15th Triennial World Congress, Barcelona, Spain

ON OPTIMIZATION OF HYPERBOLIC SYSTEMS WITH SMOOTH CONTROLS AND INTEGRAL CONSTRAINTS

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Abstract: A non-classic optimality condition and numerical algorithm for smooth boundary controls in semi-linear first-order hyperbolic systems are presented. Additional integral control constraints are considered. The suggested approach is based on special variations of admissible continuously differentiable controls. These variations lead to a new necessary optimality condition which is a base for a numerical method. The method is applied to an optimal control problem for an age-structured population. *Copyright ©2002 IFAC*

Keywords: Optimal control, partial differential equations, distributed-parameter systems, boundary conditions, computational methods.

1. INTRODUCTION

A necessary optimality condition and numerical algorithms for smooth boundary controls in semi-linear first-order hyperbolic systems are considered in this paper. The classic second-order hyperbolic equations and the systems with orthogonal characteristic families (Goursat-Darboux type) can be reduced to considered systems. The problems of such kind arise in modelling a number of phenomena of aero- and hydrodynamics, chemical technology, population dynamics, transfer processes.

In (Arguchintsev and Vasiliev, 1996a, 1996b) necessary optimality conditions and successive approximations algorithms were derived for distributed controls in the right-hand side of hyperbolic operators. Extension of these results to multi-dimensional hyperbolic systems is proved to be a non-trivial operation. Arguchintsev (1988) proved the maximum principle for the hyperbolic boundary value problem in which boundary conditions are determined from ordinary differential equations. In (Choo, *et al.*, 1981; Brokate, 1987) a differential (linearized) maximum principle was derived for boundary controls.

The special feature of the general optimization problem for boundary conditions is non-validity of the classic optimality condition of Pontryagin's type. For instance, a counter example was constructed in (Wolfersdorf, 1980) for systems with two orthogonal characteristic families and distributed control as a function of only one independent variable.

The proposed approach is based on special variations of admissible smooth controls. The idea of using this approach for distributed controls in the right-hand side of hyperbolic operator was supposed in (Arguchintsev and Vasiliev, 1996b). The variations can be applied for continuously differentiable controls satisfying integral restrictions. Such unusual class of admissible controls appeared while researching inverse problems of mathematical physics. It turns out that the suggested approach is efficient also for solving inverse problems of optimal control. The unknown parameters of controlled dynamic system can be considered as new smooth controls. In this paper analysis of a cost functional increment formula on non-classic variations of admissible boundary controls brought a new necessary optimality condition and efficient numerical methods.

The final part of the paper is devoted to a possibility of application of the method to an optimal birth control problem for an age-structured population. This problem can be considered as an optimal control problem by the boundary conditions of a first-order equations with integral constraint. The results of computational experiments are given.

2. PROBLEM STATEMENT

Consider in some given region $P = S \times T$, $S = [s_0, s_1]$, $T = [t_0, t_1]$ of independent variables $(s,t) \in P$ the semilinear hyperbolic system

$$x_t + \mathbf{A}(s,t)x_s = f(x,s,t), \qquad (s,t) \in P. \quad (1)$$

Here $x = x(s,t)$ is n-dimensional state vector-function, the given $(n \times n)$ matrix-function $\mathbf{A}(s,t)$ is diagonal with sign-constant, continuously differentiable elements $a_i = a_i(s,t)$, $i = 1, 2, \ldots, n$, the vector function f is continuous with respect to its variables together with partial derivatives f_x.

In accordance with this, define controlled initial-boundary conditions for the system (1) as

$$x(s,t_0) = p(u(s),s), \quad s \in S; \quad (2)$$

$$x^+(s_0,t) = M(t)x^-(s_0,t) + g^{(1)}(t), \quad t \in T,$$
$$x^-(s_1,t) = N(t)x^+(s_1,t) + g^{(2)}(t), \quad t \in T, \quad (3)$$

here the signs $(+)$ and $(-)$ denotes state subvectors $x(s,t)$ corresponding to the positive and negative diagonal elements $a_i(s,t)$, $i = 1, 2, \ldots, n$. $M(t)$ and $N(t)$ are rectangular matrices.

Any first-order semilinear hyperbolic system with sign-constant in P eigenvalues of the matrix of coefficients can be transformed to the diagonal form (1).

The admissible controls are r-dimensional continuously differentiable vector-functions $u = u(s)$ satisfying

$$u(s) \geq 0, \qquad s \in S, \quad (4)$$

$$\int_S \Phi_i(u(s))ds = L_i, \quad i = 1, 2, \ldots, m, \quad (5)$$

here

$$\Phi_i(\lambda u) = \lambda^\alpha \Phi_i(u), \quad \alpha \geq 1. \quad (6)$$

The objective of the control for the process described by (1)-(6) is to minimize the functional

$$\mathcal{J}(u) = \int_S \varphi(x(s,t_1),s)ds + \int\int_P F(x,s,t)dtds \rightarrow min. \quad (7)$$

The problem (1)-(7) is considered under the following assumptions.

1) Vector functions $p(u,s)$, $g^{(1)}(t)$, $g^{(2)}(t)$ and matrix functions $M(t)$ and $N(t)$ are continuous and continuous differentiable with respect to their arguments.

2) Scalar functions $F = F(x,s,t)$, $\varphi = \varphi(x,s)$ and vector function $f = f(x,s,t)$ are continuous with respect to their arguments and have continuous and bounded partial derivatives with respect to x.

3. AN OPTIMALITY CONDITION AND ITERATIVE METHODS

Consider non-classic variations of controls. Let $u = u(s)$ be an admissible control. Give the variated control by the formula

$$u_\varepsilon(s) = \lambda(s)u(s + \varepsilon\delta(s)), \quad (8)$$
$$\lambda(s) = \left(1 + \varepsilon\dot{\delta}(s)\right)^{1/\alpha}.$$

Here $\delta(s)$ is a continuously differentiable function, $s_0 \leq s + \delta(s) \leq s_1, s \in S$. It is easy to verify that the function (8) is an admissible control too, if $|\dot{\delta}(s)| \leq 1$, $\delta(s_0) = \delta(s_1) = 0$ (Vasiliev, 1995). Originally, the inner variation has been used for proving a necessary optimality condition in optimal control problems by ordinary differential equations with delay (Zabello, 1989). However this variation turned out to be very efficient for the problems with integral restrictions on control functions.

We consider the objective functional increment formula for two admissible controls:

$$\Delta_\varepsilon \mathcal{J}(u) = -\int\limits_S < \psi(s,t_0), \Delta p(u(s),s) > ds + \int\limits_S o_\varphi\left(\|\Delta x(s,t_1)\|\right)ds - \int\limits_P o_H\left(\|\Delta x(s,t)\|\right)dsdt. \quad (9)$$

Here $< \cdot, \cdot >$ is a designation of a scalar product in E^n, $o(\varepsilon)/\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

The function $\psi = \psi(s,t), \psi(s,t) \in E^n$ satisfies the conjugate problem

$$\psi_t + [A(s,t)\psi]_s = -H_x(\psi, x, s, t),$$
$$\psi(s,t_1) = -\varphi_x(x(s,t_1),s),\ s \in S;$$

$$\psi^+(s_1,t) = N_1(t)\psi^-(s_1,t),$$

$$\psi^-(s_0,t) = M_1(t)\psi^+(s_0,t),\ t \in T;$$

$$N_1(t) = -(A^+)^{-1}N^T A^-,$$

$$M_1(t) = -(A^-)^{-1}M^T A^+;$$

$$H(x,s,t) =< \psi(s,t), f(x,s,t) > -F(x,s,t).$$

Here A^+ and A^- are submatrices of the matrix A corresponding to the positive and negative diagonal elements $a_i(s,t),\ \ i = 1,2,\ldots,n$. Let

$$h(u(s),s) =< \psi(s,t_0), p(u(s),s) > .$$

Then it follows from (9):

$$\Delta_\varepsilon \mathcal{J}(u) = -\varepsilon \int\limits_S (< h_u, \dot{u}(s) > - \qquad (10)$$

$$\frac{1}{\alpha} < h_u, u >_s)\delta(s)ds + o(\varepsilon).$$

where the remainder term in (10) is estimated on the base of the energy inequality (Arguchintsev and Vasiliev, 1996a).

By the increment formula (10) we have the following theorem.

Theorem 1 *Consider problem (1)-(7). For an optimal control u^* the following necessary optimality condition holds:*

$$< h_u, \dot{u^*}(s) > -\frac{1}{\alpha} < h_u, u^* >_s = 0,\ s \in S. \quad (11)$$

Remark *If integral constraints (5) are linear with respect to control ($\alpha = 1$), then (11) can be written in a simpler form:*

$$< h_{us}, u^*(s) >= 0,\ s \in S.$$

This optimality condition allows to construct an iterative procedure.

Let an admissible control u^0 be given and u^k be calculated by the method. Calculate

$$\omega_k(s) =< h_u(u^k,s), \dot{u^k}(s) > -\frac{1}{\alpha} < h_u, u^k(s) >_s .$$

Construct an admissible function $\delta_k(s)$ which has the coinciding sign with $\omega_k(s)$. The ways of constructing such functions were considered in (Vasiliev, 1995). Then we determine one-parametric collection of controls u^k_ε (see (8)) and solve a problem

$$\varepsilon_k = argmin_{\varepsilon \in [0,1]} \mathcal{J}(u^k_\varepsilon).$$

The next approximation is given by the formula

$$u^{k+1} = u^k_{\varepsilon_k}, k = 0,1,2,\ldots.$$

The corresponding convergence theorem to the necessary optimality condition (11) is formulated by the same way as in (Vasiliev, 1995). The given iterative process is relaxative and convergent in the sense

$$\mu(u^k) = \int_S \delta_k(s)\omega_k(s)ds \to 0, \quad k \to \infty$$

under additional standard conditions (Arguchintsev and Vasiliev, 1996a, 1996b) (boundedness from below of a functional and Lipschitz inequality for partial derivatives with respect to x from right-hand sides of a system and objective functions).

4. APPLICATION TO POPULATION DYNAMICS

The results described above were applied to solving an optimal control problem of age-dependent populations.

Consider the control of the following population distributed parameter system (Song and Yu, 1987)

$$\frac{\partial p(s,t)}{\partial t} + \frac{\partial p(s,t)}{\partial s} = -\mu(s)p(s,t),\ s \in S\ , t \in T; \quad (12)$$

$$p(s,0) = p_0(s),\ s \in S, \quad (13)$$

$$p(0,t) = \beta(t) \int\limits_{s_i}^{s_e} k(s)u(s)p(s,t)\,ds,\ t \in T. \quad (14)$$

Here $p(s,t)$ is the population density, $s \in S = [0,s_1]$denotes age, s_1 is the maximum age, $t \in T = [0,t_1]$ represents time, $[s_i,s_e]$ is the fertility interval, $\beta(t)$ is the specific fertility rate of females at time t, $k(s)$ denotes the female ratio. The initial population density $p_0(s)$ and the mortality rate $\mu(s)$ are given.

Admissible controls are smooth functions $u(s)$ satisfying

$$\int_{s_i}^{s_e} u(s)\,ds = 1,\ u(s) \geq 0,\ u(s_i) = u(s_e) = 0. \quad (15)$$

Control variables denote the fertility pattern of the female.

The cost functional is

$$J(u) = \int_S \varphi(p(s,t_1),s)\,ds. \tag{16}$$

Particularly, if

$$\varphi(p,s) = \frac{1}{2}(p(s,t_1) - \bar{p}(s))^2, \tag{17}$$

where $\bar{p}(s)$ is a given function, then the objective is to reach a given density $\bar{p}(s)$ in a final time moment t_1.

The boundary conditions (14) are not classic. They are determined by integrating an expression containing phase variable $p(s,t)$. The problem of (12)-(16) type was considered in (Chan and Guo, 1989, 1990) for control variables $\beta(t)$. A case of controls in the right-hand side of the equations (12) was discussed in (Brokate, 1985). The main results of these authors were necessary optimality conditions of linearized Pontryagin's maximum principle type.

We outline the approach described above. Suppose that functions $k = k(s)$, $p_0 = p_0(s)$, $\beta = \beta(t)$ are continuously differentiable, function $\mu(s)$ is continuous, and function $\varphi(p,s)$ is continuous with respect to their arguments and have continuous and bounded partial derivatives with respect to p. Strongly speaking, solution of the boundary value problem (12)-(14) is a solution of a corresponding integral equation which is constructed on characteristics of (12). Under our assumptions the generalized solution will have classic partial derivatives with respect to s and t excepting points of $s = t$.

The conjugate problem is

$$\frac{\partial\psi(s,t)}{\partial t} + \frac{\partial\psi(s,t)}{\partial s} = \mu(s)\psi(s,t) -$$

$$\psi(0,t)\beta(t)k(s)u(s);$$

$$\psi(s,t_1) = -\varphi_p(p(s,t_1),s),\ s \in S;$$

$$\psi(s_1,t) = 0,\ t \in T.$$

A necessary optimality condition may be formulated as the following.

Theorem 2 *Let control $u^*(s)$ be optimal in the problem (12)-(16), $p^*(s,t)$ denotes the solution of (12)-(14), $\psi^*(s,t)$ is a solution of the conjugate problem. Then almost everywhere in $[s_i, s_e]$*

$$u^*(s)\int_T \psi^*(0,t)\beta(t)\left[k(s)p^*(s,t)\right]_s\,dt = 0.$$

The iterative method described in the previous chapter was applied for solving (12)-(16). In this case

$$\omega_k = u^k(s)\int_T \psi^k(0,t)\beta(t)\left[k(s)p^k(s,t)\right]_s\,dt.$$

$$\delta_k(s) = (s - s_i)(s_e - s)\frac{\omega_k}{M_k},$$

where

$$M_k = (s_i - s_e)\max_{s\in[s_i,s_e]}|\omega_k(s)|.$$

Equation (12) and the conjugate equation were solved by the numerical method of characteristics. The results of computational experiments are given for a quadratic cost functional (17) and

$$\bar{p}(s) = e^{5-s-\frac{1}{5-s}}, \qquad p_0(s) = e^{-s-\frac{1}{5-s}},$$

$$\mu(s) = \frac{1}{(5-s)^2},\ k(s) = s\,e^{s+\frac{1}{5-s}},\ \beta(t) = \frac{1}{2}e^{-\frac{1}{5}},$$

$$s_1 = 5, \qquad t_1 = 5, \qquad s_i = 1, \qquad s_e = 4.$$

Table 1 contains values of a cost functional for each iteration l and the following initial controls:

$$\tilde{u}^0(s) = \frac{1}{3}, \hat{u}^0(s) = \frac{2}{9}(s-1), \breve{u}^0(s) = \frac{1}{3}(\sin 2\pi s + 1).$$

Table 1. Results of computational experiments

l	$J_1(\tilde{u}^l)$	$J_1(\hat{u}^l)$	$J_1(\breve{u}^l)$
0	$2.827 \cdot 10^7$	$1.143 \cdot 10^4$	$1.329 \cdot 10^3$
1	$2.605 \cdot 10^1$	$2.315 \cdot 10^3$	$2.236 \cdot 10^2$
2	$2.16 \cdot 10^{-7}$	$2.646 \cdot 10^2$	$1.052 \cdot 10^2$
3		$2.26 \cdot 10^{-7}$	$4.593 \cdot 10^1$
6			$2.19 \cdot 10^{-7}$

Note that the approach developed in this paper can be successfully applied for a wide class of inverse problems: heat equations (Arguchintsev and Vasiliev, 2000), problems of determination of gravitational waves initial parameters (Arguchintsev and Krutikova, 2001), etc.

5. ACKNOWLEDGEMENTS

This work was supported in part by the Russian Foundation for Basic Research (Projects 02-01-00243, 02-01-81001).

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