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FUZZY MODELS: ENHANCING REPRESENTATION OF DYNAMIC SYSTEMS

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Abstract: The work presents an innovative interpretation of Mamdani and Takagi-Sugeno fuzzy models, that allows a better representation of systems' dynamics. It is shown, with illustrative examples, that, while Mamdani model is better for static features, Takagi-Sugeno model is better for dynamic ones, although only around the linearization points. Nevertheless, Mamdani model would be a perfect approximator for dynamic systems if new conditions are taken into account. Copyright (c) 2002 IFAC.

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1. INTRODUCTION

Mamdani and Takagi-Sugeno (T-S) fuzzy models have been used for more than two decades in systems' modeling. In control (analysis and design) theory, dynamic systems are the focus of attention. In this work it is explained why, from the authors point of view, non of these models are suitable for perfectly modelling systems' dynamics.

Firstly, sections 2,3 and 4 establish the basis of the problem. Then, in sections 5 and 6, both models, with their advantages and drawbacks in function approximation and system identification, are shown, by using a simple but clear example. Finally, in section 7, a better manner in which fuzzy models can be applied is explained.

All the conclusions are applied to continuos and discrete systems.

2. NOMENCLATURE AND ASSUMPTIONS

In the following sections, n^{th} order non-linear dynamic models of the form

$$x_{n+1} = f(x_1, x_2, \ldots, x_n) \tag{1}$$

are used, where f can be a continuos or discrete model. In the continuos case,

$$x_1 = x(t), x_2 = \frac{dx}{dt}, \ldots, x_{n+1} = \frac{d^n x}{dt^n} \tag{2}$$

while in the discrete case,

$$x_1 = x_{k-n+1}, \ldots, x_n = x_k, x_{n+1} = x_{k+1} \tag{3}$$

It is supposed that equilibrium holds at

$$x_1 = x_2 = \ldots = x_{n+1} = 0 \tag{4}$$

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At the equilibrium point, if $x_1 = x(t) \neq 0$ or $x_n = x_k \neq 0$ (in the continuos or discrete case, respectively), it is trivial to make a variable change.

Some assumptions are made along the work:

- the t-norm to be used for the and operator is $t(a,b) = a \cdot b$
- the s-norm to be used for the or operator is $s(a,b) = max(a+b,1)$
- the operator for the implication is also $t(a,b) = a \cdot b$
- let there be $X_l^{(i_l)}$ the fuzzy sets for input x_l, $\forall i_l = \{1,\ldots,r_l\}, \forall l = \{1,\ldots,n\}$ (being r_l the number of fuzzy sets for x_l). Let be also $\mu_{X_l^{(i_l)}}(x_l)$ the corresponding membership functions and $x_l^{(i_l)}$ their highest value. Then
$$\sum_{i_l=1}^{r_l} \mu_{X_l^{(i_l)}}(x_l) = 1 \quad (5)$$
$\forall x_l^{(i_l)} \leq x_l \leq x_l^{(i_l+1)}$, with $\mu_{X_l^{(i_l)}}(x_l) = 1$ and $\mu_{X_l^{(i_l+1)}}(x_l) = 0$.
- let
$$A_{n+1}^{(i_1\ldots i_n)} = \int \mu_{X_{n+1}^{(i_1\ldots i_n)}}(x_{n+1}) dx_{n+1} \quad (6)$$
be the area of x_{n+1} membership functions. Then,
$$A_{n+1}^{(i_1\ldots i_n)} = A_{n+1}^{(j_1\ldots j_n)} \quad (7)$$
$\forall i_1,\ldots,i_n, \forall j_1,\ldots,j_n$.

3. LINEAR SYSTEMS APPROACH

The first step in a controller design is to obtain a good model of the system under control. Two cases exist:

- a differential or discrete equation is available (the f function). In such a case, a fuzzy model may be obtained by function approximation techniques
- a model function is not available. In this case, identification techniques are necessary.

Approximation and identification issues are addressed in the further discussion about the two fuzzy models kindness.

Approximation in classic control theory is done just by linearizing f at a point $(x_1^{(0)},\ldots,x_{n+1}^{(0)})$:

$$\begin{aligned} x_{n+1} &= f(x_1^{(0)},\ldots,x_{n+1}^{(0)}) \\ &+ \left.\frac{\partial f}{\partial x_1}\right|_{(0)}(x_1 - x_1^{(0)}) + \ldots \\ &+ \left.\frac{\partial f}{\partial x_n}\right|_{(0)}(x_n - x_n^{(0)}) \end{aligned}$$

which leads to $x_{n+1} = a_0^{(0)} + a_1^{(0)}x_1 + \ldots + a_n^{(0)}x_n$.

If the linearization is done at the equillibrium point, then $a_0^{(0)} = 0$. Model's error increases as we deviate from the linearization point.

Identification in classic control theory is done by exciting the system around the equillibrium point and compare its response with that of a linear one, obtaining a model like in the previous case:

$$x_{n+1} = a_0^{(0)} + a_1^{(0)}x_1 + \ldots + a_n^{(0)}x_n \quad (8)$$

4. FUZZY MODELS

4.1 *Mamdani's Model*

The system is represented by rules like follows (Kickert and Mamdani, 1978):

$$\begin{aligned} R^{(i_1\ldots i_n)} &: IF(x_1 is X_1^{(i_1)}) AND \ldots \\ &AND(x_n is X_n^{(i_n)}) \\ &THEN(x_{n+1} is X_{n+1}^{(i_1\ldots i_n)}) \end{aligned}$$

where $X_{n+1}^{i_1\ldots i_n}$ are fuzzy sets for $x_{n+1}, \forall i_l = \{1,\ldots,r_l\} \forall l = \{1,\ldots,n\}$

The output of the system is computed as:

$$x_{n+1} = \frac{\int x_{n+1} \cdot \mu_{X_{n+1}}(x_{n+1}) dx_{n+1}}{\int \mu_{X_{n+1}(x_{n+1})} dx_{n+1}} \quad (9)$$

where

$$\mu_{X_n}(x_{n+1}) = s(t(w^{(i_1\ldots i_n)}, \mu_{X_{n+1}}^{(i_1\ldots i_n)}(x_{n+1}))) (10)$$

$\forall il = \{1,\ldots,r_l\}, \forall l = \{1,\ldots,n\}$ and

$$w^{(i_1\ldots i_n)}(x_1,\ldots,x_n) = t(\mu_{X_l}^{(i_l)}(x_l)) \quad (11)$$

$\forall i_l = \{1,\ldots,r_l\}, \forall l = \{1,\ldots,n\}$ is the weight of the rule $R^{(i_1\ldots i_n)}$. By using the t and s-norms described in section 2, it follows that

$$\begin{aligned} & x_{n+1} = \\ &= \frac{\int x_{n+1} \sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{(i_1\ldots i_n)} \cdot \mu_{X_{n+1}}^{(i_1\ldots i_n)}(x_{n+1}) dx_{n+1}}{\int dx_{n+1}} \\ &= \frac{\sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{(i_1\ldots i_n)} \cdot A_{n+1}^{(i_1\ldots i_n)} \cdot x_{n+1}^{(i_1\ldots i_n)}}{\sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{(i_1\ldots i_n)} \cdot A_{n+1}^{(i_1\ldots i_n)}} (12) \end{aligned}$$

where $x_{n+1}^{(i_1\ldots i_n)}$ is the centre of gravity of $\mu_{X_{n+1}}^{(i_1\ldots i_n)}(x_{n+1})$.

Furthermore, it was proved in (Matía and Jiménez, 1996) that

$$\sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{(i_1\ldots i_n)} = 1$$

provided that

$$\sum_{i_l=1}^{r_l} \mu_{X_l^{(i_l)}}(x_l) = 1$$

$\forall l = \{1, \ldots, n\}$. So, finally,

$$x_{n+1} = \sum_{i_1=1}^{r_1} \ldots \sum_{i_n=1}^{r_n} w^{(i_1 \ldots i_n)} \cdot x_{n+1}^{(i_1 \ldots i_n)} \quad (13)$$

The f function is just an interpolation between the points $x_{n+1}^{(i_1 \ldots i_n)}$, centre of gravity of the output's membership functions. This means that there could be used rules like

$$\begin{gathered} R^{(i_1 \ldots i_n)} : IF(x_1 is X_1^{(i_1)}) \\ AND \ldots AND(x_n is X_n^{(i_n)}) \\ THEN x_{n+1} = x_{n+1}^{(i_1 \ldots i_n)} \end{gathered}$$

and apply T-S centre of gravity calculation, as is described in next subsection.

4.2 *Takagi-Sugeno's Model*

The system is represented by rules as follows (Sugeno, 1985; Takagi and Sugeno, 1985):

$$\begin{gathered} R^{(i_1 \ldots i_n)} : IF(x_1 is X_1^{(i_1)}) \\ AND \ldots AND(x_n is X_n^{(i_n)}) \\ THEN x_{n+1} = f^{(i_1 \ldots i_n)}(x_1, \ldots, x_n) \end{gathered}$$

being the most used $f^{(i_1 \cdots i_n)}$ functions in control applications linear expressions such as:

$$\begin{aligned} f^{(i_1 \ldots i_n)} &= a_0^{(i_1 \ldots i_n)} + a_1^{(i_1 \ldots i_n)} x_1 \\ &+ \ldots + a_n^{(i_1 \ldots i_n)} x_n \end{aligned} \quad (14)$$

Then, the system's output is computed as

$$\begin{aligned} x_{n+1} &= \frac{\sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{(i_1 \ldots i_n)} \cdot f^{(i_1 \ldots i_n)}}{\sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{(i_1 \ldots i_n)}} \\ &= 1 \end{aligned} \quad (15)$$

Attention must be paid to the fact that, under section 2 suppositions, when

$$f^{(i_1 \ldots i_n)}(x_1, \ldots, x_n) = a_0^{i_1 \ldots i_n} \quad (16)$$

this means, is constant, Mamdani and Sugeno's models are equivalent, since $a_0^{i_1 \ldots i_n}$ may be considered the centre of gravity of the output x_{n+1} membership functions.

5. FUZZY MODELS AS FUNCTION APPROXIMATORS

Fuzzy models have been used in the literature (Buckley and Hayashi, 1993; Wang, 1992) as function approximators. We will comment the behaviour of both models regarding this concept.

5.1 *Mamdani's Model*

Theorem 1

The first order function $x_2 = f(x_1)$ may be exactly approximated by a Mamdani-like fuzzy model in the range $x_1^{(i_1)} \leq x_1 \leq x_1^{(i_1+1)}$, with two rules:

$$\begin{gathered} R^{(i_1)} : IF(x_1 is X_1^{(i_1)}) THEN x_2 = f(x_1^{(i_1)}) \\ R^{(i_1+1)} : IF(x_1 is X_1^{(i_1+1)}) THEN x_2 = f(x_1^{(i_1+1)}) \end{gathered}$$

provided that f is strictly monotonous (increasing or decreasing) in that range. The fuzzy sets are given by

$$\mu_{X_1^{i_1}}(x_1) = \frac{f(x_1^{i_1+1}) - f(x_1)}{f(x_1^{i_1+1}) - f(x_1^{i_1})} \quad (17)$$

$$\begin{aligned} \mu_{X_1^{i_1+1}}(x_1) &= \frac{f(x_1) - f(x_1^{i_1})}{f(x_1^{i_1+1}) - f(x_1^{i_1})} \\ &= 1 - \mu_{X_1^{i_1}}(x_1) \end{aligned} \quad (18)$$

μ functions do not belong to $[0, 1]$ when f is not monotonous.

Proof will be given in an extended version.

Example 1

The system $x_2 = sin x_1$ may be approximated by a fuzzy model in $0 \leq x_1 \leq \frac{\pi}{2}$ as

$$\begin{gathered} R^{(1)} : IF(x_1 is SMALL) THEN x_2 = 0 \\ R^{(2)} : IF(x_1 is BIG) THEN x_2 = 1 \end{gathered}$$

being

$$\mu_{SMALL}(x_1) = 1 - sin x_1 \quad (19)$$

$$\mu_{BIG}(x_1) = sin x_1 \quad (20)$$

so $x_2 = (1 - sin x_1) \cdot 0 + sin x_1 \cdot 1 = sin x_1$.

Theorem 2

The second order function $x_3 = f(x_1, x_2) = a + b g_1(x_1) + c g_2(x_2) + d g(x_1) g_2(x_2)$ may be exactly approximated by a Mamdani-like fuzzy model in the range $x_l^{(i_l)} \leq x_1 \leq x_l^{(i_l+1)}, \forall i = \{1, 2\}$, with four rules:

$$R^{(i_1 i_2)}: IF(x_1 is X_1^{(i_1)}) AND (x_2 is X_2^{(i_2)})$$
$$THEN x_3 = f(x_1^{(i_1)}, x_2^{(i_2)})$$

$$R^{(i_1 i_2+1)}: IF(x_1 is X_1^{(i_1)}) AND (x_2 is X_2^{(i_2+1)})$$
$$THEN x_3 = f(x_1^{(i_1)}, x_2^{(i_2+1)})$$

$$R^{(i_1+1 i_2)}: IF(x_1 is X_1^{(i_1+1)}) AND (x_2 is X_2^{(i_2)})$$
$$THEN x_3 = f(x_1^{(i_1+1)}, x_2^{(i_2)})$$

$$R^{(i_1+1 i_2+1)}: IF(x_1 is X_1^{(i_1+1)}) AND (x_2 is X_2^{(i_2+1)})$$
$$THEN x_3 = f(x_1^{(i_1+1)}, x_2^{(i_2+1)})$$

provided that g_l are strictly monotonous (increasing or decreasing) in that range. The fuzzy sets are given by

$$\mu_{X_l^{i_l}}(x_l) = \frac{g_l(x_l^{i_l+1}) - g_l(x_l)}{g_l(x_l^{i_l+1}) - g_l(x_l^{i_l})} \tag{21}$$

$$\mu_{X_l^{i_l+1}}(x_l) = \frac{g_l(x_l) - g_l(x_l^{i_l})}{g_l(x_l^{i_l+1}) - g_l(x_l^{i_l})} = 1 - \mu_{X_l^{i_l}}(x_l) \tag{22}$$

μ_l functions do not belong to $[0,1]$ when g_l is not monotonous.

Proof will be given in an extended version.

5.2 *Takagi-Sugeno's Model*

T-S- model can not be properly used as function approximator as is shown in the next example.

Example 2

For $x_2 = sin x_1$, T-S model $\forall 0 \leq x_1 \leq \frac{\pi}{2}$ follows

$$R^{(1)}: IF(x_1 is SMALL)$$
$$THEN x_2 = x_2^{(1)} + cos x_1^{(1)} \cdot (x_1 - x_1^{(1)}) = x_1$$

$$R^{(2)}: IF(x_1 is BIG)$$
$$THEN x_2 = x_2^{(2)} + cos x_1^{(2)} \cdot (x_1 - x_1^{(2)}) = 1$$

Any kind of interpolation (this means, membership function shape) in $0 \leq x_1 \leq \frac{\pi}{2}$ will produce a function which is completely different from the original function.

5.3 *Comparison*

It seems that Mamdani's model is better than T-S one for function approximation, although T-S model represents better system's dynamics around the linearization points (rules).

6. FUZZY MODELS FOR SYSTEM INDENTIFICATION

6.1 *Mamdani's Model*

Lets have a system to be identified at some points $(x_1^{(i_1)}, \ldots, x_n^{(i_n)})$. A Mamdani's fuzzy model may be built if we obtain information about x_{n+1} at those points (lets say $x_{n+1}^{(i_1 \ldots i_n)}$). In such a case, the model will be:

$$R^{(i_1 \ldots i_n)}: IF(x_1 is X_1^{(i_1)})$$
$$AND \ldots AND (x_n is X_n^{(i_n)})$$
$$THEN x_{n+1} = f(x_1^{(i_1)}, \ldots, x_n^{(i_n)}) = x_{n+1}^{(i_1 \ldots i_n)}$$

Inference on Mamdani's model will provide an interpolation method for the system.

Example 3

$x_2 = sin x_1$ can be identified at $x_1 = 0$ and $x_1 = \frac{\pi}{2}$, as $x_2 = 0$ and $x_2 = 1$, respectively. Then,

$$R^{(1)}: IF(x_1 is SMALL) THEN x_2 = 0$$

$$R^{(2)}: IF(x_1 is BIG) THEN x_2 = 1$$

If we choose triangular membership funtions in $0 \leq x_1 \leq \frac{\pi}{2}$, this means

$$\mu_{SMALL}(x_1) = 1 - \frac{2x_1}{\pi} \tag{23}$$

$$\mu_{BIG}(x_1) = \frac{2x_1}{\pi} \tag{24}$$

then

$$x_2 = (1 - \frac{2x_1}{\pi}) \cdot 0 + \frac{2x_1}{\pi} \cdot 1 = \frac{2x_1}{\pi} \tag{25}$$

where it is clear that $x_2 \neq sin x_1$.

6.2 *Takagi-Sugeno's Model*

Now the goal is to identify a system at some points $(x_1^{(i_1)}, \ldots, x_n^{(i_n)}), \forall i_l = \{1, \ldots, r_l\}, \forall l = \{1, \ldots, n\}$, but using linear subsystems. With T-S model, rules like follows are obtained:

$$R^{(i_1\dots i_n)} : IF(x_1 is X_1^{(i_1)})$$
$$AND \dots AND(x_n is X_n^{(i_n)})$$

$$THEN x_{n+1} = a_0^{(i_1\dots i_n)} + a_1^{(i_1\dots i_n)} x_1 + \dots + a_1^{(i_1\dots i_n)} x_n$$

Between the linear subsystems, T-S model also provides an interpolation method for the dynamics of the system.

Example 4

$x_2 = sin x_1$ may be identified at $x_1 = 0$ and $x_1 = \frac{\pi}{2}$ as $x_2 = x_1$ and $x_2 = 1$, respectively, as was seen in example of subsection 5.2. Then,

$$R^{(1)} : IF(x_1 is SMALL) THEN x_2 = x_1$$

$$R^{(2)} : IF(x_1 is BIG) THEN x_2 = 1$$

If we choose again triangular membership functions,

$$\mu_{SMALL}(x_1) = 1 - \frac{2x_1}{\pi} \quad (26)$$

$$\mu_{BIG}(x_1) = \frac{2x_1}{\pi} \quad (27)$$

then $x_2 = (1 - \frac{2x_1}{\pi}) \cdot x_1 + \frac{2x_1}{\pi} \cdot 1 = (1 + \frac{2}{\pi})x_1 + \frac{2}{\pi}x_1^2$. The static approximation is not fine. Note that

$$f'(x_1) = (1 + \frac{2}{\pi}) + \frac{4}{\pi}x_1 \quad (28)$$

so the derivatives at 0 and $\frac{\pi}{2}$ do not correspond with those of the original system.

6.3 *Comparison*

Mamdani's model does not take into account system dynamics, but T-S model does not provide a good static approximation.

7. FUZZY MODELS DISCUSSION AND CONCLUSION

7.1 *A new approach*

Although T-S model has been widely used in fuzzy modeling for control applications, because it includes valuable information about the system dynamics, we will prove that Mamdani's model can achieve a better approximation, without loosing its static approximation capabilities.

The way to do that is just to increase membership functions information, using linear subsystems in the identification process as in T-S case.

Theorem 3

Lets suppose the case of identifying a first order function $x_2 = f(x_1)$ in $x_1^{(i_1)} \leq x_1 \leq x_1^{(i_1+1)}$, with Mamdani's model, and using two rules:

$$R^{(i_1)} : IF(x_1 is X_1^{(i_1)}) THEN x_2 = f(x_1^{(i_1)})$$

$$R^{(i_1+1)} : IF(x_1 is X_1^{(i_1+1)}) THEN x_2 = f(x_1^{(i_1+1)})$$

provided that f is strictly monotonous in that range. Lets suppose that we have information (obtained from the identification process), not only about $f(x_1)$, but also about its derivative $f'(x_1)$ at $x_1^{(i_1)}$ and $x_1^{(i_1+1)}$. Then, a couple of fuzzy sets which guarrantees that the model approximates both statics and dynamics in the previous range, is given by:

$$\mu_{X_1^{(i_1+1)}}(x_1) = q_0 + q_1 x_1 + q_2 x_1^2 + q_3 x_1^3 \quad (29)$$

$$\mu_{X_1^{(i_1)}}(x_1) = 1 - \mu_{X_1^{(i_1+1)}}(x_1) \quad (30)$$

being

$$\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \mathbf{X_1^{-1}} \begin{bmatrix} 0 \\ 1 \\ \dfrac{f'(x_1^{(i_1)})}{f(x_1^{(i_1+1)}) - f(x_1^{(i_1)})} \\ \dfrac{f'(x_1^{(i_1+1)})}{f(x_1^{(i_1+1)}) - f(x_1^{(i_1)})} \end{bmatrix} \quad (31)$$

with

$$\mathbf{X_1} = \begin{bmatrix} 1 & x_1^{(i_1)} & {x_1^{(i_1)}}^2 & {x_1^{(i_1)}}^3 \\ 1 & x_1^{(i_1+1)} & {x_1^{(i_1+1)}}^2 & {x_1^{(i_1+1)}}^3 \\ 0 & 1 & 2x_1^{(i_1)} & 3{x_1^{(i_1)}}^2 \\ 0 & 1 & 2x_1^{(i_1+1)} & 3{x_1^{(i_1+1)}}^2 \end{bmatrix} \quad (32)$$

Proof

$$\begin{aligned} x_2 &= \left[1 - \mu_{X_1^{(i_1+1)}}(x_1)\right] f(x_1^{(i_1)}) + \\ &+ \mu_{X_1^{(i_1+1)}}(x_1) f(x_1^{(i_1+1)}) = \\ &= f(x_1^{(i_1)}) + \left[f(x_1^{(i_1+1)}) - f(x_1^{(i_1)})\right] \mu_{X_1^{(i_1+1)}}(x_1) \\ &= f(x_1) \end{aligned} \quad (33)$$

This gives us four conditions:

$$\mu_{X_1^{(i_1+1)}}(x_1^{(i_1)}) = 0$$

$$\mu_{X_1^{(i_1+1)}}(x_1^{(i_1+1)}) = 1$$

$$\mu'_{X_1^{(i_1+1)}}(x_1^{(i_1)}) = \frac{f'(x_1^{(i_1)})}{f(x_1^{(i_1+1)}) - f(x_1^{(i_1)})}$$

$$\mu'_{X_1^{(i_1+1)}}(x_1^{(i_1+1)}) = \frac{f'(x_1^{(i_1+1)})}{f(x_1^{(i_1+1)}) - f(x_1^{(i_1)})} \quad (34)$$

Choosing, for instance, $\mu_{X_1^{(i_1+1)}}(x_1) = q_0 + q_1x_1 + q_2x_1^2 + q_3x_1^3$, the four conditions are expressed by the above matrix equallity.

Example 5

Lets try to identify $x_2 = sinx_1$, at $x_1 = 0$ and $x_1 = \frac{\pi}{2}$. We have that $f(0) = 0$, $f(\frac{\pi}{2}) = 1$, $f'(0) = 1$ and $f'(\frac{\pi}{2}) = 0$. So,

$$\mathbf{X_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{\pi}{2} & \left(\frac{\pi}{2}\right)^2 & \left(\frac{\pi}{2}\right)^3 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & \pi & \frac{3\pi^2}{4} \end{bmatrix} \quad (35)$$

$$\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \mathbf{X_1^{-1}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0.0574 \\ 0.1107 \end{bmatrix} \quad (36)$$

$$\mu_{SMALL}(x_1) = x_1 - 0.0574x_1^2 - 0.1107x_1^3 \quad (37)$$

$$\mu_{BIG}(x_1) = 1 - x_1 + 0.0574x_1^2 + 0.1107x_1^3 \quad (38)$$

with

$$R^{(1)} : IF(x_1isSMALL)THENx_2 = 0$$

$$R^{(2)} : IF(x_1isBIG)THENx_2 = 1$$

Note that now dynamics are exactly included in the model since $x_2 = f(x_1) = 1 - x_1 + 0.0574x_1^2 + 0.1107x_1^3$ and so $f'(0) = 1$ and $f'(\frac{\pi}{2}) = 0$, as happens with $x_2 = sinx_1$.

Theorem 4

Lets suppose the case of identifying a second order function $x_3 = f(x_1, x_2)$ in $x_l^{(i_l)} \le x_l \le x_l^{(i_l+1)}$, $\forall l = \{1, 2\}$, with Mamdani's model. Let's suppose that f can be decomposed in the form $f(x_1, x_2) = a + bg_1(x_1) + cg_2(x_2) + dg_1(x_1)g_2(x_2)$. Then, it can be easily proved that the fuzzy sets that must be used are given by:

$$\mu_{X_l^{(i_l+1)}}(x_l) = q_{0l} + q_{1l}x_l + q_{2l}x_l^2 + q_{3l}x_l^3 \quad (39)$$

$$\mu_{X_l^{(i_l)}}(x_l) = 1 - \mu_{X_l^{(i_l+1)}}(x_l) \quad (40)$$

with

$$\begin{bmatrix} q_{0l} \\ q_{1l} \\ q_{2l} \\ q_{3l} \end{bmatrix} = \mathbf{X_l^{-1}} \begin{bmatrix} 0 \\ 1 \\ \frac{g_l'(x_l^{(i_l)})}{g_l(x_l^{(i_l+1)}) - g_l(x_l^{(i_l)})} \\ \frac{g_l'(x_l^{(i_l+1)})}{g_l(x_l^{(i_l+1)}) - g_l(x_l^{(i_l)})} \end{bmatrix} \quad (41)$$

with

$$\mathbf{X_l} = \begin{bmatrix} 1 & x_l^{(i_l)} & {x_l^{(i_l)}}^2 & {x_l^{(i_l)}}^3 \\ 1 & x_l^{(i_l+1)} & {x_l^{(i_l+1)}}^2 & {x_l^{(i_l+1)}}^3 \\ 0 & 1 & 2x_l^{(i_l)} & 3{x_l^{(i_l)}}^2 \\ 0 & 1 & 2x_l^{(i_l+1)} & 3{x_l^{(i_l+1)}}^2 \end{bmatrix} \quad (42)$$

Proof will be given in an extended version.

7.2 *Conclusion*

The new model presented above allows a better identification of dynamic systems than T-S and Mamdani's models with conventional membership functions. In the simple example that has been shown, $x_2 = f(x_1) = sinx_1$, when we have information about its identification at some points, Mamdani's model with extended membership functions shows better results ($x_2 = 1 - x_1 + 0.0574x_1^2 + 0.1107x_1^3$) than traditional Mamdani's model ($x_2 = x_1$) and T-S model ($x_2 = (1 + \frac{2}{\pi})x_1 - \frac{2}{\pi}x_1^2$).

Extension of the example shown along the work, can be easily extended to n^{th} order systems as well as to cases with r_l identification points, this means, with more than two membership functions per input variable.

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