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PARTIAL STABILITY, STABILIZATION AND CONTROL: A SOME RECENT RESULTS

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Abstract: A some recent results concerning partial stability, stabilization and control are considered. Its applications to problems of partial stability, stabilization and control for some mechanical systems (solid, gyrostat, point in gravitational field) are given. *Copyright © 2002 IFAC*

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1. INTRODUCTION

The partial stability and stabilization problems (or, briefly, the PSt and PSb–problems) naturally arise in applications either from the requirement of proper performance of a system or in assessing system capability. In addition, a lot of actual (or desired) phenomena can be formulated in terms of these problems and be analyzed with these problems taken as the basis. The following multiaspect phenomena and problems can be indicated:

- adaptive stabilization;
- spacecraft stabilization (especially stabilization by rotors);
- drift of the gyroscope axis;
- Lotka-Volterra ecological principle, e.t.c.

Also very effective is the approach to the problem of stability (stabilization) with respect to *all* variables based on preliminary analysis of partial stability (stabilization).

A.M.Lyapunov (1893), the founder of the modern theory of stability, was the first to formulate the problem of partial stability. Later, work by V.V.Rumyantsev (1957) drew the attention of many mathematicians around the world to this problem. Some idea of the present state in this area may be obtained, for example, from Rumyantsev and Oziraner (1987), Vorotnikov (1993,1998), Fradkov et.al (1999), Vorotnikov and Rumyantsev (2001).

The PSt–problem is closely connected with two intensively investigated stability problems: stability with respect to *two measures* (Lakshmikantham and Liu, 1993) and *polystability* (polystability with respect to part of the variables) (Martynyuk, 1998). In addition, in the last years essentially more common and difficult problem of *input to output stability* have been singled out (Sontag and Wang, 1999, 2001).

The *partial control* problems, including game-theoretic control problems with respect to part of the variables are to the same (and even greater) extent natural for theory and applications. Scientifically and methodically, it is not devoid of interest to treat these problems together with the PSt and PSb–problems; see Vorotnikov (1998).

The paper has the following structure:

- general situations leading to investigation PSt and PSb–problems;
- classification of the PSt–problems;
- the basic methods for solving of the PSt(PSb)–problems;
- partial stabilization of the steady motions of a rigid body by rotor;
- distingvishing characteristics of PSt–problems;
- partial control problems.

2. GENERAL SITUATIONS LEADING TO INVESTIGATION PSt- PROBLEMS

Summing up the approaches to research within the scope of the PSt-problems and weighing the prospective ones, we shall attempt to distinguish the basic motives for investigating the problems. We consider it reasonable that the following general problems should be primarily numbered among them.

(1) Investigation of the stability of systems with so-called "superfluous" variables. This problem is closely related to the practically always inevitable problem of finding "*essential*" variables for a dynamic system under investigation.

(2) Investigation of those systems whose PSt-property is *sufficient* for them to operate adequately.

(3) Analysis of situations in which the PSt-property is either *natural*, desirable, or even necessary.

(4) Investigation of the PSt-problems when a system is inherently unstable with respect to *all* the variables. *Assessment of capabilities* of a system being designed in "*non-nominal*" situations is often meant in this case.

(5) Solution of the PSt-problems when investigation of stability with respect to *all* the variables *presents difficulties* or when it is necessary to estimate (with respect to certain variables) the transient of a system stable with respect to all the variables.

(6) Solution of the PSt-problems as an *auxiliary* one. Here, stability with respect to certain variables often implies stability with respect to the remaining variables. In other cases, stability with respect to all the variables may turn out to be conveniently proved through successive investigation (possibly by different methods) of several PSt-problems.

Similar reasons also cause the investigation of the PSb-problems.

3. CLASSIFICATION OF THE PSt-PROBLEMS

The following basic types of PSt-problems are presently considered:
– the Lyapunov-Rumyantsev PSt-problem;
– the problem of stability of "partial" equilibrium positions (with respect to all the variables);
– the PSt-problem of stability of "partial" equilibrium positions;
– the problem (PSt-problem) of polystability.

3.1. Lyapunov–Rumyantsev PSt-problem

Let there be given a nonlinear system of ordinary differential equations of *perturbed motion*

$$\dot{\mathbf{x}} = \mathbf{X}(t,\mathbf{x}),\ \ \mathbf{X}(t,\mathbf{0}) \equiv \mathbf{0}. \qquad (1)$$

The variables constituting the phase vector $\mathbf{x}$ of system (1) are divided into two groups $\mathbf{x}^{\mathrm{T}} = (\mathbf{y}^{\mathrm{T}}, \mathbf{z}^{\mathrm{T}})^{\mathrm{T}}$:
- the $\mathbf{y}$-variables with respect to which the stability of the unperturbed motion $\mathbf{x} = \mathbf{0}$ is to be investigated;
- the remaining $\mathbf{z}$-variables.

Specifically, this partitioning depends on the nature of the problem under study. As a rule, we assume that the choice of $\mathbf{y}$-variables *has already been made* by the time that a partial stability problem should be analyzed. This means that the PSt–problem is a problem of stability with respect to a *prescribed* part of the variables; $\mathbf{z}$-variables are correspondingly called the "*uncontrollable*" variables.

We denote by $\mathbf{x}(t) = \mathbf{x}(t; t_0, \mathbf{x}_0)$ the solution of system (1) subject to the initial conditions $\mathbf{x}_0 = \mathbf{x}(t_0; t_0, \mathbf{x}_0)$.

In the theory of stability with respect to part of the variables, the following assumptions are usually made (Rumyantsev and Oziraner, 1987):

(a) the right-hand sides of system (1) in the domain

$$t \geq 0,\quad \|\mathbf{y}\| \leq h,\quad \|\mathbf{z}\| < \infty, \qquad (2)$$

are *continuous* and satisfy *conditions of uniqueness* of solutions (for example, the local *Lipschitz condition*).

(b) solutions of system (1) are $\mathbf{z}$-*continuable* (Corduneanu, 1964), i.e., any solution $\mathbf{x}(t)$ is defied for all $t \geq 0$ for which $\|\mathbf{y}(t)\| \leq h$.

Definition 1 (Lyapunov, 1983; Rumyantsev, 1957; Corduneanu, 1964). The unperturbed motion $\mathbf{x} = \mathbf{0}$ of system (1) is said to be

(1) $\mathbf{y}$-*stable* ($\mathbf{y}$-*St)*, if for any numbers $\varepsilon > 0$, $t_0 \geq 0$, there is a number $\delta(\varepsilon, t_0) > 0$ such that from $\|\mathbf{x}_0\| < \delta$ it follows that $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t > t_0$;

(2) *uniformly* $\mathbf{y}$-*stable*, if in definition (1) number δ does not depend on t_0;

(3) *asymptotically* $\mathbf{y}$-*stable*, if it is $\mathbf{y}$-St and, besides, for each $t_0 \geq 0$, there is a number $\Delta(t_0) > 0$ such that each solution $\mathbf{x}(t; t_0, \mathbf{x}_0)$ with $\|\mathbf{x}_0\| < \Delta$ satisfies the condition

$$\lim \|\mathbf{y}(t; t_0, \mathbf{x}_0)\| = 0, \quad t \to \infty; \qquad (3)$$

domain $\|\mathbf{x}_0\| < \Delta$ being contained in the *domain of* $\mathbf{y}$-*attraction* of the point $\mathbf{x} = \mathbf{0}$ for the initial time t_0;

(4) *uniformly asymptotically* $\mathbf{y}$-*stable*, if in definition (3) number Δ does not depend on t_0 and relationship (3) holds uniformly with respect to $t_0, \mathbf{x}_0$ from the domain $t_0 \geq 0$, $\|\mathbf{x}_0\| < \Delta$;

(5) *exponentially asymptotically* $\mathbf{y}$-*stable* (Corduneanu, 1971), if there exist constants $\Delta > 0$, $M > 0$, and $\alpha > 0$ such that each solution $\mathbf{x}(t; t_0, \mathbf{x}_0)$ of system (1) satisfies the inequality

$$\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| \leq M(\|\mathbf{x}_0\|)e^{-\alpha(t-t_0)}, \quad t \geq t_0.$$

When studying the **y**-stability of the unperturbed motion $\mathbf{x} = \mathbf{0}$ of system (1), in principle, one need not monitor the behavior of **z**-variables (provided certain general conditions are observed). In the coupled system (1), however, they exert an important influence on the "main" **y**-variables.
Let us distinguish the factors that determine the admissibility of the "uncontrollable" **z**-variables (Vorotnikov, 1998).

(1) *Allowance for the "worst" case scenario* (*general conditions being the same*) *in the variation of "uncontrollable" variables.* This entails the assumption $\|\mathbf{z}\|< \infty$ and, consequently, the study of **y**-stability of the unperturbed motion $\mathbf{x} = \mathbf{0}$ of system (1) in domain (2).

Such considerations may prove overcautious. Indeed, one does not use inequalities $|z_j| < h$ that are valid (or admissible) for certain **z**-components or relationships like $|f_i(t,\mathbf{x})| < h$. Such relationships may considerably facilitate examining the system for **y**-stability. In a sense, allowance for the "worst" case scenario is comparable with the game-theoretic approach.

(2) *Allowance for specification of requirements imposed on the "uncontrollable" variables.* An alternative to the "worst" case scenario. This approach has various meanings.

Rationalizing the formulation of the PSt–problem. This requires "subjecting" the system to certain general estimates (possibly including integral estimates) for the "uncontrollable" variables. This significantly simplifies the solution. An example is the study of the stability of the motion of bodies containing cavities filled with liquid (Moiseev and Rumyantsev, 1968).

"Built-in" possibilities for facilitating the solution of PSt–problems. Put differently, the use of additional relationships linking the components of the phase vector of system (1). The validity of such relationships must somehow be confirmed when solving the problem. This approach provides the basis, for example, for the method of solving PSt–problems by constructing auxiliary systems being developed in Vorotnikov (1991, 1998).

(3) *Allowance for availability of estimates (even though rough) for "uncontrollable" variables.* In such cases the PSt–problem for system (1) can be reduced to a problem of stability with respect to all the variables in an auxiliary system of differential equations of the *same* dimensions (Zubov, 1962).

3.2. Stability of "partial" equilibrium position

Let there be given a nonlinear system of ordinary differential equations

$$\dot{\mathbf{y}} = \mathbf{Y}(t,\mathbf{y},\mathbf{z}), \quad \dot{\mathbf{z}} = \mathbf{Z}(t,\mathbf{y},\mathbf{z}), \quad \mathbf{Y}(t,\mathbf{0},\mathbf{z}) \equiv \mathbf{0}. \tag{4}$$

Definition 2 (Rumyantsev, Oziraner, 1987). The set $\mathbf{y} = \mathbf{0}$ of system (4) is said to be

(1) *stable,* if for any numbers $\varepsilon > 0$, $t_0 \ge 0$, there is a number $\delta(\varepsilon, t_0) > 0$ such that from $\|\mathbf{y}_0\| < \delta$, $\|\mathbf{z}_0\| < \infty$ it follows that $\|\mathbf{y}(t;t_0,\mathbf{x}_0)\| < \varepsilon$, $t \ge t_0$;

(2) *uniform stable*, if in definition (1) number δ does not depend on t_0;

(3) *asymptotically stable,* if it is stable and, besides, for each $t_0 \ge 0$, there is a number $\Delta(t_0) > 0$ such that each solution with $\|\mathbf{y}_0\| < \Delta$, $\|\mathbf{z}_0\| < \infty$ satisfies the condition (3).

(4) *uniform asymptotically stable*, if in definition (3) number Δ does not depend on t_0 and relationship (3) holds uniformly from the domain $t_0 \ge 0$, $\|\mathbf{y}_0\| < \Delta$, $\|\mathbf{z}_0\| < \infty$.

4.THE BASIC METHODS FOR SOLVING OF PSt(PSb)-PROBLEMS

These methods are following:
- Lyapunov's functions method (LFM);
- analysis of PSt (PSb)-problems in linear approximation.

The applicability of the LFM to the PSt-problem has been considerably extended along the following lines:
- by introducing various types of "limiting" systems (Andreev, 1991, Hatvani, 1991);
- by constructing various types of auxiliary systems (Zubov, 1962, Vorotnikov, 1991, 1998);
- by refining the notion of a *V*-function sign-definite with respect to part of the variables (Vorotnikov,1993,1998) and by reducing the admissible domain of variation of "uncontrolled" variables (Vorotnikov,1999);
- by using the LFM in conjunction with *asymptotic averaging* (Khapaev, 1993).

An analysis of PSt–problems in linear approximation can be found in Vorotnikov (1998, 1999).

4.1. The basic theorems about partial stability in context of MLF

In Sections 4.1-4.3 we will consider following functions: 1) functions $a(r)$, $b(r)$, $c(r)$ which are continuous, monotone increasing for $r \in [0,h]$, and such that $a(0) = b(0) = c(0) = 0$; 2) a scalar function $V(t,\mathbf{x})$, $V(t,\mathbf{0}) \equiv 0$ which is continuously differentiable in domain (2).

Theorem 1 (Rumyantsev, 1957). *Suppose that for system* (1) *a scalar function V exist such that the following conditions hold in domain* (2)

$$V(t,\mathbf{x}) \geq a(\|\mathbf{y}\|), \quad \dot{V}(t,\mathbf{x}) \leq 0.$$

Then the unperturbed motion $\mathbf{x} = \mathbf{0}$ *of system* (1) *is* **y**-*stability. If, in addition* (Corduneanu, 1964)

$$V(t,\mathbf{x}) \leq b(\|\mathbf{x}\|),$$

then the unperturbed motion $\mathbf{x} = \mathbf{0}$ *of system* (1) *is uniformly* **y**-*stability.*

Theorem 2 (Rumyantsev, Oziraner, 1987). *Suppose that for system* (4) *a scalar function V exist such that the following conditions hold in domain* (2)

$$a(\|\mathbf{y}\|) \leq V(t,\mathbf{x}) \leq b(\|\mathbf{y}\|), \quad \dot{V}(t,\mathbf{x}) \leq 0.$$

Then the set $\mathbf{y} = \mathbf{0}$ *of system* (4) *is uniformly stability.*

4.2. The unified conditions of solving PSt-problems

Conditions of solving the Lyapunov-Rumyantsev PSt-problem and the problem stability "partial" equilibrium positions can be made *the same*, if the notions of stability in these problems are modified in the following away.

Definition 3 (Vorotnikov, 1998, 2002). The unperturbed motion $\mathbf{x} = \mathbf{0}$ of system (1) is said to be *uniformly* **y**- *stable for large* $\mathbf{z}_0$, if for any numbers $\varepsilon > 0$, $t_0 \geq 0$ and *a given* number $L > 0$, there is a number $\delta(\varepsilon, t_0, L) > 0$ such that from $\|\mathbf{y}_0\| < \delta$, $\|\mathbf{z}_0\| < L$ it follows that $\|\mathbf{y}(t;t_0,\mathbf{x}_0)\| < \varepsilon$, $t \geq t_0$.

Definition 4 (Vorotnikov, 2002). The set $\mathbf{y} = \mathbf{0}$ of system (4) is said to be *uniformly stable for large* $\mathbf{z}_0$, if for any numbers $\varepsilon > 0$, $t_0 \geq 0$ and a given number $L > 0$, there is a number $\delta(\varepsilon, t_0, L) > 0$ such that from $\|\mathbf{y}_0\| < \delta$, $\|\mathbf{z}_0\| < L$ it follows that $\|\mathbf{y}(t;t_0,\mathbf{x}_0)\| < \varepsilon$, $t \geq t_0$.

Theorem 3 (Vorotnikov, 2002). *Suppose that for system* (1) *and* (4) *a scalar function V exist such that the following conditions hold in domain* (2)

$$V(t,\mathbf{x}) \geq a(\|\mathbf{y}\|), \quad V(t,\mathbf{0},\mathbf{z}) \equiv 0, \quad V(t,\mathbf{x}) \leq b(\|\mathbf{x}\|),$$
$$\dot{V}(t,\mathbf{x}) \leq 0.$$

Then: 1) *the unperturbed motion* $\mathbf{x} = \mathbf{0}$ *of system* (1) *is uniformly* **y**-*stability for large* $\mathbf{z}_0$;

2) *the set* $\mathbf{y} = \mathbf{0}$ *of system* (4) *is uniformly stability for large* $\mathbf{z}_0$.

4.3. A method of reducing the admissible domain of variation of "uncontrolled" variables

One modification (Vorotnikov, 1999) of LFM for solving PSt–problems reduces to adjusting the structure of the domain in which the Lyapunov functions are constructed. To elucidate: the domain (2) usually considered in studying $\mathbf{y}$ – stability of the position $\mathbf{x} = (\mathbf{y}^{\mathrm{T}}, \mathbf{z}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{0}$ of system (1) is *contracted,* being replaced by a domain

$$t \geq 0, \ \|\mathbf{y}\| + \|\mathbf{W}(t,\mathbf{x})\| \leq h, \ \|\mathbf{z}\| < \infty, \quad (5)$$

where $\mathbf{W}(t,\mathbf{x})$ is some vector function, which depends on t and the phase variables of system (1). In this case, naturally, the new condition $\|\mathbf{y}\| + \|\mathbf{W}(t,\mathbf{x})\| \leq h$ must be verified while the problem is being solved.

The main point in studying the problem of $\mathbf{y}$ – stability in domain (5) is that the $\mathbf{y}$ – stable position $\mathbf{x} = (\mathbf{y}^{\mathrm{T}}, \mathbf{z}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{0}$ of system (1) is always actually stable not only with respect to $\mathbf{y}$ but also with respect to certain functions $W_i = W_i(t,\mathbf{x})$. However, it is not always clear in advance just what W_i – functions are involved. In such a situation, suitable W_i – functions are naturally treated as an additional vector-valued Lyapunov $\mathbf{W}$ – function for the most rational substitute (5) for domain (2). When that is done it is not necessary to analyze the derivative of the $\mathbf{W}$ – function along trajectories of system (1), which is an added argument in favor of this approach.

Such an approach not only facilitates the construction of Lyapunov functions with appropriate properties, but also enables one to prove $\mathbf{y}$ – stability using functions which, even when $\dim(\mathbf{y}) = \dim(\mathbf{z}) = 1$, need not be of fixed sign (Vorotnikov, 1998) either with respect to $\mathbf{y}$ in Rumyantsev's sense (Rumyantsev, 1957; Rumyantsev and Oziraner, 1987) or in Lyapunov's sense.

The following theorem is developing of main Rumyantsev theorem (Rumyantsev, 1957) about partial stability.

Theorem 4 (Vorotnikov, 1998). *Suppose that for system* (1) *a scalar function V and vector function* $\mathbf{W}$ *exist such that the following conditions hold in domain* (5)

$$V(t,\mathbf{x}) \geq a(\|\mathbf{y}\| + \|\mathbf{W}(t,\mathbf{x})\|); \quad (6)$$

$$\dot{V}(t,\mathbf{x}) \le 0. \qquad (7)$$

Then the unperturbed motion $\mathbf{x}=\mathbf{0}$ *of system* (1) *is* **y**-*stability.*

Discussion of Theorem 4. A *V*-function satisfying inequality (6) in domain (5) also satisfies inequality $V(t,\mathbf{x}) \ge a(\|\mathbf{y}\|)$ in this domain, but not in domain (2). Consequently, it is not **y**-sign-definite in the sense of Rumyantsev (1957), Rumyantsev and Osiraner (1987).

Example 1. Consider the motion of a point of unit mass in a constant gravitational field, constrained to move on the surface

$$x_3 = f(x_1, x_2),\ f = x_1^2\left(1 + x_2^2\right)\left(1 + x_1^4 x_2^4\right)^{-1} \qquad (8)$$

in three-dimensional x_1, x_2, x_3 – space, with the x_3-axis pointing vertically upward.

The kinetic and potential energies are

$$T = \frac{1}{2}\left\{\dot{x}_1^2 + \dot{x}_2^2 + \left[\left(\frac{\partial f}{\partial x_1}\right)\dot{x}_1 + \left(\frac{\partial f}{\partial x_2}\right)\dot{x}_2\right]^2\right\},$$
$$\Pi = g f(x_1, x_2),\ g = \text{const} > 0 .$$

Putting $\mathbf{y} = (x_1, \dot{x}_1, \dot{x}_2), z = x_2$ and introducing auxiliary functions $V = T + \Pi$, $W = x_1 x_2$, we obtain

$$V \ge \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) + g\left(x_1^2 + W^2\right)\left(1 + W^4\right)^{-1}, \qquad (9)$$
$$\dot{V} \equiv 0 .$$

Consequently, conditions (6) and (7) hold in (5) for sufficiently small *h***.** As a result, the equilibrium position of the point

$$x_i = \dot{x}_i = 0,\ i = \overline{1,3} \qquad (10)$$

is **y** – stable by virtue of Theorem 4.

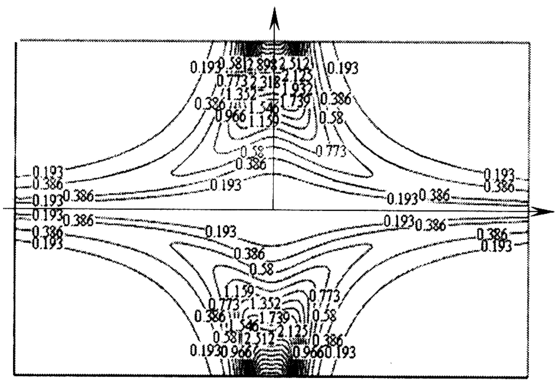


Fig. 1. The level surfaces $f = c$ of f – function (8).

At the same time, the V –function is not sign-definite whether relative to **y** in the sense of Rumyantsev (1957), Rumyantsev and Oziraner (1987) ($V \to 0$ for $\dot{x}_1 = \dot{x}_2 = 0$, $|x_2| \to \infty$ and any fixed x_1) or in Lyapunov's sense ($V = 0$ for $x_1 = \dot{x}_1 = \dot{x}_2$ and any x_2).

By (9), the position (10) is also W – stable. Summing up in view of (8), we conclude that this position is stable with respect to $x_1, x_2, \dot{x}_1, \dot{x}_2$ (including the case of large x_{20}).

The level surfaces $f = c$ of f – function (8) under $\dot{x}_1 = \dot{x}_2 = 0$ are shown in Figure 1.

Theorem 5 (Vorotnikov, 1999). *Suppose that for system* (1) *a scalar function V and two vector functions* **U** *and* **W** *exist such that the following conditions hold in domain* (5)

1. $V(t,\mathbf{x}) \ge a\left(\|\mathbf{y}\| + \|\mathbf{W}(t,\mathbf{x})\|\right);$
2. $\dot{V}(t,\mathbf{x}) \le -b\left(\|\mathbf{U}(t,\mathbf{x})\|\right);$
3. $\|\mathbf{U}(t,\mathbf{x})\| \ge c\left(\|\mathbf{y}\|\right);$
4. *a number* $M(t_0, \mathbf{x_0}) > 0$ *exists such that, for each of the functions* $\dot{U}_i$ *either* $\dot{U}_i \le M$ *or* $\dot{U}_i \ge -M$.

Then the unperturbed motion $\mathbf{x}=\mathbf{0}$ *of system* (1) *is asymptotically* **y** – *stable.*

Discussion of Theorem 5. 1) If $\mathbf{W} = \mathbf{0}$ Theorem 5 extends the corresponding results in Salvadori (1974), Rumyantsev and Oziraner (1987).

2) If $\mathbf{W} \ne \mathbf{0}$, then not only V and $\dot{V}$ but also $\|\mathbf{U}\|$ need not be sign-definite, either with respect to **y** (in Rumyantsev's sense) or in Lyapunov's sense. In addition, condition 4 may be verified in domain (5) but not in domain (2), and this extends the possibilities for using the theorem.

3) If $\mathbf{U} = \mathbf{y}$, condition 4 reduces to the requirement that each component of the vector function **Y,** defining the right-hand side of the first group of equations in system (1), should be bounded above or below. Therefore, when $\mathbf{U} = \mathbf{y}$, $\mathbf{W} = \mathbf{0}$, Theorem 5 reduces to a theorem Peiffer and Rouche (1969) which extends the classical result of Marachkov to the case of partial asymptotic stability.

4) The approach of using an additional Lyapunov vector function has also been used in Vorotnikov (1997a, 1998) to strengthen a number of theorems (Risito, 1970; Rumyantsev and Oziraner, 1987) on partial asymptotic for an autonomous system (1) (of the type of the Barbashin - Krasovskii theorem)

Example 2. Let system (1) be

$$\begin{aligned} \dot{y}_1 &= -y_1 + 2y_2 + e^t y_1 (z_1 z_2)^2 + \\ &\quad + y_1 z_1 z_2^2 z_3, \\ \dot{y}_2 &= -2y_1 - y_2 - e^t y_2 (z_2 z_3)^2, \\ \dot{z}_1 &= 2z_3 - 2e^t y_1^2 z_1, \quad \dot{z}_2 = e^t y_1^2 z_2, \\ \dot{z}_3 &= -2z_1. \end{aligned} \tag{11}$$

Let us consider the problem of asymptotic (y_1, y_2)–stability of the unperturbed motion $y_1 = y_2 = z_i = 0$, $i = \overline{1,3}$ of system (11). To do this, we introduce Lyapunov functions

$$\begin{gathered} V = y_1^2 + y_2^2 + (z_1 z_2)^2 + (z_2 z_3)^2, \\ \mathbf{W} = (W_1, W_2), \quad W_1 = z_1 z_2, \quad W_2 = z_2 z_3, \\ \mathbf{U} = (U_1, U_2), \quad U_1 = y_1^2, \quad U_2 = y_2^2. \end{gathered}$$

Positive constants l, M_1 and M_2 exist such that the following relations hold in domain (5)

$$\begin{gathered} V = y_1^2 + y_2^2 + W_1^2 + W_2^2 \geq a(\|\mathbf{y}\| + \|\mathbf{W}\|), \\ \dot{V} = -2(y_1^2 + y_2^2 - y_1^2 W_1 W_2) \leq -l(y_1^2 + y_2^2) \leq \\ \leq -b(\|\mathbf{U}\|), \quad \|\mathbf{U}\| \geq c(\|\mathbf{y}\|), \\ -M \leq \dot{U}_1 = 2y_1(-y_1 + 2y_2 + e^t y_1 W_1^2 + y_1 W_1 W_2), \\ \dot{U}_2 = 2y_2(-2y_1 - y_2 - e^t y_2 W_2^2) \leq M_2. \end{gathered}$$

Consequently, the functions $V, \mathbf{W}$ and $\mathbf{U}$ satisfy all the conditions of Theorem 5. Hence the equilibrium position $y_1 = y_2 = z_i = 0$, $i = \overline{1,3}$ of system (11) is asymptotically (y_1, y_2)–stable.

Note that the relation $\dot{V} \leq -l(y_1^2 + y_2^2)$ is not guaranteed in domain (2), that is, the function $\dot{V}$ need not be $\mathbf{y}$-sign-definite in Rumyantsev's sense.

4.4. Generalization of the Lyapunov - Malkin theorem

Let us we present system (1) as two groups of equations

$$\begin{aligned} \dot{\mathbf{y}} &= A(t)\mathbf{y} + B(t)\mathbf{z} + \mathbf{Y}(t, \mathbf{y}, \mathbf{z}), \\ \dot{\mathbf{z}} &= C(t)\mathbf{y} + D(t)\mathbf{z} + \mathbf{Z}(t, \mathbf{y}, \mathbf{z}), \end{aligned} \tag{12}$$

where A, B, C and D are matrix functions of t appropriate dimensions, whose elements are functions continuous in $t \in [0, +\infty)$. The non-linear perturbations $\mathbf{Y}$ and $\mathbf{Z}$ are continuous and satisfy the conditions of the existence and uniqueness theorems in the domain $t \geq 0, \|\mathbf{x}\| \leq h = \text{const} > 0$.

We assume that the following conditions are satisfied (Rumyantsev and Oziraner, 1987)

$$\begin{gathered} \mathbf{Y}(t, \mathbf{0}, \mathbf{0}) \equiv \mathbf{Y}(t, \mathbf{0}, \mathbf{z}) \equiv \mathbf{0}, \\ \mathbf{Z}(t, \mathbf{0}, \mathbf{0}) \equiv \mathbf{Z}(t, \mathbf{0}, \mathbf{z}) \equiv \mathbf{0}, \\ \frac{\|\mathbf{Y}(t, \mathbf{y}, \mathbf{z})\| + \|\mathbf{Z}(t, \mathbf{y}, \mathbf{z})\|}{\|\mathbf{y}\|} \Rightarrow 0 \\ \text{as } \|\mathbf{y}\| + \|\mathbf{z}\| \to 0. \end{gathered} \tag{13}$$

Theorem 6 (Vorotnikov, 1999). *Let the trivial solution of the linear system*

$$\dot{\mathbf{y}} = A(t)\mathbf{y} + B(t)\mathbf{z}, \; \dot{\mathbf{z}} = C(t)\mathbf{y} + D(t)\mathbf{z} \tag{14}$$

be uniformly stable in Lyapunov's sense and (simultaneously) exponentially asymptotically $\mathbf{y}-$*stable. Then, if conditions* (13) *are satisfied, the unperturbed solution* $\mathbf{y} = \mathbf{0}$, $\mathbf{z} = \mathbf{0}$ *of the non-linear system* (12) *has the same stability property.*

Discussion of Theorem 6. 1) Theorem 6 extends certain results in Rumyantsev and Oziraner (1987); Vorotnikov (1998), Malkin (1966). In Vorotnikov (1998) the matrix functions *A, B, C* and *D* are independent of *t;* also in Malkin (1966) additionally $B \equiv 0$, $D \equiv 0$ (all elements of the matrices *B* and *D* vanish identically). In Rumyantsev and Oziraner (1987) the matrix functions *A*, *C* and *D* depend on t, but $B \equiv 0$ and, in addition, all elements of the matrix functions *A* and *C* are bounded for $t \in [0, +\infty)$.

2) Confining themselves to the case $B \equiv 0$, in Rumyantsev and Oziraner (1987) considered the more general class of $\mathbf{Z}-$non-linearities, while in Vorotnikov (1998) the more general class of $\mathbf{Y}-$non-linearities is considered. In Vorotnikov (1998), however, the matrix functions *A*, *C* and *D* do not depend on *t*.

Example 3. The equations of angular motion of a rigid body about its center of mass under the action of linear torques are

$$\begin{gathered} \dot{\mathbf{x}} = L(t)\mathbf{x} + \mathbf{X}^*(\mathbf{x}), \; \mathbf{x} = (y_1, y_2, z_1)^{\mathrm{T}}, \\ \mathbf{X}^* = [(J_2 - J_3)J_1^{-1} y_2 z_1, (J_3 - J_1)J_2^{-1} y_1 z_1, \\ (J_1 - J_2)J_3^{-1} y_1 y_2]^{\mathrm{T}}, \end{gathered} \tag{15}$$

where y_1, y_2, z_1 are the projections of the angular velocity vector $\mathbf{x}$ of the body onto the principal central axes of inertia, J_i are the principal central moments of inertia, and L is a 3×3 matrix whose elements are functions of $t \in [0, +\infty)$ characterizing the action of linear torques of dissipative and accelerating forces on the body.

Suppose the trivial solution $\mathbf{x} = (y_1, y_2, z_1)^{\mathrm{T}} = \mathbf{0}$ of the linear system

$$\dot{\mathbf{x}} = L(t)\mathbf{x} \qquad (16)$$

is uniformly stable in Lyapunov's sense and (simultaneously) exponentially asymptotically (y_1, y_2) – stable.

The structure of the non-linear terms in system (15) is such that they satisfy conditions (13). Therefore, we conclude from Theorem 6 that the aforementioned stability property for linear system (16) also holds for the equilibrium position $\mathbf{x} = (y_1, y_2, z_1)^{\mathrm{T}} = \mathbf{0}$ of non-linear system (15).

Note that system (15) does not satisfy all the conditions of the Lyapunov - Malkin theorem as stipulated in Rumyantsev and Oziraner (1987); Vorotnikov (1998), Malkin (1966).

5. PARTIAL STABILIZATION OF THE STEADY MOTIONS OF A RIGID BODY

In applications, stabilization of the steady motions of a rigid body (such as a spacecraft) is frequently achieved by means of rotating masses attached to the body: flywheels and or power gyroscopes. In the stabilization process, these masses "take upon themselves" perturbations which occur as a result of the body's deviation from a given state (Junkins and Turner, 1986; Vorotnikov, 1998).

We will show, however, that if the steady motions of the rigid body are stabilized *only partially* (that is, with respect to part of the variables), which is sufficient in many cases of practical importance, then the masses attached to the body may *only "transfer"* (without "taking upon themselves") perturbations to the part of the variables not controlled by the stabilization.

Suppose we have an asymmetric rigid body, with the axis of rotation of a uniform symmetric flywheel attached along one of the principal central axes of inertia of the body. The angular motion of the (gyrostat) system about its center of mass is described by the equations (Vorotnikov, 1998)

$$\begin{gathered} (J_1 - A_1)\dot{x} = (J_2 - J_3)x_2 x_3 - u_1, \\ J_2 \dot{x}_2 = (J_3 - J_1)x_1 x_3 - A_1 x_3 \dot{\varphi}, \\ J_3 \dot{x}_3 = (J_1 - J_3)x_1 x_2 - A_1 x_2 \dot{\varphi}, \\ A_1(\ddot{\varphi} + \dot{x}_1) = u_1, \end{gathered} \qquad (17)$$

where J_i are the principal central moments of inertia of the gyrostat, x_i are the projections of the angular velocity vector of the main body onto the principal central axes of inertia s_i of the gyrostat, A_1 and $\dot{\varphi}$ are the axial moment of inertia and angular velocity of the flywheel's own motion and u_1 is the controlling torque applied to the flywheel.

Equations (17) have the solution

$$\begin{gathered} x_1 = x_2 = 0,\ x_3 = \omega = \text{const} > 0, \\ \dot{\varphi} = 0,\ u_1 = 0 \end{gathered} \qquad (18)$$

corresponding to permanent rotation ("twist") of the main body of the gyrostat at a constant angular velocity co about the s_3 axis. In this motion the flywheel, whose axis of rotation is attached along the s_1 axis, is fixed relative to the main body, while the direction of the vector $\mathbf{K}$ of angular momentum of the gyrostat coincides with the direction of the s_3 axis.

Introducing new variables $y_j = x_j (j = 1,2)$, $y_3 = \dot{\varphi}$, $z_1 = x_3 - \omega$, we set up a system of equations for the deviations from solution (18)

$$\begin{gathered} (J_1 - A_1)\dot{y}_1 = (J_2 - J_3)y_2(z_1 + \omega) - u_1, \\ J_2 \dot{y}_2 = [(J_3 - J_1)y_1 - A_1 y_3](z_1 + \omega), \\ (J_1 - A_1)\dot{y}_3 = (J_3 - J_2)y_2(z_1 + \omega) + J_1 A_1^{-1} u_1, \\ J_3 \dot{z}_1 = [(J_1 - J_2)y_1 + A_1 y_3]y_2 . \end{gathered} \qquad (19)$$

Let us consider the problem of *partial stabilization* of the motion $\mathbf{x} = (y_1, y_2, z_1)^{\mathrm{T}} = \mathbf{0}, z_1 = 0$ of system (19): $\mathbf{y}$ – stabilization by means of the control u_1. In this context, stabilization with respect to y_1, y_2 means that one must suppress small precessional and nutational oscillations of the angular momentum vector $\mathbf{K}$ of the gyrostat about the s_i axes attached to the body. Additional stabilization with respect to y_3 means that in the process of the (y_1, y_2)-stabilization, the flywheel only *"transfers"* the small perturbations to the "additional rotation" of the gyrostat about the s_3 axis of rotation.

Proposition (Vorotnikov, 1998). *If* $J_2 \neq J_3$ *solution of the* $\mathbf{y}$ – *stabilization problem for unperturbed motion* $\mathbf{y} = \mathbf{0}$, $z_1 = 0$ *of system* (19) *yields the control law*

$$u_1 = P\mathbf{y}, \qquad (20)$$

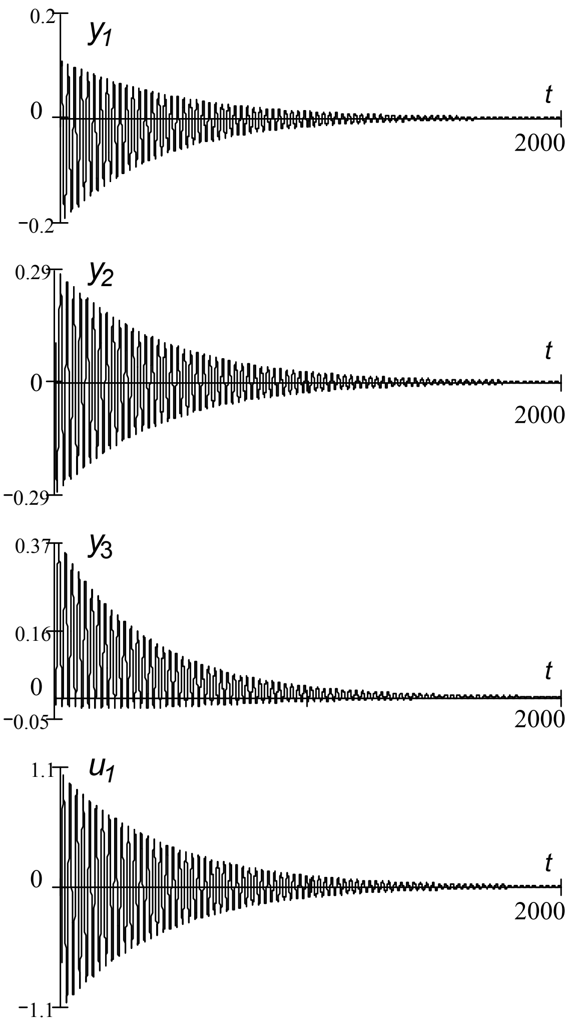
where P is some constant 1×3 row-vector.

Let us consider the linear subsystem describing the behavior of the $\mathbf{y}$ – variables of the linear part of system (19). If $J_2 \neq J_3$, this subsystem is completely controllable. Therefore the coefficients of the vector P in (20) may be chosen so that the trivial solution $\mathbf{y} = \mathbf{0}$, $z_1 = 0$ of the non-linear part of system (19) will be uniformly Lyapunov-stable and (simultaneously) exponentially $\mathbf{y}$ – stable.

The right-hand sides of system (19) vanish at $\mathbf{y} = \mathbf{0}$. Therefore, by Theorem 6, the stability property specified for the linear part of system (19) will also hold for the unperturbed motion $\mathbf{y} = \mathbf{0}$, $z_1 = 0$ of the non-linear system (19).

Remark. In technical terms, implementation of control law (20) reduces to the following. As long as the gyrostat is performing the given motion (18), the flywheel is at rest (control drive switched off). In the event of small perturbations, special devices produce a control torque (20) and transmit it to the flywheel. As a result, the main body of the gyrostat returns in time to its original steady rotation, and the flywheel to its state of rest.

Example 4. Let us consider a computer simulation system (19), (20) in the case $J_1 = 900$, $J_2 = 600$, $J_3 = 500$, $A_1 = 100$ $(kg \cdot m^2)$, $\omega = 0.4$ (s^{-1}), $y_{10} = y_{20} = 0.1\,(s^{-1})$, $y_{30} = z_{10} = 0$. We suggest that $|u_1| \le 1$ $(N \cdot m)$. In this case $P = (-3.5;\ -3.5;\ -1.1)$ $(N \cdot m \cdot s)$.



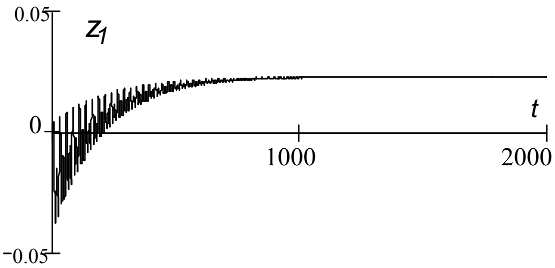


Fig. 2-6. Function y_i ($i = \overline{1,3}$), u_1, z_1.

Practical full damping of disturbances with respect to variables y_1, y_2, as well as practical full stopping of flywheel takes about 1500 (s).

Function y_i ($i = \overline{1,3}$), u_1, z_1 are shown in Figures 2-6.

6. DISTINGVISHING CHARACTERISTICS OF PSt-PROBLEMS

The point is that the PSt-theory deals with rather delicate properties of the system. The necessary "persistence" of these properties depends on a larger number of factors than do the properties of "overall" stability. What is required is a deeper understanding of the nature of PSt-problems, of the laws governing the performance of PSt-systems, and of the mechanisms through which PSt-properties may be achieved (or lost). One needs also to recognize pitfalls along the path to practical application of alluring possibilities of PSt-problems.

Now we formulate a few assumptions (Vorotnikov, 1993, 1998) that afford a deeper understanding of the special features of PSt-system performance.

1. *The predictability of structural changes as a precondition for normal PSt-system performance.* This assumption is motivated by the greater sensitivity of the PSt-property (as compared with stability with respect to all the variables) to changes in system structure. The implication is that the idea of "robustness" in the PSt-theory cannot be as general as it is in the theory of stability with respect to all the variables. This is natural, because PSt-theory is concerned with more "delicate" cases, in which "improved" or "better" stability is simply impossible. In addition, as was already noted, PSt-properties are sometimes not just desirable but absolutely necessary, whereas universal auxiliary function of the PSt-problem can be revealed in establishing various properties of the system regarding its "robustness."

A better understanding of the problem may also be gained by clarifying the nature of the relationships among notions that determine whether PSt-properties are preserved. Among the latter are PSt-properties in the presence of constantly acting perturbations (CAP) and parametric perturbations.

2. The *PSt-problem in the presence of CAP is not generally equivalent to the PSt-problem of preserving stability even under small parametric perturbations.* This is even the case for linear autonomous systems. Though partially stable in the presence of CAP, such systems may lose their stability when even a slight "stir" is given to certain coefficients. This does not occur in the problem of stability with respect to all the variables.

Owing to this conclusion, any decision about whether to use the results of PSt-theory must be made at the design stage in each specific case. Here it is important to have constructively verifiable conditions for preserving PSt-properties at the researcher's disposal

At the other end of the "fragility" scale for PSt-properties one has the following.

3. *A system that loses the PSt-property is nevertheless frequently "coarse" in the Andronov-Pontryagin sense.* In principle, the phase portrait of a "coarse" system is invariant under minor "stirs" of the parameters. Hence the loss of PSt-properties in such cases implies only a certain *"rotation"* of the phase portrait in the corresponding phase plane.

4. *The possibility of the invariance of PSt-properties in the presence of arbitrarily large CAP* in certain channels of system (1). This question is related to the general problem of invariance

7. THE PARTIAL CONTROL PROBLEMS

The problematics of PSt- and PSb-problems are closely related to the problem of *control with respect to part of the variables* (partial control) in a finite time interval. Being rather natural for many controlled systems, this problem has long been studied in the literature. Apart from, the term "control with respect to part of the variables" (Krasovskii, 1968; Vorotnikov, 1998; Kovalev, 1994) there are a number of widely used terms:

– "control in output" (Desoer and Vidyasagar, 1975);
– "control with free endpoints" (Pontryagin et al., 1962);
– "control in configuration space" (Roitenberg 1987);
– "control on manifolds" (Zubov, 1980).

Besides, the control problems of the "*hard encounter*" type also relate to problems of control with respect to part of the variables. In such problems it is required to ensure a rendezvous of two controlled objects with arbitrary velocities at the time of encounter. In particular, these problems relate to military applications: hitting mobile and immobile targets. A new class of the problems of the "hard encounter" type is distinguished in Vorotnikov (1997b, 1998). These are the problems of reorienting a spacecraft without damping its final angular velocity. As a result, a spacecraft just "passes" through a given angular position without making a stop in it.

We also note that the problems of control with respect to part of the variables (and the problems of partial stabilization) naturally arise when a controller falls into the class of dynamic regulators. In this case, the variables characterizing the state of the main object are taken as the controlled variables (or the variables being stabilized).

Two principal situations that provide motives stimulating the research of these problems can distinguished:

(1) when it is *sufficient* to solve control problems only with respect to part of the variables characterizing the state of a system and

(2) when controllability of a system with respect to all the variables is *not possible* at all (for example, because the system possesses some first integrals).

A number of methods are known for investigating the problems of control including those of optimal control) with respect to part of the variables for nonlinear systems, among which are the following:

– Pontryagin's maximum principle for problems with free endpoints (Pontryagin et. al., 1962).
– methods of the theory of games in the case of control in the presence of uncontrollable interference (Krasovskii and Subbotin, 1988);
– the asymptotic method (Akulenko, 1994);
– the method of oriented manifolds (Kovalev, 1994);
– the method of nonlinear transformations of the variables combined with a special choice of control structure (Vorotnikov, 1997b, 1998).
– Lyapunov function method (Fradkov, et. al., 1999).

8. THE PROBLEM OF REORIENTATING A SPACECRAFT

Consider Euler dynamic equations

$$\begin{aligned} J_1\dot{x}_1 &= (J_2 - J_3)x_2x_3 + u_1, \\ J_2\dot{x}_2 &= (J_3 - J_1)x_1x_3 + u_2, \\ J_3\dot{x}_3 &= (J_1 - J_2)x_1x_2 + u_3, \end{aligned} \tag{21}$$

which describe the angular motion of a solid body with respect to its center of mass. Here, x_i are the projection of angular velocity on major central axes of inertia of the body, u_i are the projection of controlling moments on the same axes, and J_i are the principal central moments of inertia. Here and below, $i = \overline{1,3}$; and summation in i from 1 to 3 is assumed.

Let us denote by **x**, **u** the vectors that consist, respectively, of x_i, u_i.

In addition to (21), let us consider the kinematic equations that determine the orientation of the body in Rodriges-Hamilton variables

$$\begin{aligned} 2\dot{\lambda}_0 &= -\sum (x_i \lambda_i), \\ 2\dot{\lambda}_1 &= x_1\lambda_0 + x_3\lambda_2 - x_2\lambda_3, \\ 2\dot{\lambda}_2 &= x_2\lambda_0 + x_1\lambda_3 - x_3\lambda_1, \\ 2\dot{\lambda}_3 &= x_3\lambda_0 + x_2\lambda_1 - x_1\lambda_2. \end{aligned} \qquad (22)$$

Variables λ_0, λ_i that constitute the vector $\boldsymbol{\lambda}$ obey the equation

$$\lambda_0^2 + \sum \lambda_i^2 = 1 . \qquad (23)$$

Controls $\mathbf{u} \in K$ are selected within the class K of functions $\mathbf{u} = \mathbf{u}(\mathbf{x}, \boldsymbol{\lambda}; \mathbf{x}^0, \boldsymbol{\lambda}^0)$ ($\mathbf{x}^0, \boldsymbol{\lambda}^0$ – initial states) with constraints

$$| u_i | \le \alpha_i = \text{const} > 0. \qquad (24)$$

Problem 1. Find the controls $\mathbf{u} \in K$ that transfer the body in a finite time from the initial state $\boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}^0$ into the final state $\boldsymbol{\lambda}(t_1) = \boldsymbol{\lambda}^1$. The initial state is the rest state $\mathbf{x}(t_0) = \mathbf{x}^0 = \mathbf{0}$. The angular velocity $\mathbf{x}^1 = \mathbf{x}(t_1)$ can be arbitrary at the moment $t = t_1$. The moment t_1 is not fixed.

The time moment $t_1 > t_0$ is not fixed. Without losing the generality, we assume $\boldsymbol{\lambda}^1 = (1,0,0,0)$.

Controls

$$\begin{aligned} u_1 &= 2J_1\lambda_0^{-1}[(\lambda_0^2 + \lambda_1^2)u_1^* + \\ &(\lambda_1\lambda_2 + \lambda_0\lambda_3)u_2^* + (\lambda_1\lambda_3 - \lambda_0\lambda_2)u_3^*] + \\ &+ \frac{1}{2}J_1\lambda_1\lambda_0^{-1}\sum x_i^2 - M_1, \\ M_1 &= (J_2 - J_3)x_2x_3 \quad (123) \end{aligned} \qquad (25)$$

are implied in Vorotnikov (1997b, 1998) for solving Problem 1. (Only one is explicitly written; the rest are obtained by a cyclic permutation of indices 1→ 2→ 3.)

As the result, a linear controlled system

$$\ddot{\lambda}_i = u_i^* \qquad (26)$$

can be constructed.

In accordance with the goals of Problem 1 as applied to system (26), we solve the control problem for a subset of variables on the fastest attainment of position

$$\lambda_i = 0 . \qquad (27)$$

The final values of $\dot{\lambda}_i$, can be arbitrary. By virtue of this fact, in view of the sense of Problem 1, controls u_i^* should *simultaneously* bring the variables λ_i, into position (27).

To solve this problem, we must first set the admissible levels for u_i^*. Following (24), we introduce the appropriate constraints in the form

$$| u_i^* | \le \alpha_i^* = \text{const} > 0 .$$

When choosing the constants α_i^*, we must take into account two circumstances: (i) the *simultaneous* attainment of position (27) by all variables λ_i; and (ii) the feasibility of constraints (24) for the initial controls u_i.

For fixed α_i^*, the solution to the optimum-response problem in a subset of variables for system (26) is provided by controls (Pontryagin et. al., 1962)

$$u_i^* = -\alpha_i^* \text{sgn}\lambda_i , \qquad (28)$$

Relations

$$\tau = \left[2 | \lambda_i^0 | (\alpha_i^*)^{-1} \right]^{1/2}$$

determine the minimum time τ, which is necessary to attain position (27) (and is *the same* for all variables λ_i).

The trajectories of the linear system (26), (28), which correspond to the case $\mathbf{x}^0 = \mathbf{0}$, have the form

$$\dot{\lambda}_i = -\alpha_i^* (\text{sgn}\lambda_i^0) t , \; \lambda_i = \lambda_i^0 - \frac{1}{2}\alpha_i^* (\text{sgn}\lambda_i^0) t^2 . \quad (29)$$

Algorithm for solving Problem 1.

1) Construct controls u_i, with u_i^* of the form (28) according to (26).

2) Assign the levels α_i^* for auxiliary controls u_i^* assuming the fixed values of λ_i^0; and *equalize* the control time for all variables λ_i. The levels α_i^* determine the corresponding value of τ.

3) Check the feasibility of the given constraints (24) as applied to original controls u_i. In order to do this, we use equalities

$$x_1 = 2\lambda_0^{-1}[(\lambda_0^2+\lambda_1^2)\dot{\lambda}_1 + (\lambda_1\lambda_2+\lambda_0\lambda_3)\dot{\lambda}_2 + (\lambda_1\lambda_3-\lambda_0\lambda_2)\dot{\lambda}_3] \qquad (123), \qquad (30)$$

which are obtained by solving equations for $\dot{\lambda}_i$ in (22) in terms of x_i. Taking into account (30), we conclude that the checking is feasible on trajectories (29) of the *linear* system (26), (28). Note that the calculations are simplified by using the relationship

$$\sum x_i^2 = 4\{[\lambda_0^{-1}\sum(\lambda_i\dot{\lambda}_i)]^2 + \sum(\dot{\lambda}_i^2)\}.$$

If the estimates (24) do not hold, or, on the contrary, hold with an excess, it is necessary to continue the search for appropriate values of α_i^*. Otherwise, the reorientation is realized in time τ.

As a result, we obtain an iterative algorithm of the solution to Problem 1. For each fixed set of values of λ_i^0, this algorithm can be easily implemented in real time.

Remark. The control structure" (25) contains a factor λ_0^{-1}, which, formally speaking, generates a "singularity". However, a more detailed subsequent analysis shows that the relation $|\lambda_0| \in [|\lambda_0^0|, 1]$ holds in the process of control when $\boldsymbol{\lambda}^1 = (1,0,0,0)$. Therefore, the virtual singularity does not actually appear. If $\boldsymbol{\lambda}^1 \neq (1,0,0,0)$ or if the value of λ_0^0 is small, then it is sufficient to generate new controls by applying the permutation of indices to (25) or to use a combination of controls obtained in this way. The final choice of controls u_i, is performed iteratively, which is characteristic for many modern methods of applied control theory. Within the approach proposed, the iterative search for controls is simple and can be performed in real-time conditions.

Theorem 7 (Vorotnikov, 1997b, 1998) *For all values* $\boldsymbol{\lambda}^0$ *and* $\boldsymbol{\lambda}^1$, *Problem* 1 *is solved by using arbitrarily small controls* u_i *of the form* (25), (28) or *controls obtained from these by a permutation of indices.*

Example 5. For a solid body for which $J_1 = 4 \times 10^4$, $J_2 = 8 \times 10^4$, and $J_3 = 5 \times 10^4$ $(kg \cdot m^2)$, we consider a triaxial reorientation from position

$$\mathbf{x}^0 = \mathbf{0}, \ \boldsymbol{\lambda}^0 = (0.701; 0.353; 0.434; 0.432)$$

to that of

$$\boldsymbol{\lambda}^1 = (1,0,0,0).$$

The value of $\mathbf{x}^1$ is arbitrary.

In the case $\mathbf{x}^0 = \mathbf{x}^1 = \mathbf{0}$, reorientation is completed in a single spatial turn with time $\tau = 70$ (s) by using controls (25) with $\alpha_1 = 32.6$, $\alpha_2 = 80.1$, $\alpha_3 = 68.0$ $(N \cdot m)$. Controls u_i, are piecewise continuous and involve a single switching at the moment $t = 35$ (s).
In the same time interval $\tau = 70$ (s), reorientation is performed within the framework of Problem 1 in a single turn, as well, and using the values $\alpha_1 = 29.9$, $\alpha_2 = 40.1$, $\alpha_3 = 24.9$ $(N \cdot m)$. The controls are *continuous* and vary within the range $-29.9 \le u_1 \le -16.3$, $-40.1 \le u_2 \le -24.4$, $-24.9 \le u_3 \le 2.4$ $(N \cdot m)$. Equal reorientation time is attained within the framework of problem 1 for the value of $\sum \alpha_i$, which is lower by 41.6% than that obtained in the case of $\mathbf{x}^0 = \mathbf{x}^1 = \mathbf{0}$.

Calculations also show that for $\max \alpha_1 \le 80.1$ $(N \cdot m)$, reorientation is performed within the framework of Problem 1 in 49.5 (s). The controls are *continuous,* and only u_2 reaches the limiting value, whereas $-32.6 \le u_1 \le -59.8$, $-49.9 \le u_3 \le 4.6$ $(N \cdot m)$. With the same "geometrical" constraints imposed on u_i, the time gain in the reorientation within the framework of Problem 1 is 29.3% (see Figure 7).

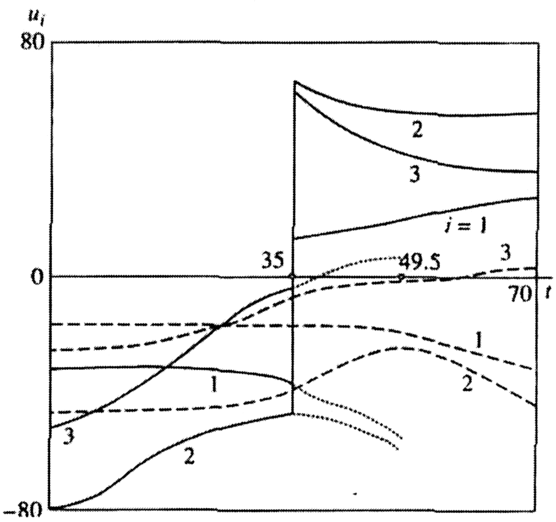


Fig. 7. Solid lines show the controls u_i, with a discontinuity at $t = 35$ (s), which solve the reorientation problem for $\mathbf{x}^0 = \mathbf{x}^1 = \mathbf{0}$ at time $\tau = 70$ (s). Broken lines show the continuous controls u_i, which solve Problem 1 at time $\tau = 70$ (s). Solid lines at $0 \le t \le 35$ (s) continued as broken lines at $35 \le t \le 49.5$ (s) show the continuous controls u_i, that solve Problem 1 at time $t = 49.5$ (s) for $\max \alpha_i \le 80.1$ $(N \cdot m)$.

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