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STABILIZING FEEDBACK CONTROLS FOR A CLASS OF SINGULARLY PERTURBED, NONLINEAR, UNCERTAIN, TIME-DELAY SYSTEMS

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Abstract: A class of nonlinear controllers is presented which guarantees the stability property of *global uniform ultimate boundedness* with respect to some known set for a class of nonlinear, singularly perturbed systems, with discrete and distributed delays, in the presence of uncertainty, provided the singular perturbation parameter is small enough. The uncertainty acting on the systems, which may be time-, state-, delayed state-, and/ or input-dependent, are modelled as additive nonlinear perturbations influencing a known nominal, singularly perturbed, time-delay system of the retarded type. Each feedback controller is designed using information based mainly on a nonlinear, affine in the control, 'reduced-order' system. A 'matched' uncertainty structural condition for the 'reduced-order' system is not presumed in this paper. *Copyright ©2002 IFAC*

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1. INTRODUCTION

Often, control systems exhibit nonlinear characteristics and are also subject to various forms of disturbances. One approach to modelling systems with uncertainty is to absorb the unknown nonlinear elements and noise disturbances within a perturbed idealized model, where the uncertainty perturbation appears additively. Other problems that may arise is the coexistence of slow and fast dynamics in the plant to be controlled. This particular problem can be addressed utilizing singular perturbation theory (for more details, see (Kokotovic *et al.*, 1986) or, for a differential-geometric approach, (Isidori, 1989)). There has been much research, over the past decades, on control of uncertain singularly perturbed systems using a deterministic approach; for example, (Binning and Goodall, 1999; Binning and Goodall, 2000; Corless, 1991; Corless *et al.*, 1993; Corless *et al.*, 1990; Corless and Ryan, 1991; Garofalo and Leitmann, 1990), and (Leitmann *et al.*, 1986), to name but a few. In addition, time-delays, which can have a significant effect on the dynamic behaviour of a system, is a phenomenon that has been investigated by a number of researchers in recent times; in particular, aspects of stability analysis using a deterministic approach. However, up to the present time, very few papers have been published on singularly perturbed systems with time-delay (see (Chen, 1995; Glizer, 1999; Glizer, 2001; Hsiao and Hwang, 1996; Shao and Rowland, 1995) and (Sun *et al.*, 1996)), and none of these papers considers distributed delays. To the authors' knowledge, there appears to be no studies on *uncertain* singularly perturbed systems with time-delays.

The main objective of this paper is to design, using a deterministic approach, a class of robust feedback controls for singularly perturbed uncertain nonlinear systems, subject to time-delays (discrete and distributed), in order to achieve the stability property: global uniform ultimate boundedness. Parametric uncertainty is not considered in this paper; instead, *a priori* bounding knowledge of the system uncertainty, in terms of growth conditions with respect to its arguments, is assumed. One feature of the controllers, employed to stabilize the class of uncertain systems, is that the gains depend explicity on upper bounds of the uncertainty and, thus, robustness of the feedback controls is a consequence. The controllers, which are composite in nature, guarantee the desired behaviour provided the singular perturbation parameter is small enough. One component of a controller assures desired behaviour of the slow dynamics, whilst a second component yields the desired stability properties for the fast dynamics in the presence of the active slow controller. The stability analysis is similar to that used in (Binning and Goodall, 1999) and (Binning and Goodall, 2000) but, here, the full-order system is a nonlinear delay system of the retarded-type, containing both discrete and distributed delays. Utilizing memoryless feedback controllers, together with a deterministic methodology based on Lyapunov theory and Lyapunov-Krasovskiĭ functionals, some stability criteria are proposed that will ensure the desired stability property for the prescribed class of singularly perturbed delay systems, provided the singular perturbation parameter is small enough. One advantage of using memoryless controllers is that past history of the states does not need to be stored.

2. FULL-ORDER SINGULARLY PERTURBED UNCERTAIN SYSTEM

Consider a singularly perturbed uncertain dynamical system consisting of two coupled subsystems with the following structure: The class of uncertain, singularly perturbed, time-delay systems to be investigated consists of two coupled subsystems with the following structure:

$$\dot{x}(t) = a(t,x(t),x(t-\rho),I(x,\tau),y(t),u(t),\mu) \quad (1)$$

$$\mu\dot{y}(t) = b(t,x(t),x(t-\rho),I(x,\tau),y(t),u(t),\mu), \quad (2)$$

subject to the initial condition $x_{t_0} = \psi_x(\theta)$, $y_{t_0} = \psi_y(\theta)$, $\theta \in [-T,0]$, with $T = \max[\rho,\tau]$ and $x_t(\theta) := x(t+\theta)$, where

$$\begin{aligned} a(t,x_1,x_2,x_3,y,u,\mu) &= f_1(x_1,x_2,x_3) \\ &\quad + F_1(x_1,x_2,x_3)y + G_1(x_1,x_2,x_3)u \\ &\quad + h_1(t,x_1,x_2,x_3,y,u,\mu), \\ b(t,x_1,x_2,x_3,y,u,\mu) &= A(t)\big[f_2(x_1)+y(t) \\ &\quad + G_2(x_1)u\big] + h_2(t,x_1,x_2,x_3,y,u,\mu), \end{aligned}$$

$(x(t),y(t)) \in \mathbb{R}^n \times \mathbb{R}^\ell$ is the state of the system, $u(t) \in \mathbb{R}^m$ is the control input, $1 \le m \le \ell \le n$, and $\mu \in \mathbb{R}_0^+ := [0,\infty)$ denotes the singular perturbation parameter, assumed to be 'small'. The discrete and distributed delays, represented by ρ and τ, respectively, are assumed to be bounded and the initial condition functions satisfy $\psi_x \in C([-T,0];\mathbb{R}^n)$ and $\psi_y \in C([-T,0];\mathbb{R}^\ell)$; moreover, $I(x,\tau) := \int_{t-\tau}^{t} x(z)\,\mathrm{d}z$. It is assumed that the vector fields $f_1 \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n;\mathbb{R}^n)$ and $f_2 \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n;\mathbb{R}^\ell)$ are known and satisfy $f_1(0,0,0) = 0$, $f_2(0) = 0$, $F_1(x_1,x_2,x_3) \in \mathscr{L}(\mathbb{R}^\ell,\mathbb{R}^n)$ (the set of all continuous linear maps from $\mathbb{R}^\ell$ into $\mathbb{R}^n$), $G_1(x_1,x_2,x_3) \in \mathscr{L}(\mathbb{R}^m,\mathbb{R}^n)$ and $G_2(x_1) \in \mathscr{L}(\mathbb{R}^m,\mathbb{R}^\ell)$ are known and $G_2(x_1)$ is full rank for all x_1. The uncertainty in the system is represented by $A(t)$, a Lebesgue measurable matrix-valued function, and the nonlinear functions $h_1,h_2 \in C^\infty(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^m \times \mathbb{R}_0^+;\mathbb{R}^n)$.

The uncertainty in subsystem (2) is characterized by the following hypothses:
For a linear map L, the notation $\|L\|$ denotes $\{\max\sigma(L^{\mathrm{T}}L)\}^{\frac{1}{2}}$, where $\sigma(\cdot)$ denotes spectrum.

H1: (a) For all t, there exists $\kappa_0 \in [0,\ \frac{1}{2}\|P_0\|^{-1})$ and $A_0 \in \mathscr{L}(\mathbb{R}^\ell,\mathbb{R}^\ell)$ such that

$$A(t) = A_0 + \hat{A}(t), \qquad \big\|\hat{A}(t)\big\| \le \kappa_0,$$

A_0 is known and $\sigma(A_0) \subset \mathbb{C}^-$, where $P_0 > 0$ is the unique, symmetric solution of the Lyapunov equation

$$P_0A_0 + A_0^{\mathrm{T}}P_0 + I_\ell = O_\ell \quad (3)$$

and I_ℓ, O_ℓ are the $\ell \times \ell$ identity and zero matrices.

(b) For all (t,x_1,x_2,x_3,y,u),
$h_2(t,x_1,x_2,x_3,y,u,0) = 0$.

3. THE REDUCED-ORDER UNCERTAIN SYSTEM

Setting $\mu = 0$, system (1-2) degenerates to the set of n functional differential equations

$$\begin{aligned} \dot{x}(t) = &\ f_1(x(t),x(t-\rho),I(x,\tau)) \\ &+ F_1(x(t),x(t-\rho),I(x,\tau))y(t) \\ &+ G_1(x(t),x(t-\rho),I(x,\tau))u(t) \\ &+ h_1(t,x(t),x(t-\rho),I(x,\tau),y(t),u(t),\mu) \end{aligned} \quad (4)$$

subject to the constraint

$$f_2(x(t)) + y(t) + G_2(x(t))u(t) = 0. \quad (5)$$

The degenerate system (4-5) is a dynamical system that can be expressed in the form

$$\begin{aligned} \dot{x}(t) = &\ \tilde{f}(x(t),x(t-\rho),I(x,\tau)) \\ &+ \tilde{G}(x(t),x(t-\rho),I(x,\tau))u(t) \\ &+ \tilde{h}(t,x(t),x(t-\rho),I(x,\tau),u(t)), \end{aligned} \quad (6)$$

where $\tilde{f}(x_1,x_2,x_3) :=$
$f_1(x_1,x_2,x_3) - F_1(x_1,x_2,x_3)f_2(x_1)$,
$\tilde{G}(x_1,x_2,x_3) :=$
$G_1(x_1,x_2,x_3) - F_1(x_1,x_2,x_3)G_2(x_1)$,
$\tilde{h}(t,x_1,x_2,x_3,u) := h_1(t,x_1,x_2,x_3,\phi(x_1,u),u,0)$,
$\phi(x_1,u) := -f_2(x_1) - G_2(x_1)u$,
and is known as the *reduced-order system*. The function ϕ is determined uniquely in view of hypothesis H1(a).

The Euclidean inner product (on $\mathbb{R}^n$ or $\mathbb{R}^\ell$ as appropriate) and the induced norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let $L_f v : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}$ denote the Lie derivative of a scalar field $x \mapsto v(x) : \mathbb{R}^n \to \mathbb{R}$ along the vector field $f \in C^\infty(\mathbb{R}^p;\mathbb{R}^p)$. In particular, if $f : \mathbb{R}^n \to \mathbb{R}^n$ and $v : x \mapsto v(x) := \langle x, P_1 x\rangle$, then $(L_f v)(x) = \langle P_1 x, f(x)\rangle$. The reduced-order system is characterized by the prescribed triple $(\tilde{f}, \tilde{G}, \tilde{h})$ for which the following hypotheses are assumed to hold:

H2: (a) There exist $x \mapsto \tilde{f}_i(x) \in \mathbb{R}^n$, with $i = 1, 2, 3$, such that

(i) $\tilde{f}(x_1, x_2, x_3) = \tilde{f}_1(x_1) + \tilde{f}_2(x_2) + I(\tilde{f}_3(x_3), \tau)$;

(i) $$\|\tilde{f}(x_1, x_2, x_3)\| \leq v_0 + v_3 \sum_{i=1}^{m} I\big(|(L_{\tilde{g}_i} v)(x_3)|, \tau\big) + v_2 \sum_{i=1}^{m} |(L_{\tilde{g}_i} v)(x_2)| + v_1 \sum_{i=1}^{m} |(L_{\tilde{g}_i} v)(x_1)|,$$

where $\tilde{g}_i(x)$ denotes the ith column of the matrix $\tilde{G}(x)$;

(iii) given any symmetric, positive definite $K_i \in \mathbb{R}^{n \times n}$ $(i = 1,\ 2)$ and continuous, symmetric matrix-valued function $x \mapsto Q(x) \in \mathbb{R}^{n \times n}$, with $Q(x) \geq 0$ for all $x \in \mathbb{R}^n$ and, if it exists, $\lim_{\|x\| \to \infty} \inf Q(x) > 0$, there exists a unique symmetric matrix $P_1 > 0$ which satisfies, for all x, the following Riccati-type matrix equation:

$$\begin{aligned} P_1(\mathrm{D}\tilde{f}_1)(x) &+ (\mathrm{D}\tilde{f}_1)^{\mathrm{T}}(x) P_1 \\ &+ K_1 + (\mathrm{D}\tilde{f}_2)^{\mathrm{T}}(x) P_1 K_1^{-1} P_1 (\mathrm{D}\tilde{f}_2)(x) \\ &+ \tau\big(K_2 + (\mathrm{D}\tilde{f}_3)^{\mathrm{T}}(x) P_1 K_2^{-1} P_1 (\mathrm{D}\tilde{f}_3)(x)\big) \\ &\quad = -Q(x), \end{aligned} \tag{7}$$

where the notation $(\mathrm{D}\tilde{f})(x)$ denotes the Fréchet derivative of $\tilde{f}$ at x, i.e. the Jacobian matrix of $\tilde{f}$.

(b) $\tilde{G}(x_1, x_2, x_3)$ is independent of x_2 and x_3, i.e. $\tilde{G}(x_1, x_2, x_3) = \tilde{G}(x_1)$, and there exist a real constant $\kappa \in \mathbb{R}^+$, with $\mathbb{R}^+ := (0, \infty)$, such that, for all $x \in \mathbb{R}^n$ and all i,

$$\kappa |(L_{\tilde{g}_i} v)(x)| \geq \|P_1 x\| .$$

(c) There exist $p : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^m \to \mathbb{R}^n$ and $q : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^m \to \mathbb{R}^m$ such that

$$\begin{aligned} \tilde{h}(t, x_1, x_2, x_3, u) &= q(t, x_1, x_2, x_3, u) \\ &\quad + \tilde{G}(x_1) p(t, x_1, x_2, x_3, u), \end{aligned}$$

and there exist real constants $\alpha_i, \hat{\alpha}_i, \beta_i, \hat{\beta}_i, \gamma_i, \hat{\gamma}_i, \underline{\delta_i}, \hat{\delta}_i \in \mathbb{R}_0^+$, known continuous functions $\xi_i : \mathbb{R} \to [0, \overline{\xi}]$ and $\hat{\xi}_i : \mathbb{R} \to [0, \tilde{\xi}]$, with $\overline{\xi} + \kappa\tilde{\xi} \in [0, 1)$, such that, for all (t, x_1, x_2, x_3, u),

(i) $$|p_i(t, x_1, x_2, x_3, u)| \leq \alpha_i + \beta_i |(L_{\tilde{g}_i} v)(x_1)| + \gamma_i \|x_2\| + \delta_i \|x_3\| + \xi_i(t) |u_i|;$$

(ii) $$\|q(t, x_1, x_2, x_3, u)\| \leq \sum_{i=1}^{m} \Big[\hat{\alpha}_i + \hat{\beta}_i |(L_{\tilde{g}_i} v)(x_1)| + \hat{\gamma}_i \|x_2\| + \hat{\delta}_i \|x_3\| + \hat{\xi}_i(t) |u_i|\Big],$$

where u_i, q_i denote the ith components of u and q, respectively.

Remark: The vector fields $p,\ q$ are said to represent the *matched* and *unmatched* components, respectively, of the uncertainty in the nonlinear reduced-order time-delay system.

4. DESIGN OBJECTIVE AND CLASS OF FEEDBACK CONTROLS

It is desired that a memoryless feedback control function, $x(t) \mapsto \tilde{c}(x(t))$, be designed so that (a) the reduced-order system has the property of global uniform ultimate boundedness (see (Michel and Wang, 1995), Chapter 3 for a definition), (b) under additional hypotheses, the full-order system has the same stability property in the presence of singular perturbations for all $\mu \in (0, \mu^*)$, where μ^* is some computable real constant.

The design of the feedback controls emulates the work in (Corless, 1991) and (Binning and Goodall, 1999). The feedback controller $x \mapsto \tilde{c}(x) = [\tilde{c}_1(x) \ldots \tilde{c}_m(x)]^{\mathrm{T}}$ is defined by

$$\tilde{c}_i(x) := -(1 - \overline{\xi} - \kappa\tilde{\xi})^{-1} [a_i \chi_i(x) + b_i \omega_i(x)], \tag{8}$$

where
$x \mapsto \chi_i(x) := (L_{\tilde{g}_i} v)(x)$,
$x \mapsto \omega_i(x) := s_i\Big((1 - \overline{\xi} - \kappa\tilde{\xi})^{-1} b_i \chi_i(x)\Big)$,
$s_i : \mathbb{R} \to \mathbb{R}$ is any C^1 function which, for every $z = [z_1, \ldots, z_m]^{\mathrm{T}}$, satisfies

$$z_i s_i(z_i) \geq |z_i| - \varepsilon_i, \qquad |s_i(z_i)| \leq 1,$$

and $a_i,\ b_i,\ \varepsilon_i \in \mathbb{R}^+$ are design parameters. The functionals ω_i are chosen to counteract the uncertainty in the reduced-order system.
Remark: An example of a function satisfying the conditions on s_i is

$$s_i(z_i) = [|z_i| + \varepsilon_i]^{-1} z_i.$$

5. BOUNDARY-LAYER SYSTEM

Consider the change of variables $\Phi = y - \tilde{\phi}(x)$, where $\tilde{\phi}(x) = \phi(x(t), \tilde{c}(x))$. Introducing the rescaled time variable $\tau = (t - t^*)/\mu$, where $t^* \in \mathbb{R}$ is fixed, and $\tilde{\Phi}(\tau) := \Phi(t^* + \mu\tau)$, it can be shown that the behaviour of the full-order system at $\mu = 0$ is characterized by the system

$$\tilde{\Phi}'(\tau) = A(t^*) \tilde{\Phi}(\tau),$$

which is known as the *boundary-layer system*. Hypothesis H1(a) guarantees that the boundary-layer system is asymptotically stable.

6. THE FEEDBACK CONTROLLED REDUCED-ORDER SYSTEM

For any two compact sets $\mathscr{A},\ \mathscr{B} \subset \mathbb{R}^n$, the notation $\mathscr{A} \subsetneq \mathscr{B}$ is introduced to denote that the compact set $\mathscr{A}$ is contained in an open set which, in turn, is contained in the compact set $\mathscr{B}$. With state feedback $u(t) = \tilde{c}(x(t))$ and $\tilde{c}$ defined by (8), the reduced-order system (6) can be expressed as

$$\begin{aligned} \dot{x}(t) &= \tilde{f}_1(x(t)) + \tilde{f}_2(x_t(-\rho)) + I(\tilde{f}_3(x), \tau) \\ &+ \sum_{i=1}^{m} \Big[p_i(t, x(t), x_t(-\rho), I(x, \tau), \tilde{c}_i(x(t))) \\ &+ \tilde{c}_i(x(t))\Big] \tilde{g}_i(x(t)) \\ &+ q(t, x(t), x_t(-\rho), I(x, \tau), \tilde{c}_i(x(t))). \end{aligned} \tag{9}$$

The desired stability property is obtained by using essentially the same approach as that of (Binning and Goodall, 2000). Examining the behaviour of the time derivative of the functional

$$\begin{aligned}
x_t \mapsto v_1(x_t) &:= v(x(t)) \\
&+ \int_0^1 \int_{t-\rho}^{t} \langle x(r), R(zx(r))x(r)\rangle \, \mathrm{d}r\,\mathrm{d}z \\
&+ \int_0^1 \int_0^{\tau} \int_{t-s}^{t} \langle x(r), S(zx(r))x(r)\rangle \, \mathrm{d}r\,\mathrm{d}s\,\mathrm{d}z \\
&+ \sum_{i=1}^{m} \eta_i \int_{t-\rho}^{t} |(L_{\tilde{g}_i} v)(x(r))|^2 \, \mathrm{d}r \\
&+ \sum_{i=1}^{m} \zeta_i \int_0^{\tau} \int_{t-s}^{t} |(L_{\tilde{g}_i} v)(x(r))|^2 \, \mathrm{d}r\,\mathrm{d}s,
\end{aligned}$$

where $R(z) := (\mathrm{D}f_2)^{\mathrm{T}}(z) P_1 K_1^{-1} P_1 (\mathrm{D}f_2)(z)$, $S(z) := (\mathrm{D}f_3)^{\mathrm{T}}(z) P_1 K_2^{-1} P_1 (\mathrm{D}f_3)(z)$, $\eta_i = \kappa \left\|P_1^{-1}\right\| (\gamma_i + \kappa\hat{\gamma}_i)$, $\zeta_i = \kappa \left\|P_1^{-1}\right\| (\delta_i + \kappa\hat{\delta}_i)$, then, along solutions to (9), one obtains, using Leibnitz's formula for differentiation of integrals, the following result.

Theorem 1. Suppose hypotheses H1-H2 hold and a_i and b_i are chosen so that

$$\begin{aligned}
a_i > \hat{a}_i &:= \beta_i + \kappa \left\|P_1^{-1}\right\| (\gamma_i + \tau\delta_i) \\
&\quad + \kappa \left[\hat{\beta}_i + \kappa \left\|P_1^{-1}\right\| (\hat{\gamma}_i + \tau\hat{\delta}_i)\right],
\end{aligned}$$

$b_i > \hat{b}_i := \alpha_i + \kappa\hat{\alpha}_i$.

Then $\overline{a} := \min_i \left[a_i - \hat{a}_i\right] > 0$, $\overline{b} := \min_i \left[b_i - \hat{b}_i\right] > 0$, and the uncertain system (9) is globally uniformly ultimately bounded within every compact set $\mathscr{A}$ satisfying $\mathscr{E}_\varepsilon \subsetneq \mathscr{A}$, where $\mathscr{E}_\varepsilon$ is the compact set $\mathscr{E}_\varepsilon := \left\{x \in \mathbb{R}^n : v(x) \le r_\varepsilon^2\right\}$,

$$r_\varepsilon := \left[\frac{\varepsilon\kappa^2}{2m\overline{a}\sigma_{min}(P_1)}\right]^{\frac{1}{2}} \quad \text{and} \quad \varepsilon := 2\sum_{i=1}^{m} \varepsilon_i.$$

Proof. Since, along solutions to (9) and for almost all t,

$$\begin{aligned}
\dot{v}_1(x_t) \le &- \left\langle x(t), \int_0^1 Q(zx(t))x(t)\,\mathrm{d}z \right\rangle + \varepsilon \\
&- 2\overline{a}\sum_{i=1}^{m} |\chi_i(x(t))|^2 - 2\overline{b}\sum_{i=1}^{m} |\chi_i(x(t))|,
\end{aligned}$$

then, as a consequence of H2(c) and standard arguments, the result follows.

□

Remark: Clearly the 'size' of the set $\mathscr{E}_\varepsilon$ can be made as small as desired by choosing the design parameters ε_i appropriately.

7. LYAPUNOV ANALYSIS FOR THE FULL-ORDER SYSTEM

In this final stage it is shown that, using the memoryless feedback (8), the full-order uncertain system (1-2) is globally uniformly ultimately bounded with respect to some compact set. The methodology follows that of (Corless and Ryan, 1991) (see, also, (Binning and Goodall, 1999)) in which a Lyapunov analysis is used for delay-free systems. Consider the functional defined by

$$(x_t, y(t)) \mapsto v_\mu(x_t, y(t)) := v_1(x_t) + \xi_\mu v_2(x_t, y(t)),$$

where $\xi_\mu \in \mathbb{R}^+$, a real constant dependent upon the singular perturbation parameter, is to be specified,

$$\begin{aligned}
v_2(x_t, y(t)) := v_0(x(t), y(t)) &+ \sum_{i=1}^{m} \int_{t-\rho}^{t} |\chi_i(x(r))|^2 \mathrm{d}r \\
&+ \sum_{i=1}^{m} \int_0^{\tau} \int_{t-s}^{t} |\chi_i(x(r))|^2 \, \mathrm{d}r\,\mathrm{d}s
\end{aligned}$$

and $v_0(x,y) := \left\langle y - \tilde{\phi}(x), P_0(y - \tilde{\phi}(x))\right\rangle$. Along solutions to (1-2) and for almost all t,

$$\dot{v}_\mu(x_t, y(t)) = V_\mu(t, x_t, y(t)),$$

where

$$\begin{aligned}
V_\mu(t, x_t, y(t)) &:= \dot{v}_1(x_t) \\
&+ \xi_\mu \Big\{ \big\langle (\nabla_x v_0)(x(t), y(t)), \\
&\quad a(t, x(t), x_t(-\rho), I(x,\tau), y(t), \tilde{c}(x(t)), \mu) \big\rangle \\
&\quad + \mu^{-1} \big\langle (\nabla_y v_0)(x(t), y(t)), \\
&\quad b(t, x(t), x_t(-\rho), I(x,\tau), y(t), \tilde{c}(x(t)), \mu) \big\rangle \\
&\quad + \sum_{i=1}^{m} |\chi_i(x(t))|^2 - \sum_{i=1}^{m} |\chi_i(x(t-\rho))|^2 \\
&\quad + \tau\sum_{i=1}^{m} |\chi_i(x(t))|^2 - \sum_{i=1}^{m} \int_{t-\tau}^{t} |\chi_i(x(r))|^2 \, \mathrm{d}r \Big\}.
\end{aligned}$$

H3: (a) For all (x_1, x_2, x_3), there exists $k_1 \in \mathbb{R}^+$ such that $\|F_1(x_1, x_2, x_3)\| \le k_1$.

(b) There exists a continuous function $w : \mathbb{R}_0^+ \to [0, \varpi]$, with $\varpi > 0$ and $w(0) = 0$, such that

$$\begin{aligned}
&\|h_1(t, x_1, x_2, x_3, y, u, \mu) - \\
&\qquad h_1(t, x_1, x_2, x_3, y, u, 0)\| \le w(\mu).
\end{aligned}$$

(c) There exists a known scalar $k_2 \in \mathbb{R}_0^+$ such that, for all $y_1, y_2 \in \mathbb{R}^\ell$ and $(t, x_1, x_2, x_3, \mu)) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_0^+$,

$$\begin{aligned}
&\|h_1(t, x_1, x_2, x_3, y_1, \tilde{c}(x_1), \mu) - \\
&\quad h_1(t, x_1, x_2, x_3, y_2, \tilde{c}(x_1), \mu)\| \le k_2 \|y_1 - y_2\|.
\end{aligned}$$

Under hypotheses H2-H3, one may deduce the following lemma.

Lemma 2. Along solutions to (1-2),

$$\begin{aligned}
\dot{v}_1(x_t) \le &- \left\langle x(t), \int_0^1 Q(zx)x\,\mathrm{d}z \right\rangle + \varepsilon \\
&- 2\sum_{i=1}^{m} (a_i - \hat{a}_i) |\chi_i(x(t))|^2 \\
&+ 2\Big(\kappa_1 \{v_0(x,y)\}^{\frac{1}{2}} + \kappa m^{-1}\varpi - \overline{b}\Big) \sum_{i=1}^{m} |\chi_i(x(t))|,
\end{aligned}$$

where $\kappa_1 = \kappa(k_1 + k_2) / \left(m\{\sigma_{min}(P_0)\}^{\frac{1}{2}}\right)$, almost everywhere.

H4: (a) The Fréchet derivative of $\tilde{\phi}$ satisfies

$$\left\|(\mathrm{D}\tilde{\phi})(x)\right\|, \quad \left\|(\mathrm{D}\tilde{\phi})(x)\tilde{g}_i(x)\right\| \le k_3, \quad k_3 \in \mathbb{R}^+,$$

for all x and for all i;
(b) There exist known scalars $\lambda_1 - \lambda_5 \in \mathbb{R}_0^+$ such that, for all $\mu \in \mathbb{R}^+$ and $(t, x_1, x_2, x_3, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\ell$,

$$\left\|h_2(t,x_1,x_2,x_3,y,\tilde{\phi}(x),\mu)\right\| \le \mu\Big[\lambda_1\left\|y-\tilde{\phi}(x_1)\right\| + \lambda_2 + \lambda_3\|x_1\| + \lambda_4\|x_2\| + \lambda_5\|x_3\|\Big].$$

Define

$$\kappa_2 := 2m^{-1}\big(\lambda_1 + k_3(k_1+k_2)\big)\left\{\frac{\sigma_{\max}(P_0)}{\sigma_{\min}(P_0)}\right\}^{\frac{1}{2}} + \{\sigma_{\max}(P_0)\}\Big(k_3\nu_2 + \kappa\left\|P_1^{-1}\right\|\big(\lambda_4 m^{-1} + k_3\max_i[\gamma_i+\hat{\gamma}_i]\big)\Big)^2 + \tau\{\sigma_{\max}(P_0)\}\Big(k_3\nu_3 + \kappa\left\|P_1^{-1}\right\|\big(\lambda_5 m^{-1} + k_3\max_i[\delta_i+\hat{\delta}_i]\big)\Big)^2,$$

$$\kappa_3 := m^{-1}(1-2\kappa_0\|P_0\|)\{\sigma_{\max}(P_0)\}^{-1},$$

$$\kappa_4 := \{\sigma_{\max}(P_0)\}^{\frac{1}{2}}\Big(\lambda_3\kappa m^{-1}\left\|P_1^{-1}\right\| + k_3\nu_1 + k_3\max_i\big[\beta_i+\hat{\beta}_i+a_i(1+\overline{\xi}+\tilde{\xi})(1-\overline{\xi}\kappa\tilde{\xi})^{1}\big]\Big),$$

$$\kappa_5 := \{\sigma_{\max}(P_0)\}^{\frac{1}{2}}\Big[\lambda_2 + k_3\Big(\varpi + \nu_0 + \sum_{i=1}^m\big\{\alpha_i+\hat{\alpha}_i+b_i(1+\overline{\xi}+\tilde{\xi})(1-\overline{\xi}-\kappa\tilde{\xi})^{-1}\big\}\Big)\Big].$$

Lemma 3. Suppose H1-H4 hold. Along solutions to (1-2),

$$\dot{v}_2(x_t,y(t)) \le (1+\tau)\sum_{i=1}^m|\chi_i(x(t))|^2 + \sum_{i=1}^m\Big[\big(\kappa_2-\kappa_3\mu^{-1}\big)v_0(x(t),y(t)) + 2\kappa_4|\chi_i(x(t))|\{v_0(x(t),y(t))\}^{\frac{1}{2}}\Big] + 2\kappa_5\{v_0(x(t),y(t))\}^{\frac{1}{2}},$$

almost everywhere.

Hence, as a consequence of Lemmas 2 and 3,

$$\begin{aligned}V_\mu(t,x_t,y(t)) \le{}& -\left\langle x(t),\int_0^1 Q(zx)x\,\mathrm{d}z\right\rangle + \varepsilon \\ &- \sum_{i=1}^m\Big[(2a_i-2\hat{a}_i-\xi_\mu(1+\tau))|\chi_i(x(t))|^2 \\ &- 2(\kappa_1+\xi_\mu\kappa_4)|\chi_i(x(t))|\{v_0(x(t),y(t))\}^{\frac{1}{2}} \\ &+ \xi_\mu(\mu^{-1}\kappa_3-\kappa_2)v_0(x(t),y(t))\Big] \\ &- 2(\overline{b}-\kappa m^{-1}\varpi)\sum_{i=1}^m|\chi_i(x(t))| \\ &+ 2\xi_\mu\kappa_5\{v_0(x(t),y(t))\}^{\frac{1}{2}}.\end{aligned}$$

Let $\xi_\mu = \pi\left(\frac{\mu}{\mu^*}\right)^{\frac{1}{2}}$, where $\pi > 0$ and $\mu < \mu^*$, then, since $\xi_\mu < \pi$,

$2a_i - 2\hat{a}_i - \xi_\mu(1+\tau) > 2a_i - 2\hat{a}_i - \pi(1+\tau)$.
Therefore, design

$$a_i > \overline{a}_i := \hat{a}_i + \tfrac{1}{2}\pi(1+\tau) \qquad (10)$$

and let $\overline{a} = \min_i[a_i - \overline{a}_i]$, then

$$\begin{aligned}V_\mu(t,x_t,y(t)) \le{}& -\left\langle x(t),\int_0^1 Q(zx)x\,\mathrm{d}z\right\rangle + \varepsilon \\ &- \sum_{i=1}^m\langle\omega, M_\mu\omega\rangle - 2(\overline{b}-\kappa m^{-1}\varpi)\sum_{i=1}^m|\chi_i(x(t))| \\ &+ 2\xi_\mu\kappa_5\{v_0(x(t),y(t))\}^{\frac{1}{2}},\end{aligned}$$

where $\omega = \big[\,|\chi_i(x(t))|\ \{v_0(x(t),y(t))\}^{\frac{1}{2}}\big]^{\mathrm{T}}$ and

$$M_\mu = \begin{bmatrix}\overline{a} & -(\kappa_1+\xi_\mu\kappa_4) \\ -(\kappa_1+\xi_\mu\kappa_4) & \xi_\mu(\mu^{-1}\kappa_3-\kappa_2)\end{bmatrix}.$$

It can be shown that there exists $\mu^* > 0$ such that $\det(M_\mu) > 0$ for all $\mu \in (0,\ \mu^*)$. Selecting real $\phi > 0$, then, since $\kappa_3 > 0$ (in view of H1(a)),

$$\mu^* := \begin{cases}\overline{a}\kappa_3\left[\overline{a}\kappa_2 + 4\kappa_1\kappa_4\right]^{-1}, & \kappa_1\kappa_4 \ne 0 \\ \overline{a}\kappa_3\left[\overline{a}\kappa_2+\phi\right]^{-1}, & \text{otherwise.}\end{cases}$$

is one such possibility (see (Corless and Ryan, 1991)), where

$$\pi := \begin{cases}\kappa_1\kappa_4^{-1}, & \kappa_1\kappa_4 \ne 0 \\ \kappa_1^2\phi^{-1}, & \kappa_1 \ne 0,\ \kappa_4 = 0 \\ \phi\kappa_4^{-2}, & \kappa_1 = 0,\ \kappa_4 \ne 0 \\ \phi, & \kappa_1 = 0,\ \kappa_4 = 0.\end{cases}$$

Thus, for μ^* and π defined above and designing b_i so that

$$\overline{b} > m^{-1}\kappa\varpi, \qquad (11)$$

it follows that

$$V_\mu(t,x_t,y(t)) \le -\overline{\eta}_\mu\overline{v}_\mu(x(t),y(t)) + \overline{\zeta}_\mu\{\overline{v}_\mu(x(t),y(t))\}^{\frac{1}{2}} + \varepsilon,$$

where $\overline{v}_\mu(x,y) := v(x) + \xi_\mu v_0(x,y)$,
$\overline{\eta}_\mu := m\left\|M_\mu^{-1}\right\|^{-1}\min[\kappa^{-2}\sigma_{min}(P_1),\ \xi_\mu^{-1}]$
and $\overline{\zeta}_\mu := 2\xi_\mu^{\frac{1}{2}}\kappa_5$.
Since

$$\overline{v}_\mu(x_t,y(t)) = \left\langle\begin{bmatrix}x \\ y-\tilde{\phi}(x)\end{bmatrix}, \tilde{P}_\mu\begin{bmatrix}x \\ y-\tilde{\phi}(x)\end{bmatrix}\right\rangle,$$

where $\tilde{P}_\mu := \begin{bmatrix}P_1 & O \\ O & \xi_\mu P_0\end{bmatrix}$, standard arguments can be employed to show that there exists $\mu^* \in \mathbb{R}^+$ such that the full-order uncertain system has the desired behaviour, provided $\mu < \mu^*$.

Theorem 4. Suppose hypotheses H1-H4 are satisfied and $\mu \in (0,\mu^*)$ is fixed. If a_i and b_i are designed so that (10) and (11), respectively, hold, then the uncertain system (1)-(2), with $u(t) = \tilde{c}(x(t))$ (defined in (8)), is globally uniformly ultimately bounded within every compact set $\mathscr{A}$ satisfying $\mathscr{W}_\mu \subsetneq \mathscr{A}$, where $\mathscr{W}_\mu$ is the compact set $\mathscr{W}_\mu := \big\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^\ell : \overline{v}_\mu(x,y) \le r_\mu^2\big\}$ and

$$r_\mu := \frac{1}{2}\overline{\eta}_\mu^{-1}\left[\overline{\zeta}_\mu + \left\{4\overline{\eta}_\mu\varepsilon + \overline{\zeta}_\mu^2\right\}^{\frac{1}{2}}\right].$$

In fact, the asymptotic behaviour of the solution of the controlled full-order system tends to that of the controlled reduced-order system, as seen in the following corollary.

Corollary 5. For each $\mu \in (0, \mu^*)$, $x(\cdot)$ is ultimately bounded within every compact set $\mathscr{B}$, with $\mathscr{E}_\mu \subsetneq \mathscr{B}$ and $\mathscr{E}_\mu := \{x \in \mathbb{R}^n : v_1(x) \leq r_\mu^2\}$, on every solution $(x(\cdot), y(\cdot))$ of (1-2) and, in addition, the 'Hausdorff' distance, $d(\mathscr{E}_\mu, \mathscr{E}_\varepsilon)$, between $\mathscr{E}_\mu$ and $\mathscr{E}_\varepsilon$ satisfies

$$\lim_{\mu \downarrow 0} d(\mathscr{E}_\mu, \mathscr{E}_\varepsilon) = 0 \, .$$

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