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ON METHODS FOR GRADIENT ESTIMATION IN IFT FOR MIMO SYSTEMS

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Abstract: Iterative feedback tuning (IFT) is a model free control tuning method using closed loop experiments. For single-input single-output (SISO) systems only 2 or 3, depending on the controller structure, closed loop experiments are required. However for multivariable systems the number of experiments increases to a maximum of $1 + m \times p$, where $m \times p$ is the dimension of the controller. In this contribution several methods are proposed to reduce the experimental time by approximating the gradient of the cost function. The local convergence for a method which uses the same technique as in IFT for SISO systems is analyzed. It is shown that even if there are commutation errors due to the approximation method, the numerical optimization may still converge to the true optimum.

Keywords: Multivariable Non-model based control

1. INTRODUCTION

In the early 1990s the interaction between identification and control was brought into focus. The issue has been how to perform the identification such that the performance for the derived model-based controller improves. As a part of this research many iterative identification and control schemes, e.g. (Zang *et al.*, 1995), have been developed. These schemes iteratively perform plant identification and controller design in closed loop.

As a continuation and alternative to those methods, Iterative Feedback Tuning (IFT) was developed and first presented in (Hjalmarsson *et al.*, 1994). It is a model-free control optimization method and in (Hjalmarsson *et al.*, 1994) it is shown that for linear time-invariant (LTI) single-input single output (SISO) systems and certain signal-based control criteria, e.g. LQG, the controller optimization with respect to the control parameters can be done only using measurements from the plant in closed loop with the current controller.

During the last years IFT has been used for many applications, e.g. robust control of a simulation model of a flexible transmission system (Hjalmarsson *et al.*, 1995), the flexible arm of the Laboratiore d'Automatique de Grenoble (Ceysens and Codrons, 1997), vibration attenuation (Meurers and Veres, 1999). It has also been applied by the chemical multinational S.A. Solvay to tune PID controllers for temperature control in furnaces, in distillation columns, flow control in evaporators etc., see (Hjalmarsson *et al.*, 1997).

Many of the processes in these applications exhibits some kind of nonlinear behavior. Even though the original IFT scheme was developed for LTI systems it seems that the method performs well for those processes as well. The author of (Hjalmarsson, 1998) tries to explain why and when the original IFT algorithm also may work for nonlinear systems.

The idea of IFT is transformed into a nonlinear context in (Sjöberg and Agarwal, 1996) and (De Bruyne *et al.*, 1997), where it is shown that the gradients can be computed from experimental

data by performing additional experiments. The drawback is that the number of experiments with these methods are proportional to the number of parameters. To reduce the number of experiments identification-based methods to approximate the gradient are proposed in (De Bruyne *et al.*, 1997) and (Sjöberg and Agarwal, 1997). A hybrid version between the ideas of the original IFT and the model-based approximations is presented in (Sjöberg and Bruyne, 1999), where a model of the linearized closed loop system is introduced in order to compensate for the errors that occurs when the original IFT is used for nonlinear systems.

The IFT scheme for LTI multi-input multi-output (MIMO) systems is thoroughly treated in (Hjalmarsson, 1999). MIMO systems have the common problem with nonlinear systems that a large number of experiments is required to compute the gradient of the control cost. In this paper, we will discuss some methods to approximate the gradient in order to make the algorithm less time consuming. The analysis will be in a linear context although the use of IFT is more interesting in the setting with a nonlinear plant and a linear controller.

The paper is organized as follows. In Section 2 a short review of IFT is given. Section 3 presents some new and some well known ways of approximating the gradient needed for the optimization in the IFT scheme. Some preliminary analysis of the different methods are performed in Section 4 Simulation examples are given in Section 5 and Section 6 offers some conclusions.

2. IFT - ITERATIVE FEEDBACK TUNING

In this section a short review of iterative feedback tuning (IFT) is given. For further details we refer to (Hjalmarsson *et al.*, 1998).

The idea in IFT is to tune a controller with known structure only using experiments on the controlled system. The unknown system is assumed to be described by the following discrete time system,

$$\boldsymbol{y}_t = \boldsymbol{G}_0 \boldsymbol{u}_t + \boldsymbol{v}_t, \tag{1}$$

where $\boldsymbol{G}_0$ is assumed to be a linear time-invariant multivariable system, $\boldsymbol{y}_t \in \mathbf{R}^p$ is the output, $\boldsymbol{u}_t \in \mathbf{R}^m$ is the input and $\boldsymbol{v}_t \in \mathbf{R}^p$ is some stochastic disturbance. The sub index t denotes the discrete time instants. To simplify the presentation the control system has only one-degree of freedom, i.e.

$$\boldsymbol{u}_t(\rho) = \boldsymbol{C}(\boldsymbol{\rho})(\boldsymbol{r}_t - \boldsymbol{y}_t(\boldsymbol{\rho})), \tag{2}$$

where the controller $\boldsymbol{C}(\boldsymbol{\rho})$ is a $m \times p$ transfer function matrix parameterized by some parameter vector $\boldsymbol{\rho} \in \mathbf{R}^f$. The reference $\boldsymbol{r}_t$ is an external vector. Notice that signals originating from measurements on the closed loop system are functions of $\boldsymbol{\rho}$. To ease the notation the time argument will from now on be omitted. The control objective is

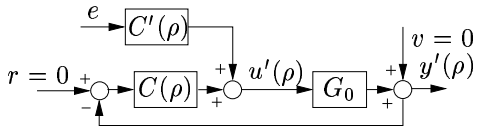

Fig. 1. Setup for exact gradient experiments

to minimize some design criterion. In IFT, almost any signal based criterion can be used. Here, for simplicity the following quadratic criterion will be considered:

$$J(\boldsymbol{\rho}) = \frac{1}{2N} E\left[\sum_{k=0}^{N} \tilde{\boldsymbol{y}}_k(\boldsymbol{\rho})^T \tilde{\boldsymbol{y}}_k(\boldsymbol{\rho})\right] \tag{3}$$

where $\tilde{\boldsymbol{y}}(\boldsymbol{\rho}) = \boldsymbol{y}(\boldsymbol{\rho}) - \boldsymbol{y}_d$ is the difference between the achieved output and the desired output $\boldsymbol{y}_d$. The expectation $E[\cdot]$ is w.r.t. the disturbance $\boldsymbol{v}_t$. The optimal controller parameterized by $\boldsymbol{\rho}^c$ is defined by

$$\boldsymbol{\rho}^c = \arg\min_{\boldsymbol{\rho}} J(\boldsymbol{\rho}), \tag{4}$$

which under convexity assumptions is the same as to find the solution to

$$0 = \frac{\partial J(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}} = \frac{1}{N} E\left[\sum_{k=1}^{N} \left\{\frac{\partial \boldsymbol{y}_k(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}\right\}^T \tilde{\boldsymbol{y}}_k(\boldsymbol{\rho})\right]. \tag{5}$$

With computed gradients the solution can be obtained by gradient based methods, e.g. the Gauss-Newton search algorithm:

$$\boldsymbol{\rho}^{j+1} = \boldsymbol{\rho}^j - \gamma_j \boldsymbol{R}_j^{-1} \frac{\partial J(\boldsymbol{\rho}^j)}{\partial \boldsymbol{\rho}}, \tag{6}$$

where $\boldsymbol{R}_j$ is an approximation of the Hessian of $J(\boldsymbol{\rho})$ and γ_j is the adjustable step-length. The key contribution in the IFT algorithm (Hjalmarsson *et al.*, 1994) is that an unbiased gradient of $\frac{\partial J(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}$ can be obtained by performing experiments on the plant in feedback with the controller.

With the achieved sensitivity function and the complementary sensitivity function, respectively defined by, $\boldsymbol{S}_0(\boldsymbol{\rho}) = [\boldsymbol{I} + \boldsymbol{G}_0\boldsymbol{C}(\boldsymbol{\rho})]^{-1}$ and $\boldsymbol{T}_0(\boldsymbol{\rho}) = \boldsymbol{S}_0(\boldsymbol{\rho})\boldsymbol{G}_0\boldsymbol{C}(\boldsymbol{\rho})$, the expression for the output $\boldsymbol{y}(\boldsymbol{\rho})$ from the system (1) in feedback with the controller (2) is

$$\boldsymbol{y}(\boldsymbol{\rho}) = \boldsymbol{T}_0(\boldsymbol{\rho})\boldsymbol{r} + \boldsymbol{S}_0(\boldsymbol{\rho})\boldsymbol{v}. \tag{7}$$

The gradient of $\boldsymbol{y}(\boldsymbol{\rho})$ w.r.t. the ith entry of $\boldsymbol{\rho}$, denoted by $\boldsymbol{y}'(\boldsymbol{\rho})$, is then

$$\boldsymbol{y}'(\boldsymbol{\rho}) = \boldsymbol{S}_0(\boldsymbol{\rho})\boldsymbol{G}_0\boldsymbol{C}'(\boldsymbol{\rho})(\boldsymbol{r} - \boldsymbol{y}(\boldsymbol{\rho})) \tag{8}$$

where $\boldsymbol{C}'(\boldsymbol{\rho}) = \frac{\partial \boldsymbol{C}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}_i}$ and $\boldsymbol{y}(\boldsymbol{\rho})$ is the output collected from the closed loop system operating under normal operating conditions. Defining the control error as $\boldsymbol{e}(\boldsymbol{\rho}) = \boldsymbol{r} - \boldsymbol{y}(\boldsymbol{\rho})$, then ideally the gradient of $\boldsymbol{y}(\boldsymbol{\rho})$ is obtained running the closed loop experiment shown in Fig. 1. In practice a perturbed estimate $\hat{\boldsymbol{y}}'(\boldsymbol{\rho}) = \boldsymbol{y}'(\boldsymbol{\rho}) + \boldsymbol{S}_0(\boldsymbol{\rho})\boldsymbol{v}$ is obtained due to the non-zero disturbance. It can be shown (Hjalmarsson *et al.*, 1994) that an unbiased estimate of $\partial\hat{J}(\boldsymbol{\rho})/\partial\boldsymbol{\rho}$, i.e. $E[\partial\hat{J}(\boldsymbol{\rho})/\partial\boldsymbol{\rho}] =$

$\partial J(\boldsymbol{\rho})/\partial \boldsymbol{\rho}$ is obtained if $\boldsymbol{v}$ is a stochastic stationary signal with zero mean. With this setup one has to perform one gradient experiment for each parameter in the vector $\boldsymbol{\rho}$. For SISO systems (8) can be rewritten as

$$\boldsymbol{y}'(\boldsymbol{\rho}) = \boldsymbol{C}(\boldsymbol{\rho})^{-1}\boldsymbol{C}'(\boldsymbol{\rho})\boldsymbol{S}_0(\boldsymbol{\rho})\boldsymbol{G}_0\boldsymbol{C}(\boldsymbol{\rho})\boldsymbol{e}(\boldsymbol{\rho}) \quad (9)$$

since the operators commute. Thus, to obtain the gradient signal $\frac{\partial \boldsymbol{y}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}$ only two experiments are needed independent of the number of parameters. The first one collects $\boldsymbol{y}(\boldsymbol{\rho})$ under normal operating conditions and in the second one the control error $\boldsymbol{e}(\boldsymbol{\rho})$ is fed as the reference and the output of this experiment is then filtered through $\boldsymbol{C}(\boldsymbol{\rho})^{-1}\boldsymbol{C}'(\boldsymbol{\rho})$, which is done off-line since $\boldsymbol{C}(\boldsymbol{\rho})$ is a known function of $\boldsymbol{\rho}$.

For MIMO systems the operators in (8) typically not commute. However, it is shown in (Hjalmarsson, 1999) that the maximum number of required gradient experiments is $m \times p$, i.e. equal to the dimension of the controller. Despite this reduction, the experiment may be prohibitive long from a practical point of view since the experiments are performed on the true plant and possibly disturbing the normal operational conditions. Hence it is of great interest to further reduce the number of experiments. It is the objective of this paper to discuss some options that exists for doing this.

3. GRADIENT APPROXIMATIONS

One way to reduce the number of experiments further is to approximate the signal $\frac{\partial \boldsymbol{y}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}$. There is a large theory of inexactness in optimization, contributions related to IFT are e.g. (Bruyne and Carrette, 1997) for linear systems and (Sjöberg and Bruyne, 1999) for nonlinear systems. In the analysis of the original IFT scheme for nonlinear systems (Hjalmarsson, 1998) the author make a remark, that in practice for many systems, IFT seems to be robust w.r.t. the gradient estimate. The most important property of the estimate is that it is a descent direction.

However, for convergence, it is more important that the quality of the estimate is good in a vicinity of the optimum than in the surroundings. We will here suggest some methods to approximate $\frac{\partial \boldsymbol{y}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}$ for MIMO systems.

Rewrite (8) as

$$\begin{aligned}\frac{\partial \boldsymbol{y}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}_i} &= \boldsymbol{S}_0(\boldsymbol{\rho})\boldsymbol{G}_0\boldsymbol{C}(\boldsymbol{\rho})\boldsymbol{C}(\boldsymbol{\rho})^{-1}\boldsymbol{C}'(\boldsymbol{\rho})\boldsymbol{e}(\boldsymbol{\rho}) \\ &= \boldsymbol{T}_0(\boldsymbol{\rho})\underbrace{\boldsymbol{C}(\boldsymbol{\rho})^{-1}\boldsymbol{C}'(\boldsymbol{\rho})}_{\boldsymbol{A}_i(\boldsymbol{\rho})}\boldsymbol{e}(\boldsymbol{\rho}) \qquad (10)\end{aligned}$$

The following approximation techniques are considered:

(1) $\frac{\partial \hat{\boldsymbol{y}}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}_i} = \hat{\boldsymbol{T}}_0(\boldsymbol{\rho})\boldsymbol{A}_i(\boldsymbol{\rho})\boldsymbol{e}(\boldsymbol{\rho})$
Here $\hat{\boldsymbol{T}}_0(\boldsymbol{\rho})$ is an identified model of the closed loop system. This idea was presented in (Bruyne and Carrette, 1997). The identification is assumed to be simplified by the assumption that the closed loop system is typically of low order and that nonlinear effects in $\boldsymbol{G}_0$ is reduced by the feedback. Some of the drawbacks is that the method relies on the identified model, possibly extra signals need to be injected to excite the system during identification and it requires more knowledge of the user to perform the identification properly.

(2) $\frac{\partial \hat{\boldsymbol{y}}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}_i} = \boldsymbol{T}_d\boldsymbol{A}_i(\boldsymbol{\rho})\boldsymbol{e}(\boldsymbol{\rho})$
Here $\boldsymbol{T}_d$ is the reference model defined by $\boldsymbol{y}_d = \boldsymbol{T}_d\boldsymbol{r}$. This method is a naive alternative to the previous presented. When $\boldsymbol{T}_0(\boldsymbol{\rho})$ is far from $\boldsymbol{T}_d$ we cannot expect the approximation to be good. However, it has the nice property that when $\boldsymbol{T}_0(\boldsymbol{\rho})$ tends to $\boldsymbol{T}_d$, the approximation error decreases.

(3) $\frac{\partial \hat{\boldsymbol{y}}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}_i} = \boldsymbol{A}_i(\boldsymbol{\rho})\boldsymbol{T}_0(\boldsymbol{\rho})\boldsymbol{e}(\boldsymbol{\rho})$
This is the same approach as was used for SISO systems. For MIMO there is almost always an error due to the commutation error between $\boldsymbol{A}_i(\boldsymbol{\rho})$ and $\boldsymbol{T}_0(\boldsymbol{\rho})$. The motivation for using this method is that if $\boldsymbol{T}_d\boldsymbol{A}_i(\boldsymbol{\rho}) = \boldsymbol{A}_i(\boldsymbol{\rho})\boldsymbol{T}_d$, then when $\boldsymbol{T}_0(\boldsymbol{\rho})$ tends to $\boldsymbol{T}_d$ the commutation error decreases.

(4) $\frac{\partial \hat{\boldsymbol{y}}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}_i} = \boldsymbol{T}_d\boldsymbol{A}_i(\boldsymbol{\rho})\boldsymbol{T}_0(\boldsymbol{\rho})\boldsymbol{T}_d^{-1}\boldsymbol{e}(\boldsymbol{\rho})$
When $\boldsymbol{T}_d\boldsymbol{A}_i(\boldsymbol{\rho}) \neq \boldsymbol{A}_i(\boldsymbol{\rho})\boldsymbol{T}_d$ this is an alternative method. When $\boldsymbol{T}_0(\boldsymbol{\rho})$ tends to $\boldsymbol{T}_d$, $\boldsymbol{T}_0(\boldsymbol{\rho})\boldsymbol{T}_d^{-1}$ tends to the identity which, obviously, commutes with all matrices. An implementation issue is that $\boldsymbol{T}_d^{-1}$ in many cases is non-causal. Then the non-causal filtering $\boldsymbol{T}_d^{-1}\boldsymbol{e}(\boldsymbol{\rho})$ can be performed off-line. An alternative is to make an all-pass approximation of $\boldsymbol{T}_d$.

(5) $\frac{\partial \hat{\boldsymbol{y}}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}_i} = \hat{\boldsymbol{T}}_0(\boldsymbol{\rho})\boldsymbol{A}_i(\boldsymbol{\rho})\boldsymbol{T}_0(\boldsymbol{\rho})\hat{\boldsymbol{T}}_0(\boldsymbol{\rho})^{-1}\boldsymbol{e}(\boldsymbol{\rho})$
This method is inspired by the previous one. The difference between this method and the first one is that the control error is filtered through $\boldsymbol{T}_0(\boldsymbol{\rho})\hat{\boldsymbol{T}}_0(\boldsymbol{\rho})^{-1}$, which also means that this method needs one more online experiment than the first one.

The list of different gradient approximations can be extended much further, e.g. we can think of estimating $\hat{\boldsymbol{T}}_0(\boldsymbol{\rho})$ based on an identified model $\hat{\boldsymbol{G}}_0$ of the plant. This approach is used in (Trulsson and Ljung, 1985). The main drawback is that the plant might be both nonlinear and of high order which complicates the identification and the following control design.

Notice that there is a difference in the number of experiments between the methods presented above. In every method at least one online experiment is needed to generate $\boldsymbol{e}(\boldsymbol{\rho})$. The last three also include a second online experiment where the

signal $\boldsymbol{e}(\boldsymbol{\rho})$, $\boldsymbol{T}_d^{-1}\boldsymbol{e}(\boldsymbol{\rho})$ or $\hat{\boldsymbol{T}}_0(\boldsymbol{\rho})^{-1}\boldsymbol{e}(\boldsymbol{\rho})$, depending on method, is filtered through the closed loop system $\boldsymbol{T}_0(\boldsymbol{\rho})$. Method 1 and 5, possibly also need some extra online experiments to carry out the identification of $\hat{\boldsymbol{T}}_0(\boldsymbol{\rho})$. Finally, the gradient signal $\frac{\partial \hat{\boldsymbol{y}}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}$ is obtained by off-line filtering through $\boldsymbol{A}_i(\boldsymbol{\rho})$, $\boldsymbol{T}_d\boldsymbol{A}_i(\boldsymbol{\rho})$ or $\hat{\boldsymbol{T}}_0(\boldsymbol{\rho})\boldsymbol{A}_i(\boldsymbol{\rho})$ for each element i in $\boldsymbol{\rho}$. This can be compared with $1+m\times p$ online experiments for the true gradient.

4. ANALYSIS OF LOCAL CONVERGENCE

The local convergence for the gradient approximation method 3. introduced in the previous section will be studied. In order to focus on the essence of the problem we will throughout the rest of the paper use the assumption that the noise is zero, which gives $\boldsymbol{y}(\boldsymbol{\rho}) = \boldsymbol{T}_0(\boldsymbol{\rho})\boldsymbol{r}$. The purpose of the analysis is to provide some insight in what the problems might be with the proposed gradient approximation method. The analysis will be based on the so-called ODE analysis (Ljung, 1977) which relate the evolution of an iterative algorithm like

$$\boldsymbol{\rho}^{j+1} = \boldsymbol{\rho}^j - \gamma_j \frac{\partial \hat{J}(\boldsymbol{\rho}^j)}{\partial \boldsymbol{\rho}} \tag{11}$$

to the trajectories of a differential equation. The corresponding ODE to (11) is

$$d\boldsymbol{\rho}/dt = -\frac{\partial \hat{J}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}. \tag{12}$$

The idea is that when the step size γ_j tends to zero the numerical iteration method will asymptotically behave as the corresponding ODE. Consider the approximation of the gradient

$$\begin{aligned}\frac{\partial \hat{J}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}} &= E\left[\frac{\partial \hat{\boldsymbol{y}}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}^T \tilde{\boldsymbol{y}}(\boldsymbol{\rho})\right] \\ &= E\left[\frac{\partial \hat{\boldsymbol{y}}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}^T (\boldsymbol{T}_0(\boldsymbol{\rho}) - \boldsymbol{T}_d)\boldsymbol{r}\right].\end{aligned} \tag{13}$$

Notice that if there exists a parameter $\boldsymbol{\rho}^c$ such that $\boldsymbol{T}_0(\boldsymbol{\rho}^c) = \boldsymbol{T}_d$ then this is a stationary point for the design criterion and furthermore $\frac{\partial \hat{J}(\boldsymbol{\rho}^c)}{\partial \boldsymbol{\rho}} = \frac{\partial J(\boldsymbol{\rho}^c)}{\partial \boldsymbol{\rho}}$. The question is whether this is a stable stationary point or not. A sufficient condition for the ODE (12) to be locally stable is that the linearized system

$$d\boldsymbol{\rho}/dt = -\frac{\partial}{\partial \boldsymbol{\rho}}\frac{\partial \hat{J}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}\boldsymbol{\rho} \tag{14}$$

is stable at the stationary point, i.e. the eigenvalues of $\frac{\partial}{\partial \boldsymbol{\rho}}\frac{\partial \hat{J}(\boldsymbol{\rho}^c)}{\partial \boldsymbol{\rho}}$ must have positive real parts.

Introduce the general linear reference model

$$\boldsymbol{T}_d(e^{i\omega}) = \begin{bmatrix} T_{11}(e^{i\omega}) & \dots & T_{1n}(e^{i\omega}) \\ \vdots & \ddots & \vdots \\ T_{n1}(e^{i\omega}) & \dots & T_{nn}(e^{i\omega}) \end{bmatrix} \tag{15}$$

and the linear controller

$$\boldsymbol{C}(\boldsymbol{\rho}, e^{i\omega}) = \begin{bmatrix} C_{11}(\boldsymbol{\rho}_{11}, e^{i\omega}) & \dots & C_{1n}(\boldsymbol{\rho}_{1n}, e^{i\omega}) \\ \vdots & \ddots & \vdots \\ C_{n1}(\boldsymbol{\rho}_{n1}, e^{i\omega}) & \dots & C_{nn}(\boldsymbol{\rho}_{nn}, e^{i\omega}) \end{bmatrix} \tag{16}$$

where

$$C_{ij}(\boldsymbol{\rho}_{ij}, e^{i\omega}) = \boldsymbol{\rho}_{ij}^T \boldsymbol{\Gamma}(e^{i\omega}), \tag{17}$$

$\boldsymbol{\rho}_{ij} = [\, \rho_{ij0}\ \rho_{ij1}\ \dots\ \rho_{ijm}\,]^T$ and $\boldsymbol{\Gamma}(e^{i\omega}) = [\, 1\ e^{-i\omega}\ \dots\ e^{-im\omega}\,]^T$. Without loss of generality, the complexity of each element in (16), m, is assumed to be equal for all elements. The controller parameters are collected in the vector $\boldsymbol{\rho} = [\, \boldsymbol{\rho}_{11}^T\ \dots\ \boldsymbol{\rho}_{n1}^T\ \boldsymbol{\rho}_{21}^T\ \dots\ \boldsymbol{\rho}_{nn}^T\,]^T$, i.e. each element in (16) is assumed to be individually parameterized. Let $\boldsymbol{\Phi}_{\boldsymbol{e}}(\omega)$ denote the spectrum of $\boldsymbol{e}(\boldsymbol{\rho}^c)$. Furthermore let $\bar{\boldsymbol{A}}$ and $\boldsymbol{A}^*$ denote the conjugate and the conjugate transpose of $\boldsymbol{A}$, respectively. From now on, the frequency argument will be omitted.

Theorem 1. Assume there exists a parameter vector $\boldsymbol{\rho} = \boldsymbol{\rho}^c$ such that $\boldsymbol{T}_0(\boldsymbol{\rho}^c) = \boldsymbol{T}_d$ where $\boldsymbol{T}_d$ is defined by (15). If $\boldsymbol{\Phi}_{\boldsymbol{e}}(\omega) = \sigma \boldsymbol{I}$ and if the controller is defined by (16) then $\boldsymbol{\rho} = \boldsymbol{\rho}^c$ is a stable stationary point to the ODE (12) using the gradient approximation $\frac{\partial \hat{\boldsymbol{y}}(\boldsymbol{\rho})}{\partial \rho_{jkl}} = \boldsymbol{C}^{-1}(\boldsymbol{\rho})\frac{\partial \boldsymbol{C}(\boldsymbol{\rho})}{\partial \rho_{jkl}}\boldsymbol{T}_0(\boldsymbol{\rho})\boldsymbol{e}(\boldsymbol{\rho})$ if the matrix

$$\boldsymbol{T}_d \otimes \boldsymbol{T}_d^* + \boldsymbol{T}_d^* \otimes \boldsymbol{T}_d > 0 \tag{18}$$

for all $\omega \in [\, -\pi,\ \pi\,]$. Here $\otimes$ denotes the Kronecker product.

Proof: A sufficient condition that the real parts of the eigenvalues of $\frac{\partial}{\partial \boldsymbol{\rho}}\frac{\partial \hat{J}(\boldsymbol{\rho}^c)}{\partial \boldsymbol{\rho}}$ are positive is that

$$\hat{\boldsymbol{J}}_{pp}(\boldsymbol{\rho}^c) = \frac{\partial}{\partial \boldsymbol{\rho}}\frac{\partial \hat{J}(\boldsymbol{\rho}^c)}{\partial \boldsymbol{\rho}} + \frac{\partial}{\partial \boldsymbol{\rho}}\frac{\partial \hat{J}(\boldsymbol{\rho}^c)}{\partial \boldsymbol{\rho}}^T \tag{19}$$

is positive definite. Introduce

$$\boldsymbol{A}_{jkl}(\boldsymbol{\rho}) = \boldsymbol{C}^{-1}(\boldsymbol{\rho})\frac{\partial \boldsymbol{C}(\boldsymbol{\rho})}{\partial \rho_{jkl}} \tag{20}$$

and let $\boldsymbol{C}^{-1} = [\, \boldsymbol{a}_1\ \dots\ \boldsymbol{a}_n\,]$ where $\boldsymbol{a}_i$ is a column vector. Since each element of (16) is individually parameterized, $\boldsymbol{A}_{jkl}$ will be of rank one and thus can be expressed as

$$\boldsymbol{A}_{jkl} = e^{-il\omega}\boldsymbol{a}_j \boldsymbol{e}_k^T \tag{21}$$

where $\boldsymbol{e}_k$ is the unit vector with the kth element equal to one. Using those expressions, an arbitrary element of $\frac{\partial}{\partial \boldsymbol{\rho}}\frac{\partial \hat{J}(\boldsymbol{\rho}^c)}{\partial \boldsymbol{\rho}}$ can be expressed as

$$\begin{aligned}&\frac{\partial}{\partial \rho_{jkl}} E\left[\frac{\partial \hat{\boldsymbol{y}}(\boldsymbol{\rho})}{\partial \rho_{rst}}^T (\boldsymbol{T}_0(\boldsymbol{\rho}) - \boldsymbol{T}_d)\boldsymbol{r}\right]\Bigg|_{\boldsymbol{\rho}=\boldsymbol{\rho}^c} \\ &= E\left[(\boldsymbol{A}_{rst}(\boldsymbol{\rho}^c)\boldsymbol{T}_d\boldsymbol{e}(\boldsymbol{\rho}^c))^T \boldsymbol{T}_d \boldsymbol{A}_{jkl}(\boldsymbol{\rho}^c)\boldsymbol{e}(\boldsymbol{\rho}^c)\right] \\ &= \frac{1}{2\pi}\,\mathrm{Tr}\int_{-\pi}^{\pi} \boldsymbol{T}_d\boldsymbol{A}_{jkl}(\boldsymbol{\rho}^c)\boldsymbol{\Phi}_{\boldsymbol{e}}(\omega)(\boldsymbol{A}_{rst}(\boldsymbol{\rho}^c)\boldsymbol{T}_d)^* d\omega,\end{aligned} \tag{22}$$

Assuming $\boldsymbol{\Phi_e}(\omega) = \sigma \boldsymbol{I}$, which means that that the reference is low-pass filtered white noise and skipping all arguments,

$$\begin{aligned} &\mathrm{Tr}\, \boldsymbol{T}_d \boldsymbol{A}_{jkl} (\boldsymbol{A}_{rst} \boldsymbol{T}_d)^* \\ &= e^{-i(l-t)\omega}\, \mathrm{Tr}(\boldsymbol{T}_d \boldsymbol{a}_j \boldsymbol{e}_k^T \boldsymbol{T}_d^* \boldsymbol{e}_s \boldsymbol{a}_r^*) \\ &= e^{-i(l-t)\omega} \boldsymbol{e}_k^T \boldsymbol{T}_d^* \boldsymbol{e}_s \boldsymbol{a}_r^* \boldsymbol{T}_d \boldsymbol{a}_j \\ &= e^{-i(l-t)\omega} \bar{T}_{ks} \boldsymbol{a}_r^* \boldsymbol{T}_d \boldsymbol{a}_j. \end{aligned} \tag{23}$$

Using the definition (19) and (23), we obtain

$$\hat{\boldsymbol{J}}_{pp}(\boldsymbol{\rho}^c) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \boldsymbol{M}_J d\omega \tag{24}$$

where $\boldsymbol{M}_J$ is the factorization

$$\boldsymbol{M}_J = \boldsymbol{\Gamma}_D \underbrace{\boldsymbol{\Xi}^* (\boldsymbol{T}_d \otimes \boldsymbol{T}_d^* + \boldsymbol{T}_d^* \otimes \boldsymbol{T}_d) \boldsymbol{\Xi}}_{\boldsymbol{H}} \boldsymbol{\Gamma}_D^* \tag{25}$$

with $\boldsymbol{\Gamma}_D$ and $\boldsymbol{\Xi}$ being block diagonal matrices with each diagonal block element equal to $\boldsymbol{\Gamma}$ and $\boldsymbol{C}^{-1}$, respectively. If $\boldsymbol{x} \in \mathbb{R}^{mn^2}$ then

$$\boldsymbol{x}^T \hat{\boldsymbol{J}}_{pp} \boldsymbol{x} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \boldsymbol{F}(\mathrm{e}^{i\omega}) \boldsymbol{H}(\omega) \boldsymbol{F}(\mathrm{e}^{i\omega})^* d\omega \tag{26}$$

where $\boldsymbol{F}(\mathrm{e}^{i\omega}) = \boldsymbol{x}^T \boldsymbol{\Gamma}_D$ is a multiple input single output filter and thus $\boldsymbol{x}^T \hat{\boldsymbol{J}}_{pp} \boldsymbol{x} \geq 0$ with equality if and only if $\boldsymbol{x} = 0$. This holds under the assumption that $\boldsymbol{H} > 0$ for all $\omega \in [-\pi, \pi]$. Furthermore $\boldsymbol{H} > 0$ if and only if $\boldsymbol{T}_d \otimes \boldsymbol{T}_d^* + \boldsymbol{T}_d^* \otimes \boldsymbol{T}_d > 0$ since $\boldsymbol{\Xi}$ is quadratic and has full rank for all $\omega \in [-\pi, \pi]$. This concludes the proof. □

This result shows that even if the operators $\boldsymbol{C}^{-1}\boldsymbol{C}'$ and $\boldsymbol{T}_0$ do not commute, the descent method (11) might still be locally convergent using this approximation method. Although, some caution has to be taken as will be illustrated in the next section. If the controller is diagonal, the structure can be exploited to make the condition in Thm. 1 less conservative.

Theorem 2. Assume there exists a parameter vector $\boldsymbol{\rho} = \boldsymbol{\rho}^c$ such that $\boldsymbol{T}_0(\boldsymbol{\rho}^c) = \boldsymbol{T}_d$ where $\boldsymbol{T}_d$ is defined by (15). If $\boldsymbol{\Phi_e}(\omega) = \sigma \boldsymbol{I}$ and if the controller is defined by the diagonal of (16) then $\boldsymbol{\rho} = \boldsymbol{\rho}^c$ is a stable stationary point to the ODE (12) using the gradient approximation $\frac{\partial \hat{\boldsymbol{y}}(\boldsymbol{\rho})}{\partial \rho_{jkl}} = \boldsymbol{C}^{-1}(\boldsymbol{\rho}) \frac{\partial \boldsymbol{C}(\boldsymbol{\rho})}{\partial \rho_{jkl}} \boldsymbol{T}_0(\boldsymbol{\rho}) \boldsymbol{e}(\boldsymbol{\rho})$ if

$$\boldsymbol{T}_d \odot \bar{\boldsymbol{T}}_d + \boldsymbol{T}_d^T \odot \boldsymbol{T}_d^* \tag{27}$$

is positive definite for all $\omega \in [-\pi, \pi]$. Here $\odot$ denotes the Hadamard product.

Proof: The outline follows the proof of Thm. 1. Using the diagonal structure of the controller, (23) becomes

$$\begin{aligned} &tr \boldsymbol{T}_d \boldsymbol{A}_{jjt} (\boldsymbol{A}_{rrl} \boldsymbol{T}_d)^* \\ &= e^{-i(l-t)\omega} T_{jr} \boldsymbol{a}_r^* \boldsymbol{T}_d \boldsymbol{a}_j \\ &= e^{-i(l-t)\omega} C_{jj}^{-1} \bar{C}_{rr}^{-1} |T_{jr}|^2 \end{aligned} \tag{28}$$

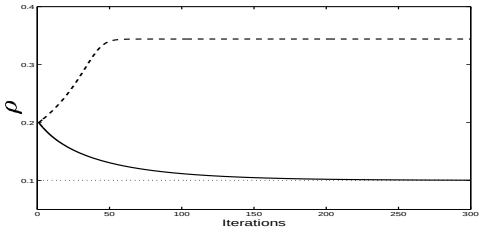

Fig. 2. Solid line-exact method, dashed line-approximation method 3, dotted horizontal line-optimal ρ.

and again $\hat{\boldsymbol{J}}_{pp}(\boldsymbol{\rho}^c)$ can be written as (24), but now

$$\boldsymbol{M}_J = \boldsymbol{\Gamma}_D \bar{\boldsymbol{C}}^{-1} (\boldsymbol{T}_d \odot \bar{\boldsymbol{T}}_d + \boldsymbol{T}_d^T \odot \boldsymbol{T}_d^*) \boldsymbol{C}^{-1} \boldsymbol{\Gamma}_D^*. \tag{29}$$

Then $\hat{\boldsymbol{J}}_{pp}(\boldsymbol{\rho}^c) > 0$ if $\boldsymbol{T}_d \odot \bar{\boldsymbol{T}}_d + \boldsymbol{T}_d^T \odot \boldsymbol{T}_d^* > 0$ for all $\omega \in [-\pi, \pi]$. This follows from the factorization (29) and the proof of Thm. 1. □

Remark: It is easy to realize that if the complexity of the controller determined by m tends to infinity, the conditions $\boldsymbol{T}_d \otimes \boldsymbol{T}_d^* + \boldsymbol{T}_d^* \otimes \boldsymbol{T}_d > 0$ and $\boldsymbol{T}_d \odot \bar{\boldsymbol{T}}_d + \boldsymbol{T}_d^T \odot \boldsymbol{T}_d^* > 0$, respectively, for all $\omega \in [-\pi, \pi]$, becomes both a necessary and sufficient condition for $\hat{\boldsymbol{J}}_{pp}(\boldsymbol{\rho}^c)$ being positive definite. When $m \to \infty$, then $\boldsymbol{x}^T \boldsymbol{\Gamma}_D$ can be any arbitrary filter. Particularly, if $\boldsymbol{x}$ is chosen such that $\boldsymbol{x}^T \boldsymbol{\Gamma}_D = \delta(\omega - \omega_0) \boldsymbol{F}(\mathrm{e}^{i\omega})$, the integral in (26) will collapse. Here δ is the Dirac delta operator. Since ω_0 is arbitrary, the conditions must hold for all $\omega \in [-\pi, \pi]$.

5. NUMERICAL ILLUSTRATION

In this simulation example we will use the approximation method 3, i.e. $\frac{\partial \hat{\boldsymbol{y}}(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}_i} = \boldsymbol{A}_i(\boldsymbol{\rho}) \boldsymbol{T}_0(\boldsymbol{\rho}) \boldsymbol{e}(\boldsymbol{\rho})$. The process to be controlled is the following two by two system

$$\boldsymbol{G}_0 = \begin{bmatrix} \dfrac{-2.25}{z-1} & \dfrac{2.25}{z-1} \\ \dfrac{-2.5z+3}{z^2-1.4z+0.4} & \dfrac{0.5z-0.6}{z^2-1.4z+0.4} \end{bmatrix} \tag{30}$$

which has a non-minimum phase zero in 1.2. The reference model is defined by

$$\boldsymbol{T}_{d1} = \begin{bmatrix} \dfrac{0.9}{z-0.1} & 0 \\ 0 & \dfrac{-0.2z+0.24}{z^2-1.6z+0.64} \end{bmatrix}. \tag{31}$$

The system is controlled with the P-controller $\boldsymbol{C}(\boldsymbol{\rho}) = \begin{bmatrix} \rho & 0.1 \\ 0.5 & 0.1 \end{bmatrix}$ which has only one parameter free to tune. The parameter is updated by the descent algorithm (11). The optimum is in $\rho^c = 0.1$. As we can see in Fig. 2, by proposing such a reference model, the optimum becomes an unstable stationary point. Theorem 1 confirms that there might be problems with this setup. The non-minimum phase zero in $\boldsymbol{T}_{d1}$ makes the

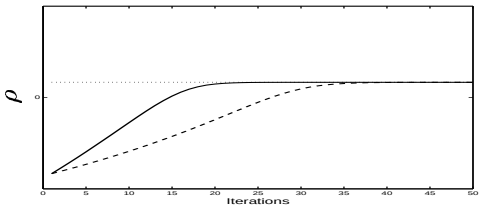

Fig. 3. Solid line-exact method, dashed line-approximation method 3, dotted horizontal line-optimal ρ.

matrix $\boldsymbol{T}_d \otimes \boldsymbol{T}_d^* + \boldsymbol{T}_d^* \otimes \boldsymbol{T}_d$ not positive definite for all ω. However, the non-minimum phase zero is not a fundamental problem for the gradient approximation method. By choosing another reference model, method 3 will converge to the true optimum, i.e. $\boldsymbol{T}_0 = \boldsymbol{T}_d$. This can be seen in Fig. 3, where $\boldsymbol{T}_d$ is chosen as

$$\boldsymbol{T}_{d2} = \begin{bmatrix} \dfrac{0.15}{z-0.85} & 0 \\ 0 & \dfrac{-0.2z+0.24}{z^2-1.6z+0.64} \end{bmatrix}. \qquad (32)$$

Since we use a P-controller, a necessary and sufficient condition that $\hat{\boldsymbol{J}}_{pp}(\boldsymbol{\rho}^c) > 0$ is that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \boldsymbol{T}_d \otimes \boldsymbol{T}_d^* + \boldsymbol{T}_d^* \otimes \boldsymbol{T}_d d\omega \qquad (33)$$

is positive definite. The expression (33) becomes indefinite for $\boldsymbol{T}_{d1}$ but positive definite for $\boldsymbol{T}_{d2}$.

6. CONCLUSIONS

In this paper we have examined several methods to approximate the gradient in IFT for MIMO systems. The local convergence for the approximation method in which operators are commuted in the same way as is done in IFT for SISO systems is further analyzed. The analysis shows that the optimum of the cost function, under certain conditions, also is a stationary point for the approximation algorithms, but not always stable. Furthermore, the numerical gradient search may still converge to the true optimum even if the commutation error is non-zero.

7. REFERENCES

Bruyne, F.De and P. Carrette (1997). "Synthetic generation of the gradient for an iterative controller optimization method". In: *Proceedings of European Control Conference*. Brussels, Belgium.

Ceysens, B. and B. Codrons (1997). Iterative identificationless control design. *Journal A* **38**(1), 26–30.

De Bruyne, F., B.D.O. Anderson, M. Gevers and N. Linard (1997). Iterative controller optimization for nonlinear systems. In: *Proc. 36th IEEE Conf. on Decision and Control*. San Diego, California.

Hjalmarsson, H. (1998). "Control of nonlinear systems using Iterative Feedback Tuning". In: *Proc. 1998 American Control Conference*. Philadelphia. pp. 2083–2087.

Hjalmarsson, H. (1999). "Efficient tuning of linear multivariable controllers using iterative feedback tuning". *International Journal on Adaptive Control and Signal Processing* **13**, 553–572.

Hjalmarsson, H., M. Gevers and O. Lequin (1997). Iterative feedback tuning: theory and applications in chemical process control. *Journal A* **38**(1), 26–30.

Hjalmarsson, H., M. Gevers and O. Lequin (1998). "Iterative Feedback Tuning: Theory and Applications". *IEEE Control Systems Magazine* **18**(4), 26–41.

Hjalmarsson, H., S. Gunnarsson and M. Gevers (1994). "A convergent iterative restricted complexity control design scheme". In: *Proc. 33rd IEEE Conf. on Decision and Control*. Orlando, FL. pp. 1735–1740.

Hjalmarsson, H., S. Gunnarsson and M. Gevers (1995). "Model-free tuning of a robust regulator for a flexible transmission system". In: *Proceedings of European Control Conference*. Rome, Italy. pp. 1975–1980.

Ljung, L. (1977). "Analysis of recursive stochastic algorithms". *IEEE Transactions on Automatic Control* **AC-22**(4), 551–575.

Meurers, T. and S.M. Veres (1999). Iterative design for vibration attenuation. *International Journal of Acoustics and Vibration* **4**(2), 79–83.

Sjöberg, J. and F. De Bruyne (1999). On a nonlinear controller tuning strategy. In: *14th IFAC World Congress*. Vol. I. Beijing, P.R. China. pp. 343–348.

Sjöberg, J. and M. Agarwal (1996). Model-free repetitive control design for nonlinear systems. In: *Proc. 35th IEEE Conf. on Decision and Control*. Vol. 4. Kobe, Japan. pp. 2824–2829.

Sjöberg, J. and M. Agarwal (1997). Nonlinear controller tuning based on linearized time-variant model. In: *Proc. of American Control Conference*. Albuquerque, New Mexico.

Trulsson, E. and L. Ljung (1985). "Adaptive control based on explicit criterion minimization". *Automatica* **21**(4), 385–399.

Zang, Z., R.R. Bitmead and M. Gevers (1995). Iterative weighted least-squares identification and weighted LQG control design. *Automatica* **31**, 1577–1594.