15th Triennial World Congress, Barcelona, Spain

A LQG CONTROLLER DESIGN FOR LINEAR CONTINUOUS-TIME SYSTEMS BASED ON INPUT-OUTPUT DATA

Masami Iwase * **Masaki Yamakita** ** **Katsuhisa Furuta** *

* *Dept. of Computers and Systems Engineering*
Tokyo Denki University, Hatoyama, Hiki-gun, Saitama,
350-0394, JAPAN, iwase@k.dendai.ac.jp
** *Dept. of Mechanical and Control Systems*
Tokyo Institute of Technology

Abstract: In this paper, a new LQG control design based on input-output data for linear time invariant continuous-time systems is proposed. Considering the laguerre series expansion of input-output signals, the controller can be synthesized using only the coefficients by an algebraic calculation. The analysis of the system transformed by the laguerre series expansion is utilized to lead this design method. Comparing the proposed controller with the model-based one through a numerical simulation, the effectiveness is illustrated.

Keywords: Sampled data, LQG Control, Continuous time systems

1. INTRODUCTION

In most control design, the mathematical model of a plant represented in the state space or the transfer function is derived, and then the controller is designed based on the model (Kalman, 1960). If the model is accurate, we can obtain the reasonable controller through this approach. On the other hand, in the modeling we usually do not have paid attention to the controller design. That is, the modeling and controller design has been done independently. However, the suggestion that the goodness of the model should be judged by closed loop performance of the system and that the modeling and controller design should be performed simultaneously is found in (Skelton, 1989). To come up to the suggestion, we consider that the optimal controller should be designed to minimize the criterion based on input-output response (Furuta and Wongsaisuwan, 1995).

There exist many works standing in this point of view, for example, the LQG controller design using the Markov parameters of the system (Skelton and Shi, 1994) (Furuta and Wongsaisuwan, 1995), the input design based on the basis array computed from an input-output data array (Fujisaki *et al.*, 1998), the optimal controller design utilizing the orthogonality of the initial response and impulse response of the optimal system (Kawamura, 2000), the convex programming algorithm for the optimal input design based on input-output data (Sugie and Hamamoto, 1998) and the LQG controller design using the subspace identification method (Favoreel, 1999). However, only the class of linear time-invariant discrete-time systems are considered in these works.

This paper proposes a new LQG control design with input-output data for continuous-time systems. We first consider the Laguerre series expansion of input-output signals of systems, then formulate the quadratic criterion with the expansion coefficients. As a result, the optimal controller minimizing the criterion is given by only the algebraic calculation of the coefficients which are obtained by the Laguerre series expansion of the measured input-output signals generated by injecting the time response of the Laguerre basis into the real system. Because the proposed algorithm consists of the algebraic operator, it can be easily implemented as a built-in controller such as DSP. Utilizing this controller the redesign can be done quickly, so it can be expected that the controller works as an auto-tuning controller in a sense, then that it is useful on industry.

2. PRELIMINARIES

First, notation and terminology are defined. R and R_+ mean the reals and the nonnegative reals, respectively. $\mathcal{L}_2$ consists of real-valued vector Lebesgue measurable functions f on R_+ such that $||f(t)|| = \sqrt{\int_{R_+} |f(t)|^2} < \infty$. $\{f_n\}_{n\geq k}$ denotes a sequence of real numbers or functions. The Laguerre function $\phi_i(t)$ is given as $\phi_i(t) = \sqrt{2p}\sum_{k=0}^{i}(-1)^{i-k}(2p)^k \frac{i!}{k!k!(i-k)!}t^k e^{-pt}$ where p is the parameter determining the mode of the function. Laguerre series expansion coefficients $\{v_i\}_{i\geq 0}$ of a continuous signal $v(t) \in R^{j\times 1}$ are obtained by the inner product: $v_i = \int_{R_+} v(t)\phi_i(t)\,dt$. Considering an expansion of $v(t)$ with N terms: $v(t) \approx \sum_{k=0}^{N-1} v_k\phi_k(t)$, *Laguerre transform* $\mathcal{L}_a[\cdot]$ is defined as $\mathcal{L}_a[v(t)] := V_N$ and $V_N := \left[v_0^T, \ \cdots, \ V_{N-1}^T\right]^T$

Proposition 1. (Szegö, 1939) The Laguerre functions $\{\phi_n(t)\}_{n\geq 0}$ form a complete orthonormal basis of $\mathcal{L}_2$. ♢

Proposition 2. The Laguerre transform operator $\mathcal{L}_a[\cdot]$ holds the linearity: $\mathcal{L}_a[\alpha v(t)] = \alpha\mathcal{L}_a[v(t)]$ and $\mathcal{L}_a[v(t)+w(t)] = \mathcal{L}_a[v(t)] + \mathcal{L}_a[w(t)]$. ♢

Proposition 3. Let $\{\phi_n(t)\}_{n\geq 0}$ be the complete orthonormal basis of $\mathcal{L}_2$. For any $v(t) \in \mathcal{L}_2$, [i] $v(t) = \sum_{k=0}^{\infty} v_k\phi_k(t)$ and [ii] $||v(t)||^2 = \sum_{k=0}^{\infty}|v_k|^2$ are hold. ♢

[ii] is called *Parseval's identity.* Because of completeness, a signal in $\mathcal{L}_2$ can be approximated with an arbitrary accuracy, and the number of coefficients required to approximate the signal can be reduced by choosing p suitably. Here, the Laguerre transform of an autonomous system is considered in the following lemma.

Lemma 1. Let us consider an autonomous system $\dot{x} = Ax$ with the initial condition $x(0)$, and the initial response $x^I(t) = e^{At}x(0)$. Then its Laguerre transform, $\mathcal{L}_a\left[x^I(t)\right] = \left[x_0^T, \ \cdots, \ x_{N-1}^T\right]^T$ is given as $x_k = \sqrt{2p}\,(pI-A)^{-(k+1)}(pI+A)^k x(0)$ where p is the pole of the Laguerre basis. ♢

Proof. For $k=0$, $x_0 = \int_{R_+} e^{At}x(0)\phi_0(t)dt = \sqrt{2p}\,(pI-A)^{-1}x(0)$. It is assumed that $x_k = \sqrt{2p}\,(pI-A)^{-(k+1)}(pI+A)^k x(0)$. Multiplying x_k by $(pI-A)^{-1}(pI+A)$ from the left-hand side and using the relations: $(pI-A)^{-1}(pI+A) = -I + 2p\,(pI-A)^{-1}$ and $(pI-A)^{-1}(pI+A) = (pI+A)(pI-A)^{-1}$, $(pI-A)^{-1}(pI+A)x_k = \sqrt{2p}\,(pI-A)^{-(k+2)}(pI+A)^{k+1}x(0)$ is obtained by straightforward calculations. The proof is completed by mathematical induction. □

Using $(pI-A)^{-i}(pI+A)^i = \sum_{k=0}^{i-1}(-1)^{i-k-1}2p \times (pI-A)^{-(k+1)}(pI+A)^k + (-1)^i I$, *Lemma 1* is transformed into an algebraic equation with the expansion coefficients by the Laguerre transform.

Theorem 1. Expand the initial response $x^I(t) = e^{At}x(0)$ by the Laguerre transform. Then the expansion coefficients $X_N = \mathcal{L}_a[x^I(t)]$ must satisfy the following algebraic equation:

$$D_N X_N - X_0 = A_N X_N \tag{1}$$

where $X_0 = \sqrt{2p}\left[I, \ \cdots, \ (-1)^{N-1}I\right]^T x(0)$, $A_N = \mathrm{diag}\underbrace{(A, \ A, \ \cdots, \ A)}_{N}$ and

$$D_N = \begin{bmatrix} pI & & & \\ -2pI & pI & & \\ \vdots & & \ddots & \\ (-1)^{N-1}2pI & \cdots & -2pI & pI \end{bmatrix} \tag{2}$$

♢

Proof. By expansion of $(pI-A)^{-k}(pI+A)^k$, $(pI-A)x_k = \sum_{i=0}^{k-1}(-1)^{k-i-1}2px_k\sqrt{2p}\,(-1)^k \times Ix(0)$. Stacking up it from $k=1$ to $N-1$, (1) is obtained. □

From *Theorem 1* and the linearity of Laguerre transform, an algebraic equation of the continuous-time system is obtained as follows.

Corollary 1. Expand the input, output and state of a time invariant continuous-time system:

$$\dot{x} = Ax + Bu, \quad y = Cx$$

with $x(0)$ by the Laguerre transform. Then, the coefficients must satisfy the algebraic equation:

$$(D_N - A_N)X_N = X_0 + B_N U_N, \ Y_N = C_N X_N \tag{3}$$
$$X_N = \mathcal{L}_a[x(t)], \ U_N = \mathcal{L}_a[u(t)], \ Y_N = \mathcal{L}_a[y(t)]$$

where B_N, C_N are defined similarly to A_N. ♢

(3) is called *Laguerre System* in this paper.

3. LQG PROBLEM FOR THE LAGUERRE SYSTEM

In this section, to design the optimal controller based on input-output signals, first of all, we show that the optimal control problem formulated with the coefficients of input-output signals is equivalent to the problem based on the state space representation, and lead the optimal controller based on the equivalent problem, using Parseval's identity.

3.1 *LQ problem*

Consider a linear time-invariant continuous-time system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \quad (4)$$

where $x(t) \in R^{n\times 1}$ is state, $u(t) \in R^{m\times 1}$ is input, $y(t) \in R^{p\times 1}$ is output, $(A,\, B)$ is controllable and $(C,\, A)$ is observable. The problem is to find, for any given $x(0)$, the control law minimizing a quadratic cost function $J = \int_{R_+} \left(y^T(t)y(t) + u^T(t)Ru(t)\right) dt$ where $R > 0$. From the Parseval's identity, the criterion is transformed to $J = \sum_{k=0}^{\infty} \left(x_k^T Q x_k + u_k^T R u_k\right)$, where $Q = C^T C$, equivalently. Then, we can restate the LQ problem for the Laguerre system.

LQ problem: *For the Laguerre system of (4), the problem is to find the control input U_N minimizing the cost function:*

$$J = Y_N^T Y_N + U_N^T R U_N = X_N^T Q_N X_N + U_N^T R_N U_N \quad (5)$$

where Q_N and R_N are defined similarly to A_N. The optimal control input U_N and the minimum value of the cost function (5) are given as follows.

Theorem 2. $\bar{A}_N := (D_N - A_N)^{-1}$. For (3), the minimum of (5) and the optimal control U_N are given as

$$U_N = -\left(R_N + B_N^T \bar{A}_N^T Q_N \bar{A}_N B_N\right)^{-1} B_N^T \bar{A}_N^T Q_N \bar{A}_N X_0 \quad (6)$$

$$\min_{U_N} J = X_0^T P_L X_0 \quad (7)$$

$$P_L = \bar{A}_N^T Q_N \bar{A}_N - \bar{A}_N^T Q_N \bar{A}_N B_N \times \left(R_N + B_N^T \bar{A}_N^T Q_N \bar{A}_N B_N\right)^{-1} B_N^T \bar{A}_N^T Q_N \bar{A}_N. \quad \diamondsuit$$

Proof. $X_N = \bar{A}_N X_0 + \bar{A}_N B_N U_N$ from (3). Substituting it into (5) leads to the following:

$$\begin{aligned} J = X_0^T P_L X_0 + &\left[U_N + \left(R_N + B_N^T \bar{A}_N^T Q_N \bar{A}_N B_N\right)^{-1}\right. \\ &\times \left. B_N^T \bar{A}_N^T Q_N \bar{A}_N X_0\right]^T \left(R_N + B_N^T \bar{A}_N^T Q_N \bar{A}_N B_N\right) \quad (8) \\ &\times \left[U_N + \left(R_N + B_N^T \bar{A}_N^T Q_N \bar{A}_N B_N\right)^{-1}\right. \\ &\times \left. B_N^T \bar{A}_N^T Q_N \bar{A}_N X_0\right]. \end{aligned}$$

Therefore, the theorem is led from (8). □

Remark 1. The optimal control (6) is not a feedback but a feedforward control law using X_0. ♢

Because of the Parseval's identity, a relation between the standard LQ and our problem is stated as follows.

Corollary 2. Let $F = -R^{-1}B^T P$ be the optimal feedback gain of the standard LQ problem, where P is a solution of the riccati equation: $A^T P + PA + Q - RBR^{-1}B^T P = O$. Then, with $N \to \infty$, P is equivalent to

$$P = I_N^T P_L I_N, \quad I_N := \sqrt{2p}\left[\, I \ \cdots \ (-1)^{N-1} I \,\right]^T \quad (9)$$

Therefore $F = -R^{-1}B^T I_N^T P_L I_N$. ♢

Proof. The statement is obvious from (7). □

3.2 *Full order observer*

Next, we consider a full order observer, $\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) + K\left[y(t) - C\tilde{x}(t)\right]$, where the initial state of the observer is assumed to be zero, $\tilde{x}(0) = 0$. Then, the observer for the Laguerre system is given as follows:

$$\begin{aligned} \tilde{X} = \bar{A}_N B_N U_N - &\left[D_N - (A_N - K_N C_N)\right]^{-1} K_N Y_N \\ &+ \left[D_N - (A_N - K_N C_N)\right]^{-1} K_N C_N \bar{A}_N B_N U_N \quad (10) \end{aligned}$$

where $\left[D_N - (A_N - K_N C_N)\right]^{-1} K_N$ can be obtained by considering the LQ problem for a dual system of (3):

$$(D_N - A_N^T)\tilde{X}_N = \tilde{X}_0 + C_N^T \tilde{U}_N, \quad \tilde{Y}_N = B_N^T \tilde{X}_N \quad (11)$$

Lemma 2. $\tilde{A}_N := \left(D_N - A_N^T\right)^{-T}$. For (11), the optimal input $\tilde{U}_N$ minimizing the quadratic cost function: $\tilde{J} = \tilde{X}_N^T \tilde{Q}_N \tilde{X}_N + \tilde{U}_N^T \tilde{R}_N \tilde{U}_N$, $\tilde{Q} \geq 0$, $\tilde{R} > 0$ and its minimum are given as

$$\tilde{U}_N = -\left(\tilde{R}_N + C_N \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T C_N^T\right)^{-1} C_N \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T \tilde{X}_0 \quad (12)$$

$$\min_{\tilde{U}_N} \tilde{J} = \tilde{X}_0^T \tilde{P}_L \tilde{X}_0 \quad (13)$$

$$\tilde{P}_L = \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T - \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T C_N^T \times \left(\tilde{R}_N + C_N \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T C_N^T\right)^{-1} C_N \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T. \quad \diamondsuit$$

Proof. This lemma is proved by replacing the coefficients in *Theorem 2* as follows: $\bar{A}_N \to \tilde{A}_N^T$, $B_N \to \tilde{C}_N^T$, $Q_N \to \tilde{Q}_N$, $R_N \to \tilde{R}_N$. □

Let $\tilde{U}_N = -K^T \tilde{X}$ denote the optimal feedback law and substitute it into (11). Then, from (12), $\tilde{U}_N = -K_N^T\left[D_N - \left(A_N^T - C_N^T K_N^T\right)\right]^{-1} \tilde{X}_0$ is equal to $-(\tilde{R}_N + C_N \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T C_N^T)^{-1} C_N \tilde{A}_N \times \tilde{Q}_N \tilde{A}_N^T \tilde{X}_0$. While $K_N^T\left[D_N - \left(A_N^T - C_N^T K_N^T\right)\right]^{-1}$ has the Toeplitz structure, $(\tilde{R}_N + C_N \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T \times C_N^T)^{-1} C_N \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T$ does not have generally. Therefore, by using the structure of X_0, an operator to fit it into the Toeplitz structure, $\mathcal{T}_{toep}$, can be defined as $\mathcal{T}_{tope}\left[\cdot\right] := \mathcal{T}_o\left[\mathcal{T}_c\left[\cdot\right]\right]$ where

$$\mathcal{T}_o\left[\left[\begin{array}{c}\phi_1\\ \vdots\\ \phi_N\end{array}\right]\right] := \left[\begin{array}{ccc}\phi_1 & & \\ \vdots & \ddots & \\ \phi_N & \cdots & \phi_1\end{array}\right]$$

$$\mathcal{T}_c\left[\Psi_N\right] := \left[\begin{array}{ccc} I & & \\ \vdots & \ddots & \\ (-1)^{N-1}I & \cdots & I\end{array}\right]\Psi_N\left[\begin{array}{c} I\\ \vdots\\ (-1)^{N-1}I\end{array}\right].$$

Because of the special structure of (2), there exists the following relation between $\bar{A}_N$ and $\tilde{A}_N$:

$$\bar{A}_N = \left[\begin{array}{ccc}\delta_1 & & \\ \vdots & \ddots & \\ \delta_N & \cdots & \delta_1\end{array}\right],\ \tilde{A}_N^T = \left[\begin{array}{ccc}\delta_1^T & & \\ \vdots & \ddots & \\ \delta_N^T & \cdots & \delta_1^T\end{array}\right]. \qquad (14)$$

Using (14), we define an operator $\mathcal{T}$ as $\mathcal{T}\left[\bar{A}_N\right] := \tilde{A}_N$. Finally, we propose the design method of the full order observer for the Laguerre system.

Theorem 3. A full order observer for (3) is given as (10). The observer gain in (10) is obtained as

$$\begin{aligned}&\left[D_N - (A_N - K_N C_N)\right]^{-1} K_N\\ &= \mathcal{T}\left[\mathcal{T}_{toep}\left[\left(\tilde{R}_N + C_N\tilde{A}_N\tilde{Q}_N\tilde{A}_N^T C_N^T\right)^{-1} C_N\tilde{A}_N\tilde{Q}_N\tilde{A}_N^T\right]\right]\end{aligned}$$

using the optimal control of the dual system. ♢

Proof. The proof is easily given from *Lemma 2* and the definitions of $\mathcal{T}$ and $\mathcal{T}_{toep}$. □

3.3 *Dynamic output feedback controller*

From *Corollary 2* and *Theorem 3*, combining the state feedback law and the full order observer leads to a dynamic output feedback controller for the Laguerre system.

Theorem 4. F_N is defined similarly to A_N. For (3), a dynamic output feedback controller is given as follows:

$$\begin{aligned}U_N = F_N\big\{&\bar{A}_N B_N U_N - \left[D_N - (A_N - K_N C_N)\right]^{-1} K_N Y_N\\ &+ \left[D_N - (A_N - K_N C_N)\right]^{-1} K_N C_N \bar{A}_N B_N U_N\big\} \quad (15)\end{aligned}$$

where F and $\left[D_N - (A_N - K_N C_N)\right]^{-1} K_N$ are designed according to *Corollary 2* and *Theorem 3*, respectively. ♢

Proof. The proof is obvious. □

Remark 2. The proposed dynamic controller is equivalent to a LQG controller minimizing the cost function:

$$J_s = \lim_{t_f\to\infty}\mathcal{E}\left\{\int_0^{t_f}\left(x^T Q x + u^T R u\right)dt\right\}$$

where $Q = C^T C \geq 0,\ R > 0,\ \mathcal{E}\{\cdot\}$ means the expectation operator, for a stochastic system: $\dot{x} = Ax + Bu + w,\ y = Cx + v$ under conditions:

$$\mathcal{E}\left\{\left[\begin{array}{c} w(t)\\ v(t)\end{array}\right]\left[\begin{array}{cc} w^T(\tau) & v^T(\tau)\end{array}\right]\right\} = \left[\begin{array}{cc}\tilde{Q} & O\\ O & \tilde{R}\end{array}\right]\delta(t-\tau).$$

$\mathcal{E}\{x(0)\} = O,\ \mathcal{E}\{w(t)\} = O,\ \mathcal{E}\{v(t)\} = O$. In this sense, we can say that (15) is a LQG controller.

4. LQG CONTROLLER WITH THE LAGUERRE SERIES EXPANSION OF INPUT-OUTPUT SIGNALS

In this section, we give the main result of this paper, that is, the design of the optimal controller minimizing the quadratic criterion (5) formulated with the coefficients of the Laguerre series expansion of an input-output signal. The coefficients used in the design is obtained by the expansion of the input-output signal generated by injecting the time response of the Laguerre basis into the real system. Therefore, we first consider the special response such as the impulse response for the Laguerre system.

4.1 *Laguerre pulse response*

Let us consider an augmented system of (4): $\dot{x}(t) = Ax(t) + Ez(t) + Bu(t),\ y(t) = Cx(t)$, and its Laguerre system is given as $D_N X_N - X_0 = A_N X_N + E_N Z_N + B_N U_N,\ Y_N = C_N X_N$. From *Theorem 1*, input-output relations of this Laguerre system with the zero initial state, $X_0 = O$, are described as

$$Y_N = \mathcal{T}_o\left[\left[\begin{array}{ccc} g_1^T & \cdots & g_N^T\end{array}\right]^T\right] Z_N \qquad (16)$$

$$g_i = \begin{cases} C\left(pI - A\right)^{-1}E, & i = 1\\ 2p\cdot C\left(pI + A\right)^{i-2}\left(pI - A\right)^{-i}E, & i > 1\end{cases}$$

$$Y_N = \mathcal{T}_o\left[\left[\begin{array}{ccc} h_1^T & \cdots & h_N^T\end{array}\right]^T\right] U_N \qquad (17)$$

$$h_i = \begin{cases} C\left(pI - A\right)^{-1}B, & i = 1\\ 2p\cdot C\left(pI + A\right)^{i-2}\left(pI - A\right)^{-i}B, & i > 1\end{cases}$$

where $\left[h_1^T,\ \cdots,\ h_N^T\right]^T,\ \left[g_1^T,\ \cdots,\ g_N^T\right]^T$ are obtained by injecting the input sequences:

$$u_i,\ z_i = \begin{cases} I & i = 0\\ O & i \neq 0\end{cases}. \qquad (18)$$

(18) can be generated practically by the Laguerre transform of the time response of the Laguerre basis $\phi_0(t)$. Therefore, $\{g_i\}_{i\geq 1}$ and $\{h_i\}_{i\geq 1}$, called *Laguerre unit pulse response* in this paper, can be generated by the Laguerre transform of responses given by injecting the time response of $\phi_0(t)$ into $z(t),\ u(t)$, respectively. (See Fig. 1, 2 for example.)

Remark 3. The continuous signals of input and output, which is needed to calculate the inner product for the Laguerre expansion, cannot be obtained actually. However, in the system where the sampling interval can be set short such as DSP system, the inner product can be approximated with enough accuracy by numerical integration of the data sampled at the interval T, for example,

$$v_k \approx \sum_i \frac{v[i+1] + v[i]}{2}T$$

where $\{v[k]\}$ are sampled data of $v(t)$.

4.2 *LQG controller with the Laguerre unit pulse response*

Now we try to represent the LQG controller (15) with the Laguerre unit pulse response. Substitute $Q_N = C_N^T C_N$ in the LQ optimal gain of *Theorem 2* and *Corollary 2*. Then,

$$U_N = -R^{-1} B^T I_N^T \left[\bar{A}_N^T C_N^T C_N \bar{A}_N - \bar{A}_N^T C_N^T C_N \bar{A}_N B_N \times \left(R_N + B_N^T \bar{A}_N^T C_N^T C_N \bar{A}_N B_N \right)^{-1} B_N^T \bar{A}_N^T C_N^T C_N \bar{A}_N \right] I_N$$

where $C_N \bar{A}_N B_N$ and $B^T I_N^T \bar{A}_N^T C_N^T$ can be represented with (17), but $C_N \bar{A}_N I_N$ cannot be individually. For the observer (10), $C_N \tilde{A}_N B_N$ can be represented with (17), but $\bar{A}_N B_N$ cannot be. Note that $[D_N - (A_N - K_N C_N)]^{-1} K_N$ is constructed by $(\tilde{R}_N + C_N \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T C_N^T)^{-1} C_N \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T$ from *Theorem 3*. Substitute $\tilde{R}$ and $\tilde{Q}_N = E_N E_N^T$ in *Lemma 2*, then

$$\begin{aligned} & K_N^T \left[D_N - \left(A_N^T - C_N^T K_N^T \right) \right]^{-1} \tilde{X}_0 \\ &= \left(\tilde{R}_N + C_N \tilde{A}_N E_N E_N^T \tilde{A}_N^T C_N^T \right)^{-1} C_N \tilde{A}_N E_N E_N^T \tilde{A}_N^T \tilde{X}_0 \end{aligned}$$

where $C_N \tilde{A}_N E_N$ are represented with (16), but $E_N^T \tilde{A}_N^T$ cannot be individually. Therefore, we need to combine $\bar{A}_N B_N$ and $E_N^T \tilde{A}_N^T$ with $C_N \bar{A}_N I_N$ of the optimal feedback gain. Actually, we can do that as shown in the following theorem.

Theorem 5. It is assumed that $R > 0$, $\tilde{R} > 0$ are given and $Q_N = C_N^T C_N$, $\tilde{Q}_N = E_N E_N^T$. Then, the LQG controller (15) can be represented with the Laguerre unit pulse response (16)-(17) as follows:

$$U_N = \left[I - \mathcal{F}_N + \mathcal{K}_N \bar{H}_N \right]^{-1} \mathcal{K}_N Y_N \tag{19}$$

$$\mathcal{F}_N = \mathcal{T}_o \left[\left[\, (F_G \gamma_1)^T \ \cdots \ (F_G \gamma_N)^T \, \right]^T \right]$$

$$\mathcal{K}_N = \mathcal{T} \left[\mathcal{T}_{toep} \left[K_G \mathcal{T}_o \left[\, F_G \tilde{\gamma}_1 \ \cdots \ F_G \tilde{\gamma}_N \, \right]^T \right] \right]$$

$$F_G = -R^{-1} \left[\bar{h}_1^T \cdots \bar{h}_N^T \right] \left[I - \bar{H}_N \left(R_N + \bar{H}_N^T \bar{H}_N \right)^{-1} \bar{H}_N^T \right]$$

$$K_G = \left[\tilde{R}_N + \bar{G}_N \bar{G}_N^T \right]^{-1} \bar{G}_N, \quad \bar{H}_N = \mathcal{T}_o \left[\left[\, h_1^T \cdots h_N^T \, \right]^T \right]$$

$$\bar{G}_N = \mathcal{T}_o \left[\left[\, g_1 \cdots g_N \, \right]^T \right]^T$$

$$\gamma_1 = \frac{\sqrt{2p}}{2p} \left[\, h_2^T \cdots h_{N+1}^T \right]^T, \quad \tilde{\gamma}_1 = \frac{\sqrt{2p}}{2p} \left[\, g_2^T \cdots g_{N+1}^T \right]^T$$

$$\gamma_i = \frac{\sqrt{2p}}{2p} \begin{bmatrix} h_{i+1} + h_i \\ \vdots \\ h_{i+N} + h_{i+N-1} \end{bmatrix}, \quad \tilde{\gamma}_i = \frac{\sqrt{2p}}{2p} \begin{bmatrix} g_{i+1} + g_i \\ \vdots \\ g_{i+N} + g_{i+N-1} \end{bmatrix}$$

$$\left[\, \bar{h}_1^T \cdots \bar{h}_N^T \right] = \sqrt{2p} \left[\, h_1^T \cdots \sum_{i=1}^{N} (-1)^{i-1} h_{N-i+1}^T \right] \qquad \diamondsuit$$

Proof. Due to the lack of space, the proof, which is complex but straightforward calculation of matrices, is omitted. □

The main point we address in this paper is that in the optimal controller design, we need only to calculate the (19) using the coefficients directly, and do not need to obtain any mathematical model or to design the feedback gain and the full-order observer separately.

4.3 *State space realization of LQG controller*

In the previous, the LQG controller has been given in the form of expansion coefficients of the unit pulse response. The realization method of (19) in the state space is given here. Let $(A_c,\ B_c,\ C_c)$ denote a state space realization of (19). (19) has the toeplitz structure:

$$U_N = \mathcal{T}_o \left[\left[\, \Phi_1^T \ \cdots \ \Phi_N^T \, \right]^T \right] Y_N \tag{20}$$

$$\Phi_i = \begin{cases} C_c \left(pI - A_c \right)^{-1} B_c, & i = 1 \\ 2p \cdot C_c \left(pI + A_c \right)^{i-2} \left(pI - A_c \right)^{-i} B_c, & i > 1 \end{cases} \tag{21}$$

Moreover, we define $\left(\bar{A}_c,\ \bar{B}_c,\ \bar{C}_c, \bar{D}_c\right)$ as follows:

$$\bar{A}_c = (pI + A_c)(pI - A_c)^{-1}, \quad \bar{B}_c = \sqrt{2p}\,(pI - A_c)^{-1} B_c$$
$$\bar{C}_c = \sqrt{2p} C_c (pI - A_c)^{-1}, \qquad \bar{D}_c = C_c (pI - A_c)^{-1} B_c,$$

then (21) is rewritten as

$$\Phi_i = \left\{ \bar{D}_c \ : \ i = 1, \ \ \bar{C}_c \bar{A}_c^{i-1} \bar{B}_c \ : \ i > 1 \right\}. \tag{22}$$

$(A_c,\ B_c,\ C_c)$ can be also described as follows.

$$A_c = p \cdot \left(\bar{A}_c + I \right)^{-1} (A_c - I), \quad B_c = \frac{1}{\sqrt{2p}} (pI - A_c) \bar{B}_c$$
$$C_c = \frac{1}{\sqrt{2p}} C_c (pI - A_c). \tag{23}$$

From (20), (22) and (23), we summarize the design algorithm of the LQG controller in the state space form.

Algorithm

Step 1: Generete input-output signals by injecting the time response of $\phi(t)$ into the real system.
Step 2: Calculate the coefficients of the generated signals by the Laguerre transform.
Step 3: Design the LQG controller according to (19).
Step 4: Extract Φ_i from the result of Step 3.
Step 5: Realize $\left(\bar{A}_c,\ \bar{B}_c,\ \bar{C}_c,\ \bar{D}_c\right)$ from Φ_i according to Ho and Kalman's realization method.
Step 6: Using (23), transform $\left(\bar{A}_c,\ \bar{B}_c,\ \bar{C}_c,\ \bar{D}_c\right)$ into the state space realization: $(A_c,\ B_c,\ C_c)$.

5. NUMERICAL EXAMPLES

To verify the effectiveness of the proposed method, we consider a LQG control design for the following 2nd order system:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -100 & -10 \end{bmatrix} x + \begin{bmatrix} 30 & 0 \\ 0 & 3 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned} \tag{24}$$

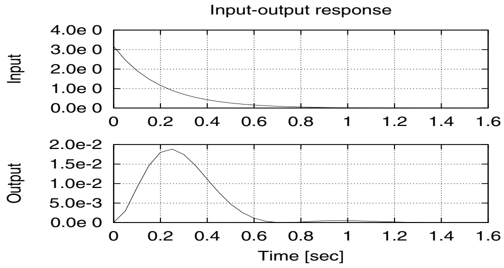

Fig. 1. Time responses used for the design

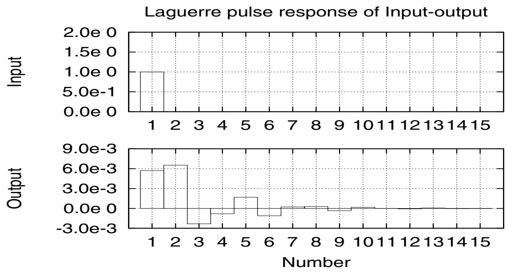

Fig. 2. Laguerre coefficients used for the design

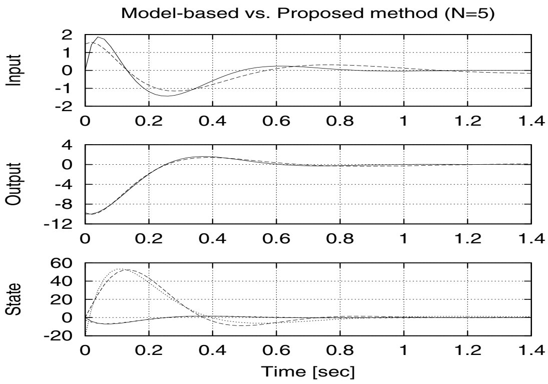

Fig. 3. Time responses in $(p,\ N) = (5,\ 5)$

and $R = [0.01]$, $\tilde{R} = I$ are given. In the above case, $Q = \mathrm{diag}(1,\ 0)$ and $\tilde{Q} = \mathrm{diag}(900,\ 9)$. The time responses and coefficients used in the design is shown in Figure 1–2. We must choose the parameter p and the length of the series N appropriately. The best choice is found in (Hof *et al.*, 1995). Here we show two cases. The result of $(p,\ N) = (5,\ 5), (5,\ 12)$ are shown in Figure 3–4. From figures, it can be seen that the proposed method gives the same controller as designed based on the state space model (24) as long as N is large enough. However, as the future work, we should investigate some left problems such as the robustness for noises and numerical calculations, and the influence of modeling error if the length N is not long enough.

6. CONCLUDING REMARKS

This paper has proposed a new LQG controller design using input-output data for continuous-time linear systems. By utilizing the Laguerre series expansion, the quadratic criterion has been formulated with the expansion coefficients. The analyses of the optimal feedback gain and full order observer minimizing the criterion have been

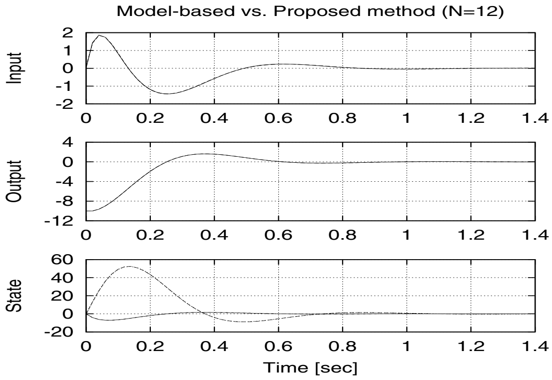

Fig. 4. Time responses in $(p,\ N) = (5,\ 12)$

given. As the result, the controller has been obtained merely by the algebraic calculation of the Laguerre transform of input-output responses generated by injecting the time response of the Laguerre basis to the real system. Moreover, the state space realization method of the controller has been proposed. Finally, the effectiveness has been verified through the numerical simulation.

7. REFERENCES

Favoreel, W. (1999). Subspace methods for identification and control of linear and bilinear systems. *Ph. Doctor dissertation, Katholiek Universiteit Leuven.*

Fujisaki, Y., Y. Duan and M. Ikeda (1998). System representation and optimal control in an input-output data space. *J. of SICE* **34**(12), 1845–1853.

Furuta, K. and M. Wongsaisuwan (1995). Discrete-time LQG dynamic controller design using plant markov parameters. *Automatica* **31**(9), 1325–1332.

Hof, P.M.J. Van Den, Peter S.C. Hauberger and József bokor (1995). System identification with generalized orthonormal basis functions. *Automatica* **31**(12), 1821–1834.

Kalman, R.E. (1960). On the general theory of control systems. *Proc. First IFAC Congr. Auto. Control* pp. 481–493.

Kawamura, Y. (2000). Lq optimal control system design based on input-output data. *J. of Systems, Control and Information* **44**(4), 169–176.

Skelton, R.E. (1989). Model error concepts in control design. *Int. J. Control* **49**(5), 1725–1753.

Skelton, R.E. and G. Shi (1994). The data-based LQG control problem. *Proc. of the 33rd Conf. on Decision and Control* pp. 1447–1452.

Sugie, T. and K. Hamamoto (1998). A construction method of optimal control input sequence based on input-output data – convex programming approach. *J. of Systems, Control and Information* **11**(2), 86–92.

Szegö, G. (1939). Orthogonal polynomials. *Amer. Math. Soc. Coll. Publ.*