t_e)) - V(x(t_0)) < \int_{t_0}^{t_e} q(x(t))dt \quad (13)$$

holds, along all solutions $x = x(t)$ of (11), where V, q are scalar-valued, sufficiently smooth functions of the state. The disturbance w is specialized to any continuous function satisfying (12). The feedback u may depend upon $\bar{\omega}$ but not on w itself and the time horizon $[t_0, t_e]$ is finite.

3.2 *The basic lemma*

Next, we wish to outline a technique for exploiting the mixed Lie brackets in order to show that a certain type of discontinuous state feedback law for the purpose of establishing dissipation inequalities in the form (13) exists. New results can be obtained only if these brackets are not linearly dependent upon the $b_j(x)$, hence we exclude from our considerations the case $m_1 = 1$ (only one control input) and $B(x) = B = const$. The idea of our approach is to solve the problem for a new system with extended control possibilities

$$\dot{z} = p(z) + \hat{B}(z)\hat{u} + G(z)w, \quad (14)$$

where $\hat{B}(z)\hat{u}$ is defined as

$$\hat{B}(z)\hat{u} = \sum_{r=1}^{m_1} b_r(z)u_r + \sum_{\nu<\mu} [b_\nu, b_\mu](z) v_{\nu,\mu}.$$

The following fundamental relation between trajectories of system (11) and (14) is a consequence - not outspoken there - of the results presented in (Knobloch and Wagner, 1984).

Lemma 1. Given a solution $z = z(t)$ of system (14) for a smooth $w = w(t)$ and $\hat{u} = \hat{u}(z)$ on some interval $[t_0, t_e]$. Given also $\varepsilon > 0$. Then there exists a certain type of discontinuous state feedback $u(t, \hat{u}(x))$ such that the solution $x = x(t)$ of (11) for $u = u(t, \hat{u}(x))$ and $w = w(t)$ with initial value $x(t_0) = z(t_0)$ satisfies $||x(t) - z(t)|| \leq \varepsilon$ for all $t \in [t_0, t_e]$.

Proof. For a sketch of the proof see Appendix A.

Remarks. (i) $u(t, \hat{u}(x))$ is in general 'high gain', i.e. if ε is small, than the amplitude of $u(t, \hat{u}(x))$ is large. (ii) One would like to have a more symmetric statement in the sense that $\hat{u}(z)$ may be time-varying and discontinuous. It would also be desirable to admit (piecewise) continuous specializations of w. These generalizations of Lemma 1 still have to be carried out by a more thorough exploitation of (Knobloch and Wagner, 1984).

3.3 *Establishing dissipation inequalities*

The application of Lemma 1 to our problem is obvious: If we have

$$V_z(z)\left(p(z) + \hat{B}(z)\hat{u}(z) + G(z)w\right) < q(z) \quad (15)$$

for the extended system (14) along a solution $z = z(t)$, $||w(t)|| \leq \bar{\omega}$, and $t_0 \leq t \leq t_e$. We have then

$$V(z(t_e)) - V(z(t_0)) - \int_{t_0}^{t_e} q(z(t))dt < -\delta \quad (16)$$

for some $\delta > 0$. Hence the inequality

$$V(x(t_e)) - V(x(t_0)) - \int_{t_0}^{t_e} q(x(t))dt < 0 \quad (17)$$

holds, if $||x(t) - z(t)|| \leq \varepsilon\ \forall t$ and ε is sufficiently small. This means, that the dissipation inequality (13) holds for our original system (11) if it holds for the extended system (14). The following examples will demonstrate the usefulness of Lemma 1.

Example 2. The example chosen here is essentially the system discussed by Brockett (Brockett, 1983), but with an additional drift term $[0\ 0\ x_3]^T$ as well as a disturbance w:

$$\dot{x} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} + \begin{bmatrix} x_1 \\ \alpha \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} \beta \\ 0 \\ x_2 \end{bmatrix} u_2 + G(x)w \qquad (18)$$

where $x = [x_1\ x_2\ x_3]^T$, $u = [u_1\ u_2]^T$. Let us consider the case where $\alpha, \beta \neq 0$. Firstly, to show that this technique makes also sense if $V(x)$ is indefinite, let us choose $V(x) = x_3$. Then $V_x(x)B(x)u = x_2u_2$ vanishes whenever $x_2 = 0$. Therefore, dissipation inequality (13) cannot be satisfied. For the extended system we get

$$\dot{z} = \begin{bmatrix} 0 \\ 0 \\ z_3 \end{bmatrix} + \begin{bmatrix} z_1 \\ \alpha \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} \beta \\ 0 \\ z_2 \end{bmatrix} u_2 + \begin{bmatrix} -\beta \\ 0 \\ \alpha \end{bmatrix} v_{1,2} + G(z)w, \qquad (19)$$

and we have

$$V_z(z)\hat{B}(z)\hat{u} = z_2u_2 + \alpha v_{1,2}. \qquad (20)$$

It is clear that one can find a $\hat{u} = \hat{u}(z)$ such that $V_z(z)\hat{B}(z)\hat{u} \neq 0$ everywhere. This is essential for our problem, because if $V_z(z)\hat{B}(z)\hat{u} \neq 0$, then the dissipation inequality (15) can be satisfied for every z. Note that this holds independently on the special form of $G(z)$. In particular, no matching condition is required. These considerations can be carried further to treat the problem of output regulation in the presence of disturbances. Let us assume that $G(x)$ is bounded for all x. If we consider $y = V(z) = z_3$ as output, then it can be controlled completely by $v_{1,2}$. To keep z_3 in an interval I around the target value (lets say $z_3 = 0$), one can apply sliding mode control. Furthermore, if we approximate the parts of the z-trajectory by a solution $x = x(t)$ of the original system (18) according to Lemma 1, then we arrive at a trajectory whose x_3-component remains for all time in an ε-neighborhood of I.

Example 3. Let us consider the same system again, but now we choose $V(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$ positive definite and $q(x)$ is an arbitrary negative definite function. This is a more practically relevant application, because the setup corresponds to a disturbance attenuation problem. Then $V_x(x)B(x)u = x_1(x_1u_1 + \beta u_2) + x_2\alpha u_1 + x_3x_2u_2$ vanishes on the set $\{x \mid x_1 = x_2 = 0\}$ identically in u. This set is unbounded, so one cannot make sure by standard state feedback that a trajectory remains for all times in a bounded subset of the state space. However, for the extended system we have

$$V_z(z)\hat{B}(z)\hat{u} = (x_1^2 + x_2\alpha)u_1 + (x_1\beta + x_2x_3)u_2 + (-\beta x_1 + x_3\alpha)v_{1,2}. \qquad (21)$$

It can be verified that all three coefficients of this linear form in $\hat{u} = [u_1\ u_2\ v_{1,2}]^T$ vanish simultaneously in at most three points $z^{(1)}, z^{(2)}, z^{(3)}$ of the state space. So let us consider a ball $B = \{x \mid \|x\| \leq R\}$ which contains $z(0)$(= initial condition) and $z^{(1)}, z^{(2)}, z^{(3)}$ in its interior. For any $z \in \partial B$(=boundary of B) the function $V_z(z)\hat{B}(z)\hat{u}$ of $\hat{u}$ is not identically zero, hence one can find $\hat{u}$ such that $V_z(z)$ times the right hand side of (19) is negative for all z and every w bounded by (12). For reasons of compactness one can find positive δ_i, $i = 1, 2$, such that for all $z(0) \in \partial B$ and all w satisfying (12), the solution $z = z(t)$ of (19) with $\hat{u} = \hat{u}(z)$ has a decreasing norm $\|z(t)\|^2$ for $0 \leq t \leq \delta_1$ and $\|z(\delta_1)\| < R - \delta_2$. Let $\varepsilon < \delta_2$ and consider the solution $x = x(t)$ of (18) - for the same w and $u = u(t, \hat{u}(x))$ - with initial value $x(0) = z(0)$ which is constructed with the help of Lemma 1. This solution satisfies $\|x(t)\| < R + \varepsilon$ for all $t \in [0, \delta_1]$ and $\|x(\delta_1)\| < R$. Thereby we succeeded in finding a control strategy for (18) which is independent upon w and has the following property: Whenever a trajectory reaches the boundary of B it is driven back by an appropriate discontinuous state feedback into the interior of B and stays in the meantime in an ε-neighborhood of B. This of course means that we can keep a trajectory starting at $t = 0$ in a point $x(0)$ in the interior of B for all times in an ε-neighborhood of B.

4. AN ALTERNATIVE FOR A DISTURBANCE ATTENUATION STRATEGY

4.1 *Local dissipation inequalities*

Next, a framework for the design of a disturbance attenuation strategy is proposed. We treat the problem formulated in Section 3.1, under certain hypothesis. First, we assume the following system structure

$$\begin{aligned} \dot{x}_1 &= p_1(x) + B_1(x)u + G_1(x)w \\ \dot{x}_2 &= p_2(x) + B_2(x)u, \end{aligned} \qquad (22)$$

where the state vector x is split up into two parts $x = [x_1\ x_2]^T$, $x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$, $n_2 > 0$. Note that it is assumed that w does not act directly on x_2. Furthermore, we assume that $V(x)$ satisfies

$$V_x(x)[p_1(x)\ p_2(x)]^T < q(x) \qquad (23)$$

i.e. the dissipation inequality (13) holds for the unperturbed and uncontrolled system. From techniques used in (Knobloch, to appear), it turns out that this hypotheses are meaningful. A control law which is a relevant form from a physical point of view and which meets the demands of Lemma 1 is given by

$$u = u(t, x(t_i)). \qquad (24)$$

This feedback can be best characterized as discontinuous ‘discretized’ state feedback (DSF), where the time interval $[t_0, t_e]$ is divided into sufficiently small subintervals $\delta = t_{i+1} - t_i$. The idea to enforce (13) by

DSF is to satisfy on each subinterval $[t_i, t_{i+1}]$ its local analog

$$V(x(t_{i+1})) - V(x(t_i)) < \int_{t_i}^{t_{i+1}} q(x(t))dt, \qquad (25)$$

and by this to satisfy (13). Finally, we arrive to a problem which fits very well into the demands we need and which one may call the local form of the dissipation inequality: To find, for sufficiently small $\delta > 0$, a $u = u(t, x(t_i))$ such that

$$V(x(t_i+\delta)) - V(x(t_i)) < \int_{t_i}^{t_i+\delta} q(x(t))dt, \qquad (26)$$

for $x = x(t)$ being a solution of (11) for some continuous w satisfying (12).

Therefore, the main problem is to find an open-loop control $u = u(t, x(t_i))$ such that the local form of the dissipation inequality (26) is satisfied. Some preliminary results on solving this problem are presented below.

4.2 *Some preliminary results*

First attempts to meet the local dissipation inequality (26) have shown, that it is meaningful to use a control law of the form

$$u = u_1(s, x(t_i), \bar{\omega}) + u_0(x(t_i), \bar{\omega}), \qquad (27)$$

where $\bar{\omega}$ satisfies equation (12), and $s = \frac{t-t_i}{\delta}$, $t \in [t_i, t_i+\delta]$. Note that all values of $u_1(s,.,.)$, $0 \le s \le 1$, contribute to the control action regardless how small δ is. Therefore, it is clear that a solution of the dissipation inequality cannot be reached by passing from the integral form (13) to a pointwise dissipation inequality

$$V_x(x)\,(p(x) + B(x)u + G(x)w) < q(x). \qquad (28)$$

The analysis leading to control law (27) is an application of the approach to dissipation inequalities for systems of the form (22) as presented in (Knobloch, to appear), see especially Chapter 1,3,5. Furthermore, it is clear that some additional conditions on system (22) are neccessary, which follow from the approach presented here and from the results in (Knobloch, to appear). Our ongoing as well as future research goal is focused on the derivation of an explicit control law on the basis of (27) by combining the results presented here with the results presented in (Knobloch, to appear). Finally, to get a feeling how control law (27) will look like, it is illustrated in Figure 1.

5. DISCUSSION AND CONCLUSION

In this paper we propose a framework for disturbance attenuation which exploits the nature of nonlinear systems with a multivariable input, in particular a concept which fits very well into the demands of Lemma 1, the backbone of our framework. Lemma 1 gives a relationship between the system under investigation and an extended system in the sense that any trajectory of the extended system can be tracked by a trajectory of the original system. Furthermore, we have shown that Lemma 1 can be applied to other problems, besides disturbance attenuation.
Of course, several problems are left open in this work. The most important one is to derive an explicit control law, which exploits the advantages of the proposed disturbance attenuation strategy. This is our future as well as ongoing research goal.

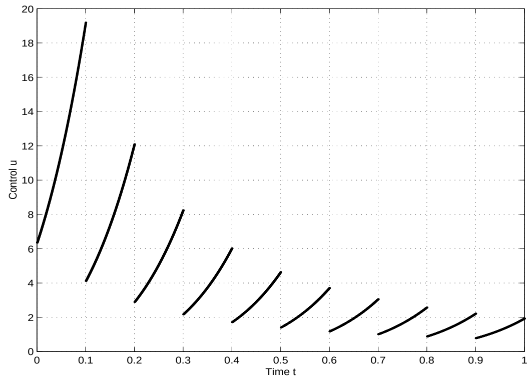

Fig. 1. An example for a control law as presented in Section 4.2.

Appendix A. A SKETCH OF THE PROOF FOR LEMMA 1.

For simplicity of exposition we only consider the case without disturbance. The more general result with disturbance can be shown along the same lines.

Proposition 1. Given a solution $z = z(t)$ of

$$\dot{z} = a(t,z) + \hat{B}(t,z)\hat{u}, \qquad (A.1)$$

where $\hat{B}(t,z)\hat{u}$ is defined as

$$\hat{B}(t,z)\hat{u} = \sum_{r=1}^{m_1} b_r(t,z)u_r + \sum_{\nu<\mu} [b_\nu, b_\mu](t,z)v_{\nu,\mu}$$

on some finite interval $[t_0, t_e]$. Given also $\varepsilon > 0$. Assume that the right hand side of (A.1) is sufficiently smooth as a function of t, z. Then there exists a discontinuous state feedback law $u = u(t, \hat{u}(t,x))$ such that the solution $x = x(t)$ of

$$\dot{x} = a(t,x) + B(t,x)u \qquad (A.2)$$

with $x(t_0) = z(t_0)$, satisfies $\|x(t) - z(t)\| \le \varepsilon$, $t_0 \le t \le t_e$.

Proof. Our basic tool is the notion of a “control variation concentrated at some point $t^\star$” (cf. (Knobloch, 1981), Definition 9.1). It combines the ‘needle-shaped’ variations used in the proof of the Pontryagin maximum principle with the standard variations used in classical calculus of variations and exhibits the same simple superposition property as the Pontryagin variations. Above all, the effect of such a variation on the state - crucial for questions of local controllability - can be described in terms of formal power series in several variables. An example has been worked out in (Knobloch and Wagner, 1984), Sec. 2. It concerns (A.2) in case $a(t,x) \equiv 0$. Note that the general case can always be reduced to this special by means of a time-dependent state transformation (see (Knobloch and Wagner, 1984), Sec. 4), so we will assume $a(t,x) \equiv 0$ from now on. It is shown that n-dimensional ($n = dim(x)$) vectors of the form

$$z + \lambda^2 \hat{B}(t,z)\hat{u} + \text{higher order terms in } \lambda \quad \text{(A.3)}$$

for arbitrary t,z, small scalar λ, and $\hat{u}$ can be interpreted as reachable points in the following sense. Denote for shortness the vector (A.3) - including the remainder term - by $c(t,z,\hat{u},\lambda)$. For fixed values $t^\star, z^\star, \hat{u}^\star, \lambda^\star$ of the variables, $c(t^\star, z^\star, \hat{u}^\star, \lambda)$ can be reached at time $t^\star$ along a trajectory $x = x(t)$ of (A.2) starting in $x(t_0) = z^\star$ at initial time $t_0 = t^\star - \kappa\lambda$. The u which is inserted in (A.2) is a time-dependent discontinuous vector-valued function depending upon $\hat{u}$. The explicit construction is documented in (Knobloch and Wagner, 1984), (2.18)-(2.21) and Theorem 7.1.

This is a ‘small-time-local-controllability’ result, the step to what one may call ‘tracking’ of $z = z(t)$ on a given (large) interval $[t_0,t_e]$ is done via iterations of the map c. Divide $[t_0,t_e]$ into sufficiently small subintervals $[t_i,t_{i+1}]$ and define recursively $z_{i+1} = c(t_{i+1}, z_i, \hat{u}_i, \lambda)$, where $\hat{u}_i = \hat{u}_i(t_i,z_i)$ is the control input - evaluated at $t = t_i$, $z = z_i$ - on the right hand side of (A.1). The initial value is $x(t_0) = z(t_0)$. The z_i's are essentially the corner points of an Euler-Cauchy polygon approximating the trajectory $z(t)$ (see (Knobloch and Wagner, 1984), Sec. 5). On the other hand z_{i+1} can be reached at time t_{i+1} along a trajectory of (A.2) (for a suitable u) starting at $t = t_i$ in the point z_i. Hence - and this it the geometric idea of the proof given in (Knobloch and Wagner, 1984) - one arrives at another approximation of $z(t)$ if one replaces the line segments connecting z_i, z_{i+1} of the standard Euler-Cauchy polygon by trajectories of (A.2) connecting z_i, z_{i+1}. □
Strictly speaking, the situation discussed here does not cover completely the one of Lemma 1 ($a(t,x) = p(x) + G(x)w(t)$) since we assume here smoothness (and not merely continuity) with respect to t and also full knowledge of $a(t,x)$ (in order to construct $u(t,x)$). So some supplementary considerations are called, which will be presented later.

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